Quantum Communication & Computation Using Spin Chains

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Quantum Computation Part:

S. C. Benjamin & S. Bose, **quant-ph/0210157 (to appear in PRL)**

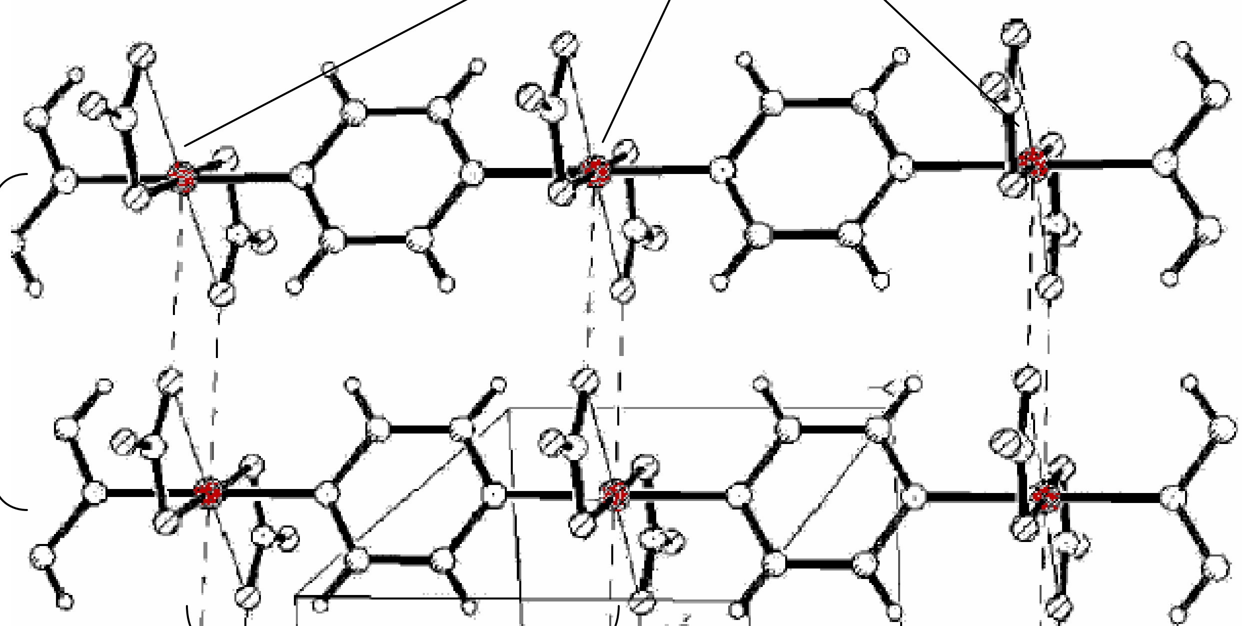
Quantum Communication Part:

S. Bose, **quant-ph/0212041**

1D Bulk Magnets are Natural Spin Chains (Examples):

$Cu(C_4H_4N_2)(NO_3)_2$

Cu (spin ½ sites)



$J'/J < 10^{-4}$

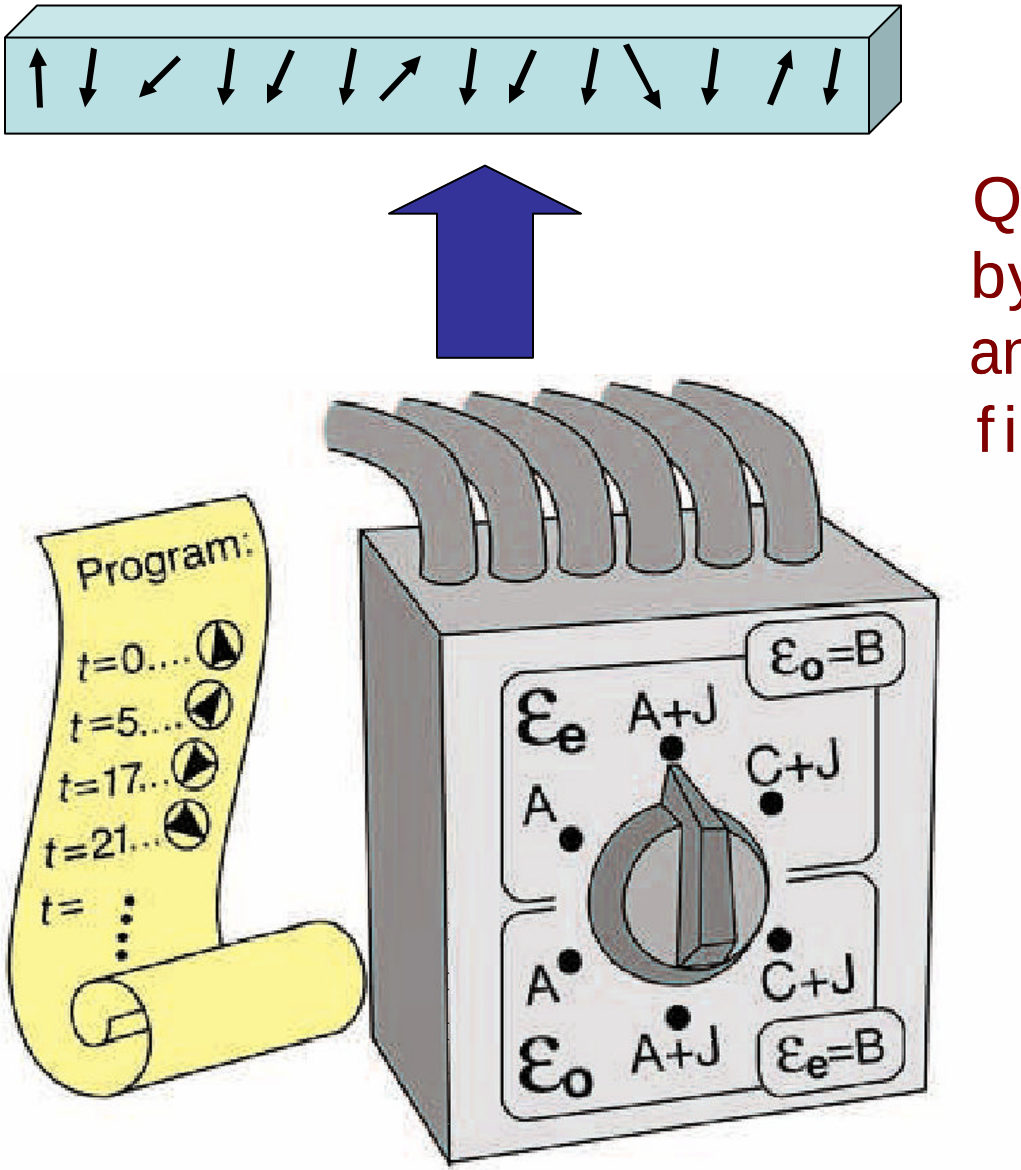
$J/k_B \approx 10.6\ \ \mathrm{K}$

Isotropic Heisenberg Antiferromagnet:

$$\mathcal{H} = \sum_i \left[J\mathbf{S}_i \cdot \mathbf{S}_{i+1} - g\mu_B \mathbf{H} \cdot \mathbf{S}_i \right]$$

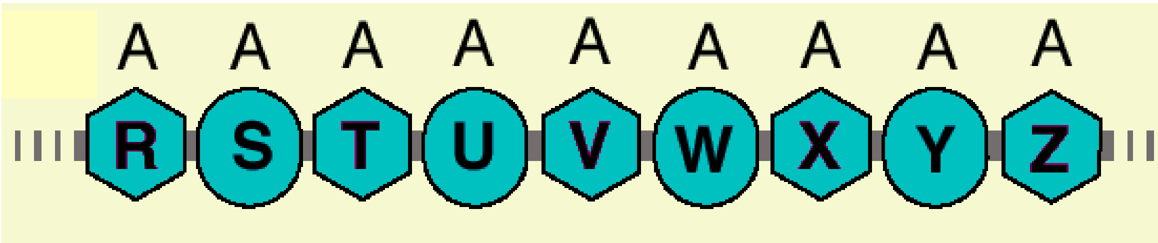
P. R. Hammar *et. al.*, PRB **59**, 1008 (1999).

Quantum computation using a 1D magnet



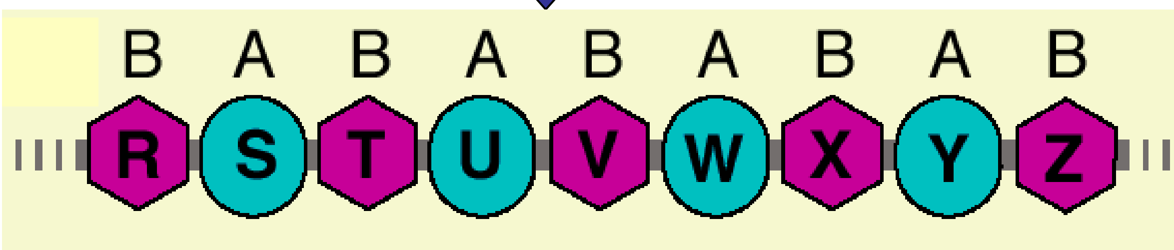
Quantum computation by applying a *time varying* and *inhomogeneous* magnetic field to a spin chain.

Heisenberg Chain to Ising Chain Conversion:



Heisenberg

A, B = Zeeman Energies,
$|A\text{-}B| >> J$



Ising

If $\hat{H} = \hat{H}_{\text{Zeeman}} + \hat{H}_{\text{int}}$

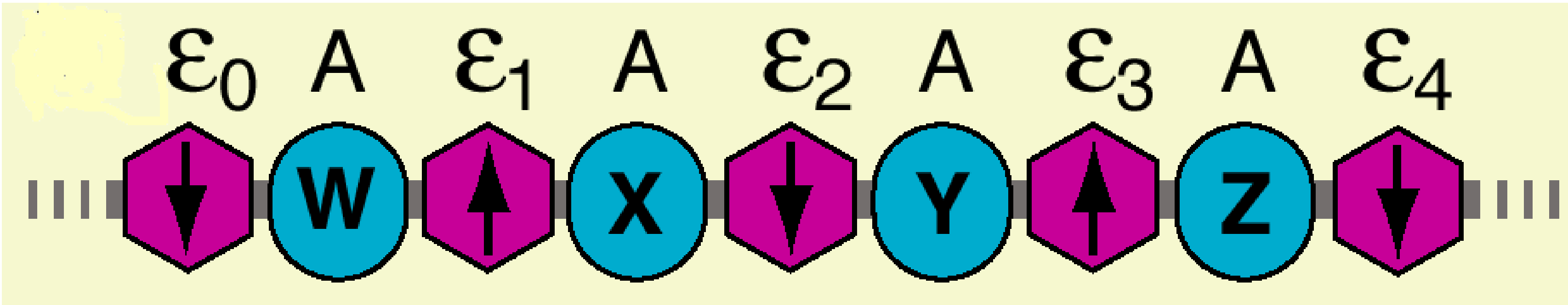
Where, $\hat{H}_{\text{Zeeman}} = \sum_{i=1}^{N} E_i(t)\hat{\sigma}_i^Z, \quad \hat{H}_{\text{int}} = J \sum_{i=1}^{N-1} \underline{\hat{\sigma}}_i \cdot \underline{\hat{\sigma}}_{i+1}$

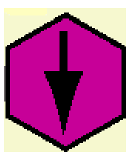
Then

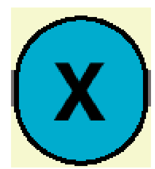
$|E_i - E_{i+1}| \gg J$ Implies $\hat{H}_{\text{int}} \approx J \sum \hat{\sigma}_i^Z \hat{\sigma}_{i+1}^Z$

Case A: When Universal Local Gates Are Possible:

Positions of Qubits & Barrier Spins



 **A barrier spin**

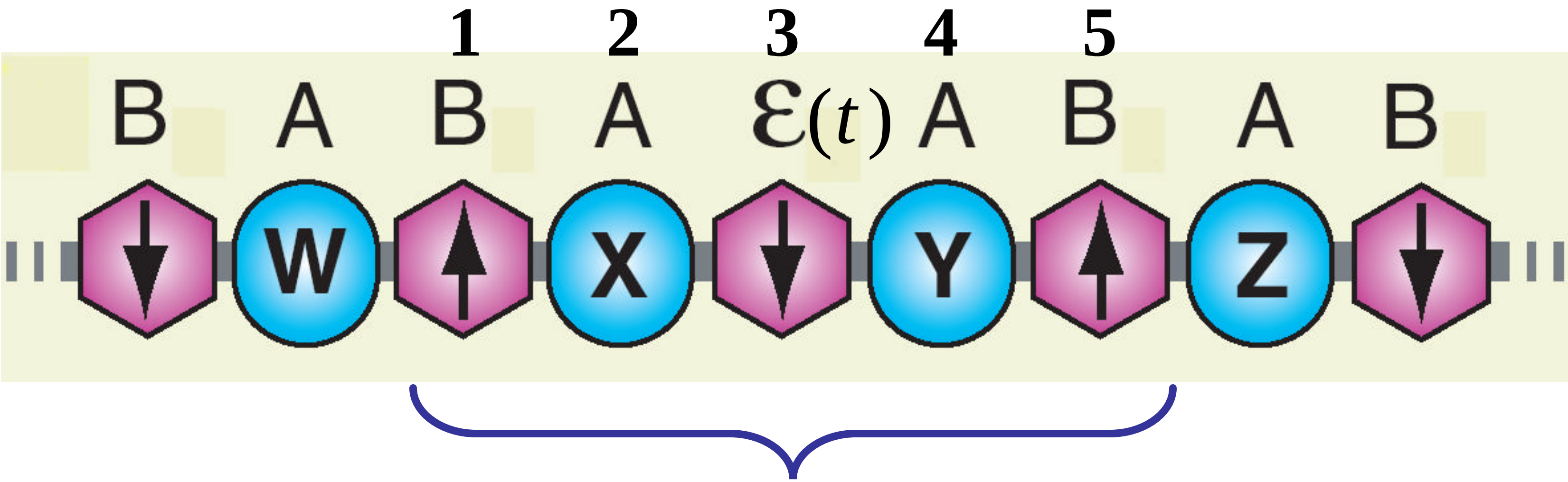


A qubit

e_i are variable energies, set to $\mathbf{B}$ in the *passive* state when single qubit gates are performed.

The Ising interaction on each qubit is then completely cancelled at all times. *Note:* Both barrier spins could be in the same state (which is easier to initialize, with periodic cancellation of Ising effects.

Case A: When Universal Local Gates Are Possible:



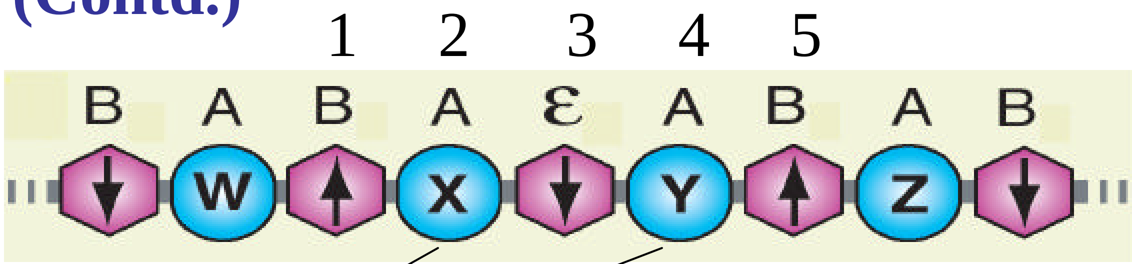
$$\hat{H} = (A+J)(\hat{\sigma}_2^z + \hat{\sigma}_4^z) + \epsilon(t)\hat{\sigma}_3 + J(\underline{\hat{\sigma}}_2 . \underline{\hat{\sigma}}_3 + \underline{\hat{\sigma}}_3 . \underline{\hat{\sigma}}_4)$$

For a Gate between X & Y, $e(t) = B$

is changed (fast) to $e(t) = A + J$

Then 3 becomes resonant with 2 & 4 (2,3,4 become a small Heisenberg chain).

Case A (Contd.)



At time $t_r = \hbar/(6J)$

2 & 4 disentangle from 3.

$$\hat{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & W & i\sqrt{3}W & 0 \\ 0 & i\sqrt{3}W & W & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } W = \frac{1}{2} e^{i\pi/3}$$

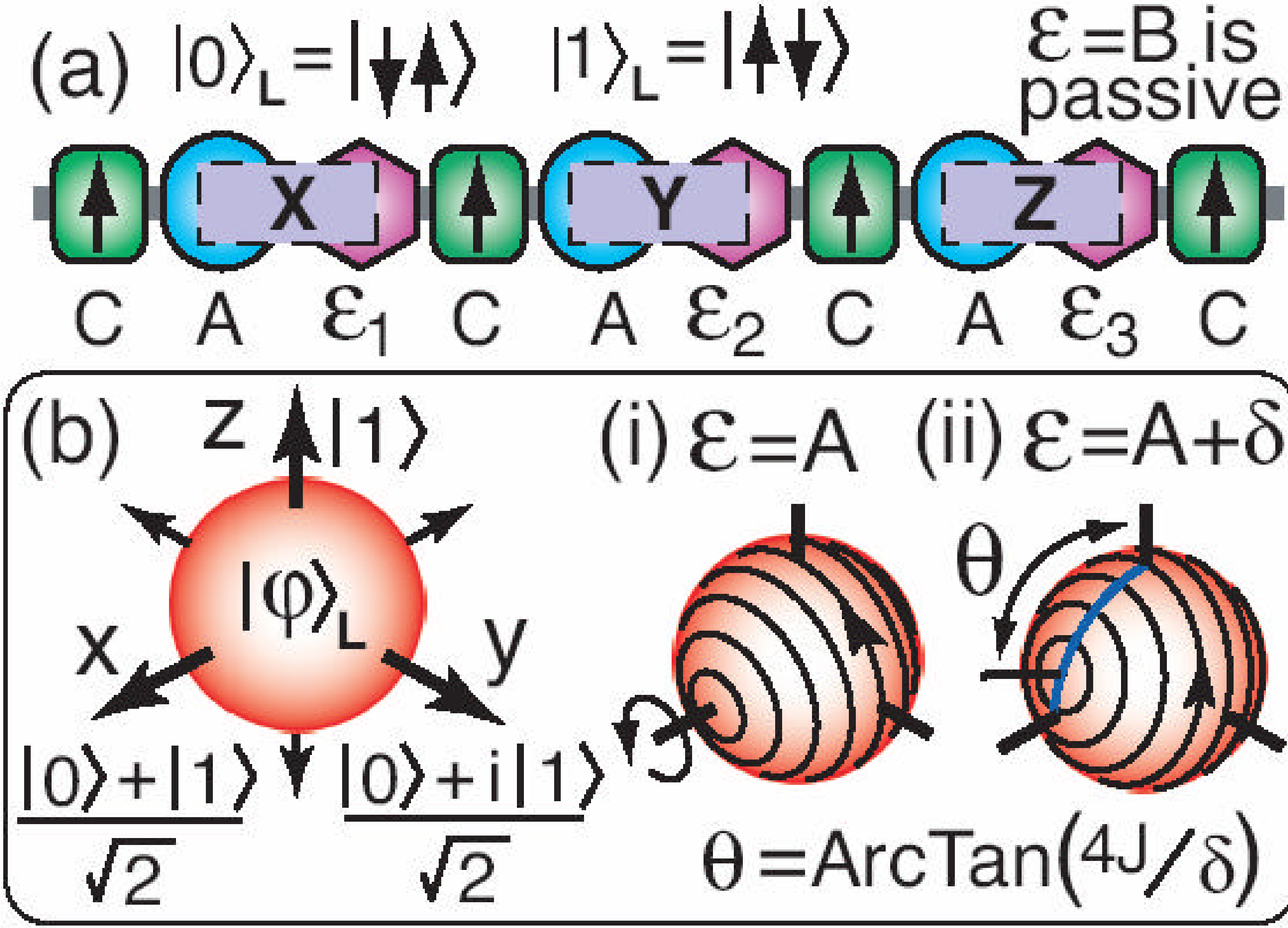
An *entangling* gate between X and Y !

Use techniques of: M. J. Bremner *et. al.*, quant-ph/0207072.

J. L. Dodd *et. al.*, PRA **65**, 040301 (2002).

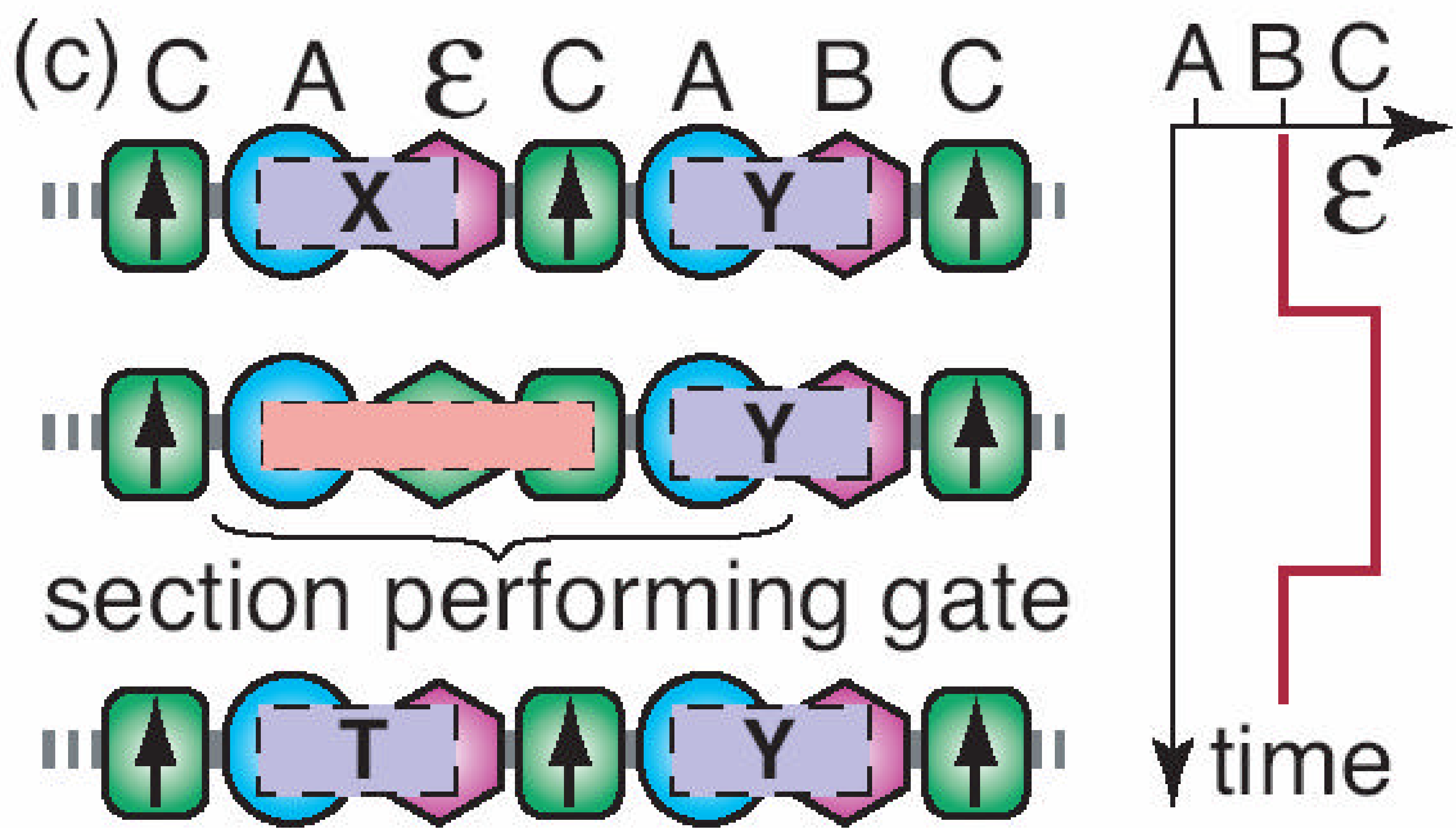
Case B: When Only Zeeman Energy Tuning is Possible Locally:

Method for one qubit gates:



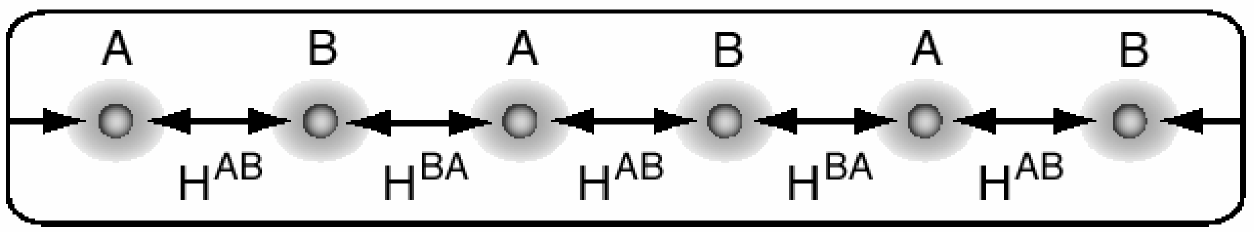
Case B: When Only Zeeman Energy Tuning is Possible Locally:

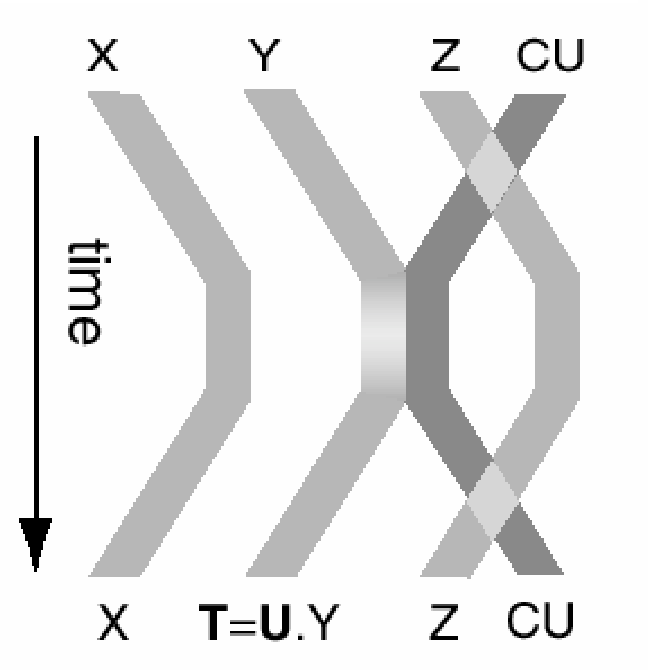
Method for two qubit gates



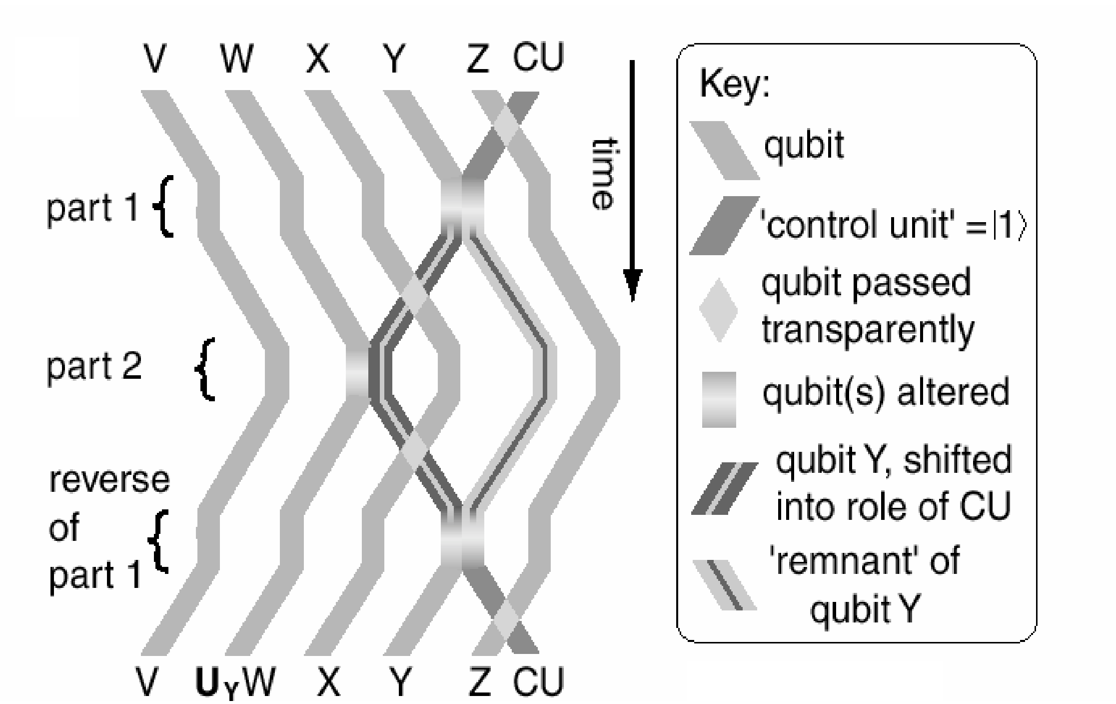
Global control quantum computation schemes of Lloyd & Benjamin

S. Lloyd, Science **261**, 1569 (1993); S. C. Benjamin, PRL **88**, 017904 (2002).



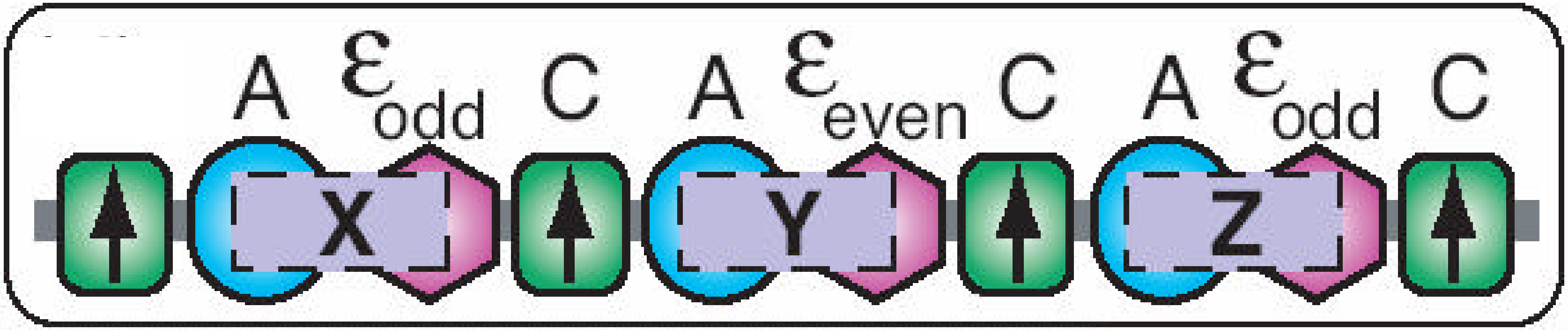


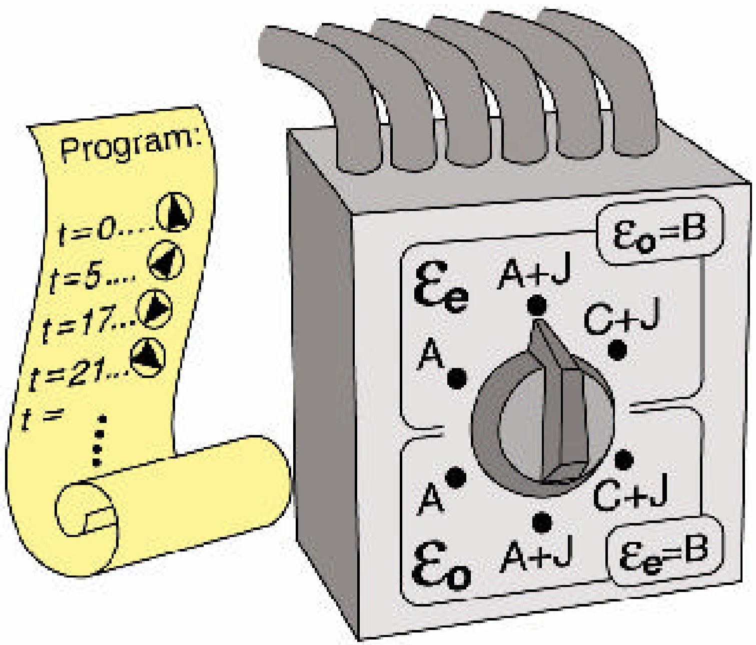
One Qubit Gates



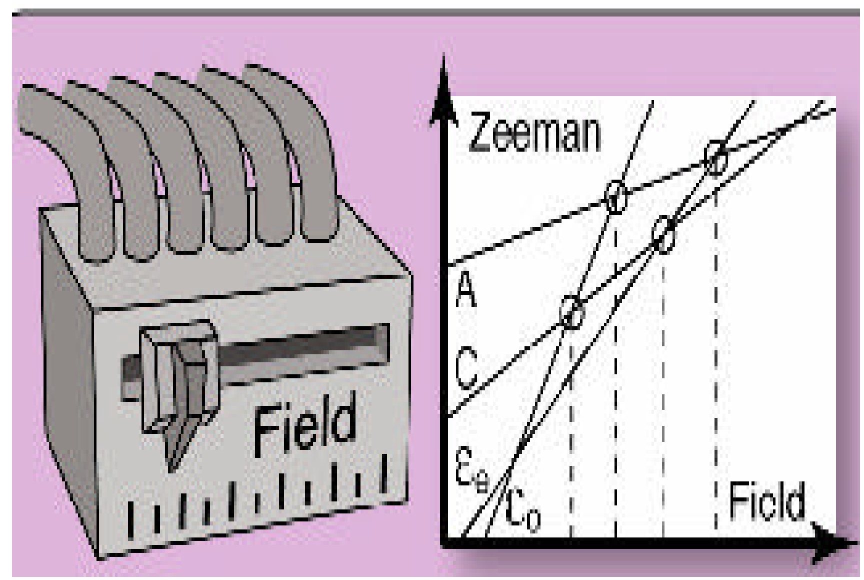
Two Qubit Gates

Case B: When No Local Ability is Present:



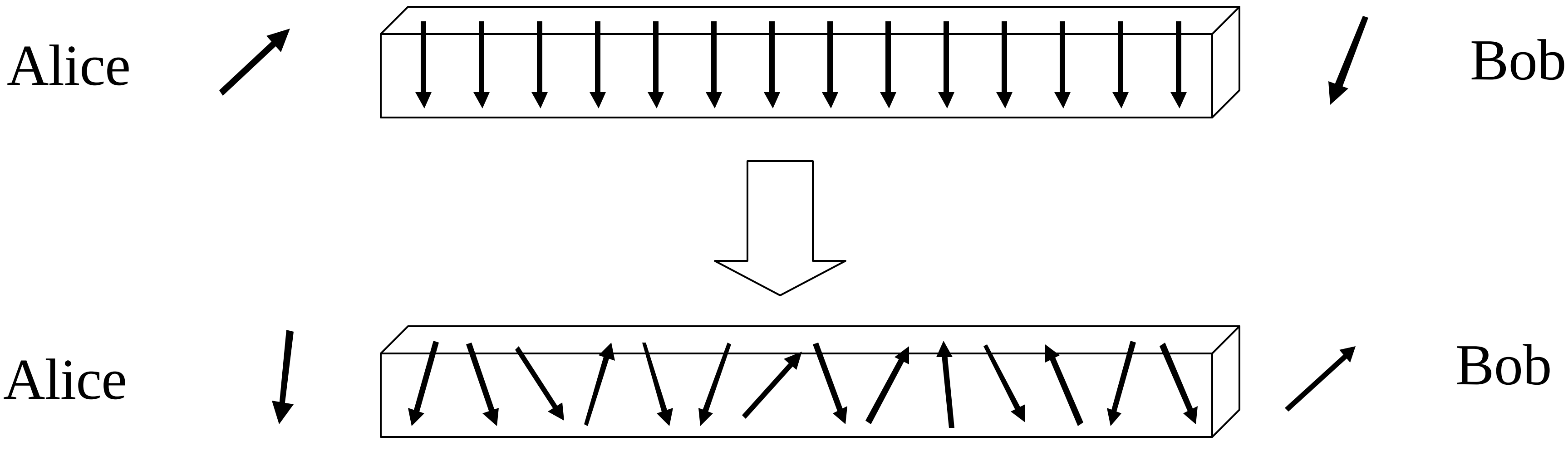


Control Switch of Six Settings

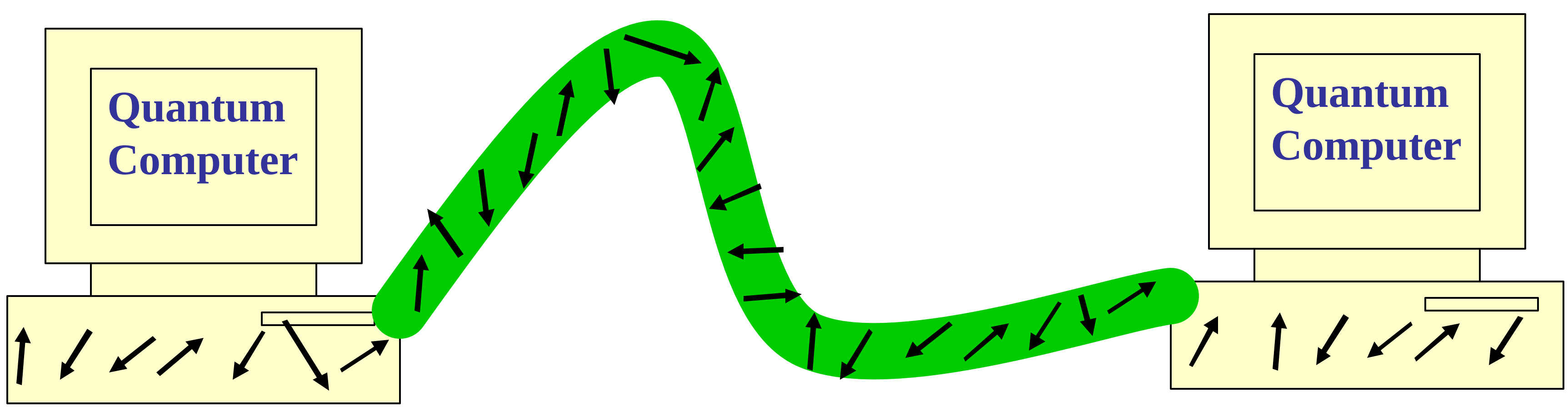


Control Through the Strength of a Single Field

Quantum Communication through a Spin Chain

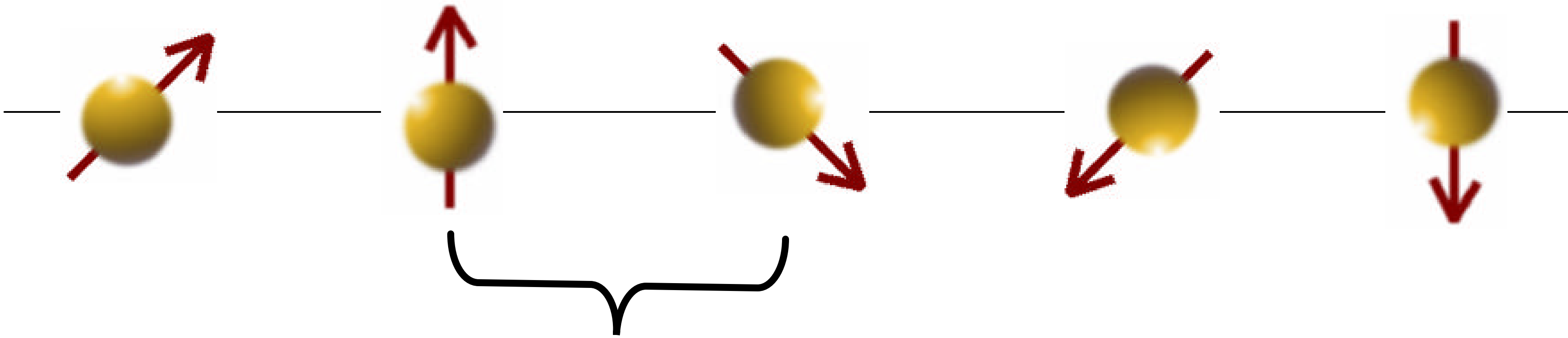


Avoids *interfacing* solid state systems with optics for the purpose of short-distance communication:



Definition of Spin-Chains:

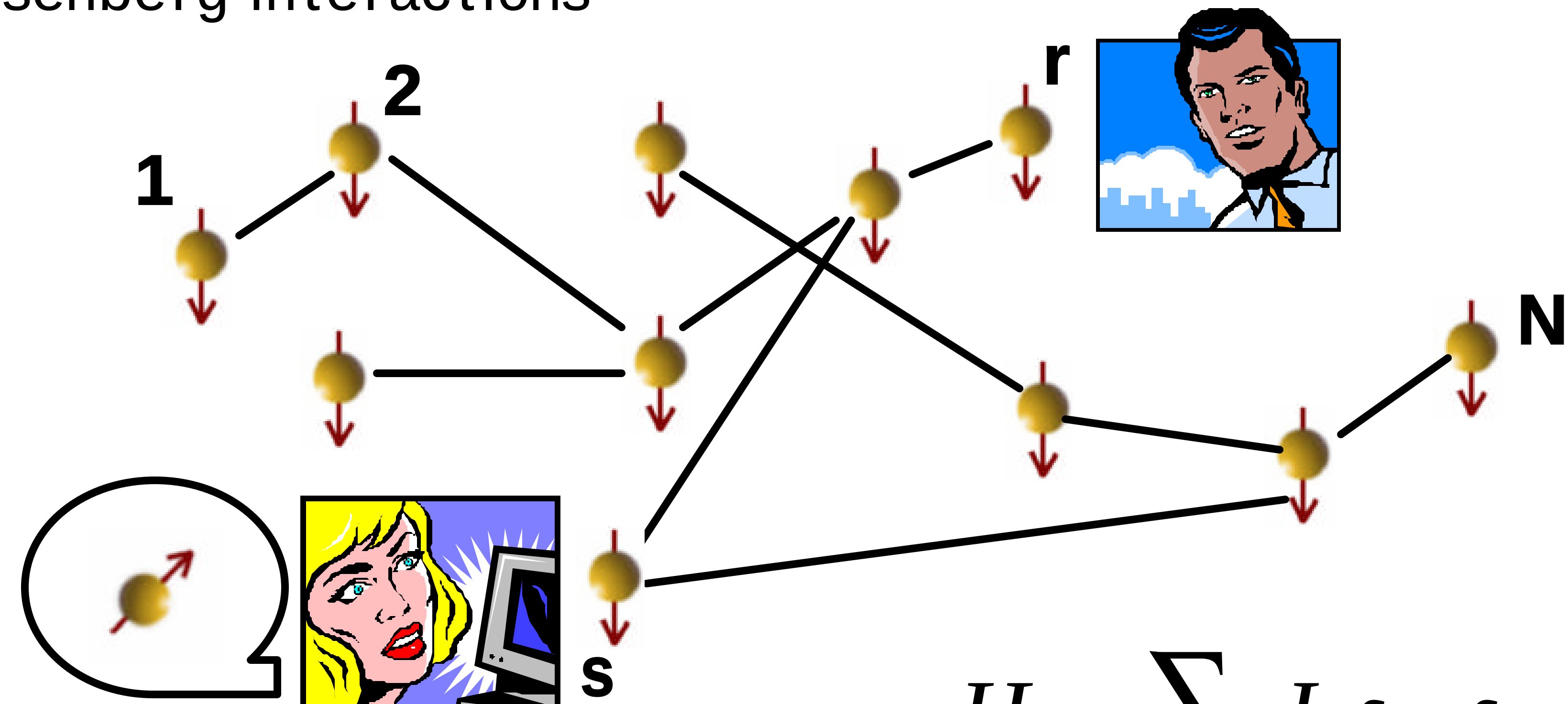
(A) 1D array of spins



(B) “Always On” (untunable) interactions

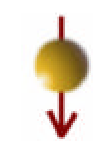
Makes it much easier to fabricate such systems with qubit arrays (especially in solid state) than to perform arbitrary quantum computations.

First consider arbitrary graphs with ferromanetic Heisenberg interactions

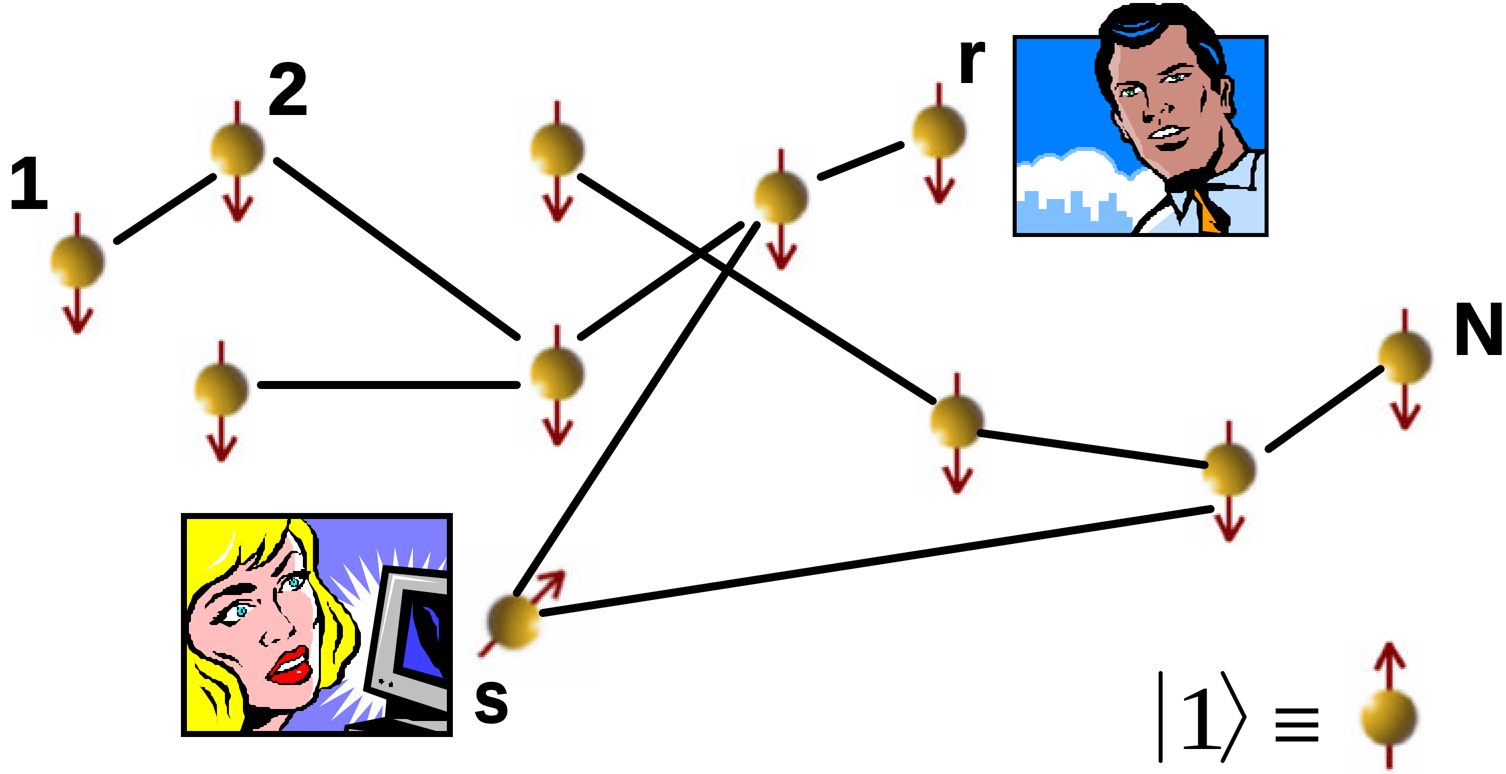


$$H = \sum_{<i,j>} J_{ij} s_i . s_j$$

Initialized in the ground state

$$|0\rangle = |000\ldots00\rangle, |0\rangle \equiv$$ 

with $$H_B = -B \sum_j s_{j,z}$$



At $t = 0$,

$$|\Psi(0)\rangle = \underbrace{|00..0\rangle}_{1,2,..,s-1}\underbrace{\left(\cos\frac{q}{2}|0\rangle + e^{if}\sin\frac{q}{2}|1\rangle\right)}_{s}\underbrace{|00..0\rangle}_{s+1,...,N}$$

Time evolution of the spin-graph:

$$\left|\Psi(t)\right\rangle = \cos\frac{q}{2}\left|0\right\rangle + e^{if}\sin\frac{q}{2}\sum_{j=1}^{N} e^{-2iBt} f_{j,s}^{N}(t)\left|\mathbf{j}\right\rangle$$

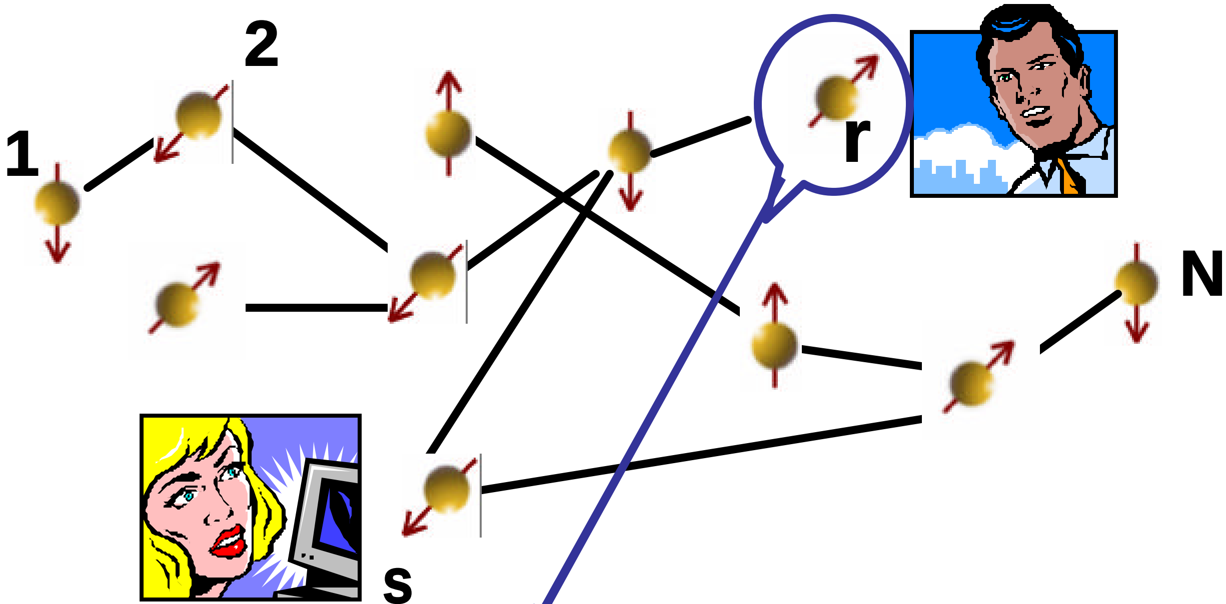
where, $\left|0\right\rangle = \left|00\ldots0\right\rangle$, $\left|\mathbf{j}\right\rangle = \left|00..010..0\right\rangle$

j th spin

and $$f_{j,s}^{N}(t) = \left\langle \mathbf{j}\right| e^{-iHt}\left|\mathbf{s}\right\rangle$$

is the transition amplitude of an excitation from the *s* th to the *j* th spin due to *H*.

Note that only the ground & N one-excitation states of the graph are invloved (because *H* does not create excitations, only propagates excitations).



$$\boxed{r_{\mathrm{r}}(t) = P(t)\left|y_{out}(t)\right\rangle\left\langle y_{out}(t)\right|_{\mathrm{r}} + (1 - P(t))\left|0\right\rangle\left\langle 0\right|_{\mathrm{r}}}$$

where, $P(t) = \cos^2\frac{q}{2} + |f_{\mathrm{r,s}}^{N}(t)|^2 \sin^2\frac{q}{2}$,

$$\left|y_{out}(t)\right\rangle = \frac{1}{\sqrt{P(t)}}\left(\cos\frac{q}{2}\left|0\right\rangle_{\mathrm{r}} + e^{if}\sin\frac{q}{2}\,e^{2iBt} f_{\mathrm{r,s}}^{N}(t)\left|1\right\rangle_{\mathrm{r}}\right)$$

B should be chosen so that $f_{\mathrm{r,s}}^{N}(t) \Rightarrow |f_{\mathrm{r,s}}^{N}(t)|$

The graph of Heisenberg interacting spins behaves as an *amplitude damping quantum channel*:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & |f_{r,s}^{N}(t)| \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{1-\left|f_{r,s}^{N}(t)\right|^2} \\ 0 & 0 \end{pmatrix}$$

Fidelity averaged over the Bloch Sphere:

$$\langle F \rangle = \frac{1}{4\pi}\int \left\langle \psi_{in} \middle| \rho_{out}(t) \middle| \psi_{in} \right\rangle = \frac{1}{2} + \frac{1}{3}\left|f_{r,s}^{N}(t)\right| + \frac{1}{6}\left|f_{r,s}^{N}(t)\right|^2$$

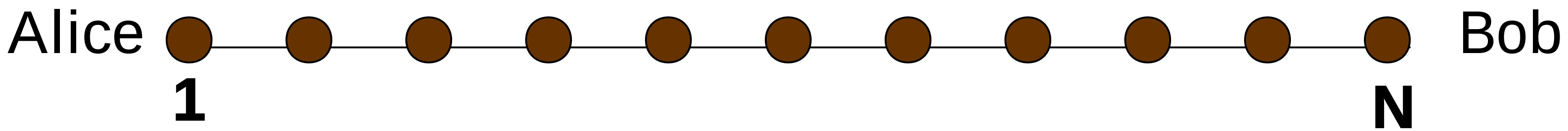
Entanglement (Concurrence) for input of one half of a $\left|\Psi^{(+)}\right\rangle$

$$E_C = \left|f_{r,s}^{N}(t)\right|$$

Exceptionally simple formulae in terms of a *single* transition amplitude

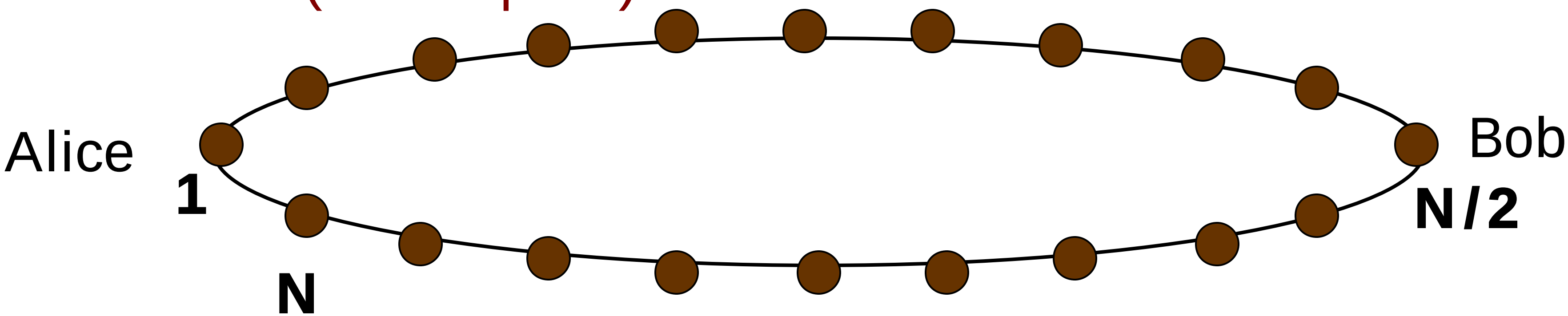
We will consider two cases:

A linear chain with communicating parties at opposite ends (most natural and readily implementable case):



$$H = -\frac{J}{2}\sum_{j=1}^{N-1} \mathbf{s}_j \cdot \mathbf{s}_{j+1}$$

A closed loop with the communicating parties at diametrically opposite ends (to compare):



$$H = -\frac{J}{2}\sum_{j=1}^{N-1} \mathbf{s}_j \cdot \mathbf{s}_{j+1} - \frac{J}{2}\mathbf{s}_N \cdot \mathbf{s}_1$$

Eigenstates in the one excitation sector:

(A) Linear (open chain) case: (A Quantum Cosine Trans.)

$$|\tilde{m}\rangle = a_m \sum_{j=1}^{N} \cos\left[\frac{p}{2N}(m-1)(2j-1)\right] |\mathbf{j}\rangle$$

for $m = 1, \ldots, N$ with $a_1 = \sqrt{\frac{1}{N}}$ and $a_{m>1} = \sqrt{\frac{2}{N}}$

(B) Closed chain case: (A Quantum Fourier Trans.)

$$|\tilde{m}\rangle = \sqrt{\frac{1}{N}} \sum_{j=1}^{N} e^{i\frac{2p}{N}(m-1)j} |\mathbf{j}\rangle$$

for $m = 1, \ldots, N$

Transition amplitudes in terms of readily computable transforms:

(A) Linear (open chain) case:

$$\boxed{|f_{r,s}^{N}| = |DCT_{s}(N, v_m(r,t))|}$$

where

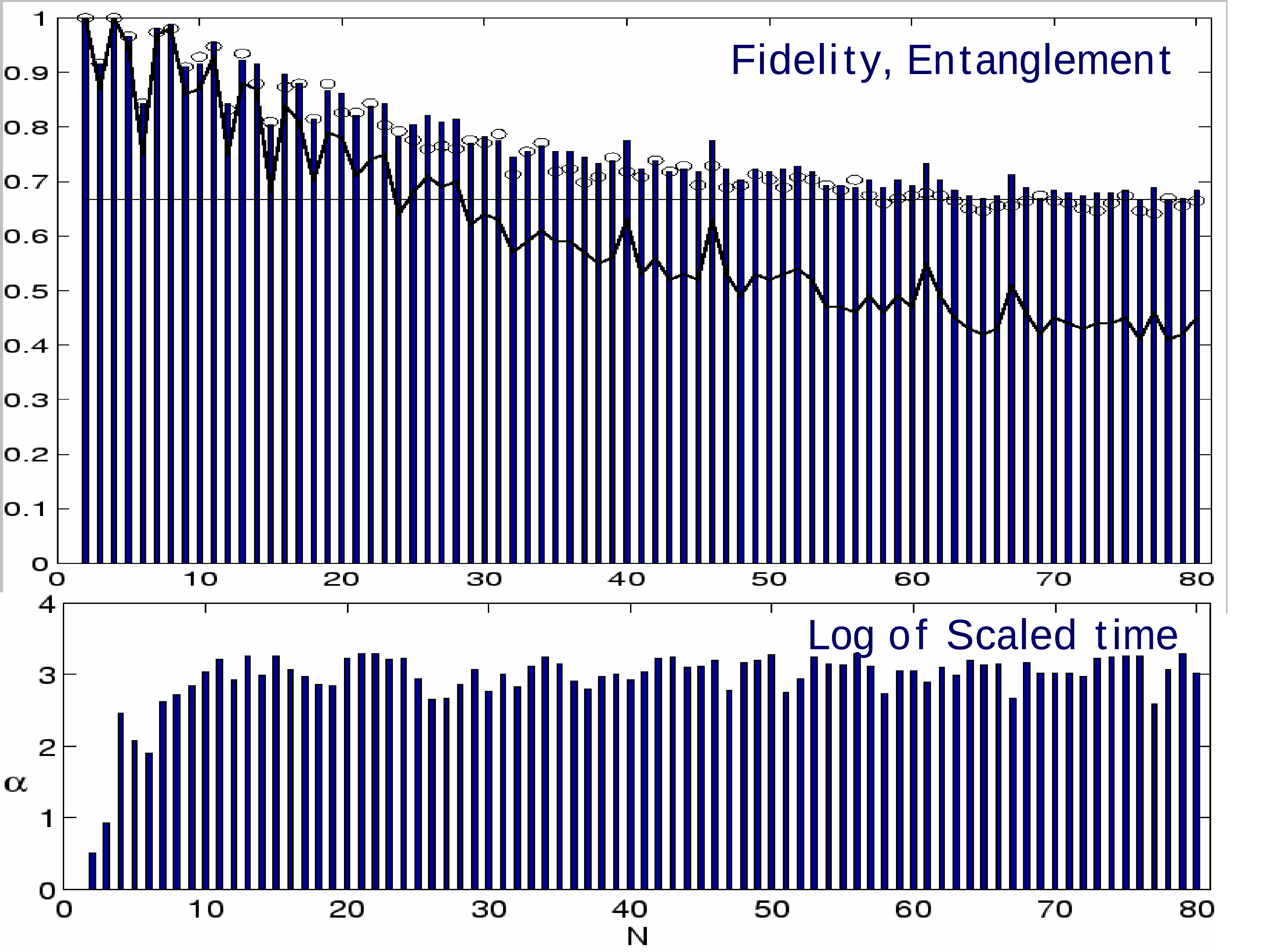
$$v_m(r,t) = a_m \cos\left[\frac{p(m-1)}{2N}(2r-1)\right] e^{i2Jt\cos\left[\frac{p(m-1)}{N}\right]}$$

$$a_1 = \sqrt{\frac{1}{N}} \quad \text{and} \quad a_{m>1} = \sqrt{\frac{2}{N}}$$

(B) Closed chain case:

$$\boxed{|f_{r,s}^{N}| = |DFT_{r-s}(N, v_m(t))|}$$

where $v_m(t) = e^{i2Jt\cos\left[\frac{2p(m-1)}{N}\right]}$



Fidelity, Entanglement
Log of Scaled time
α
N

Alternative formulas in terms of Bessel functions:

Open chain:

$$\left|f_{N,1}^{N}\right| = 2\sqrt{\left(\sum_{k=0}^{\infty}(-1)^{Nk} J_{N(k+1)}(2Jt)\right)^2 + \left(\sum_{k=0}^{\infty}(-1)^{Nk} J'_{N(k+1)}(2Jt)\right)^2}$$

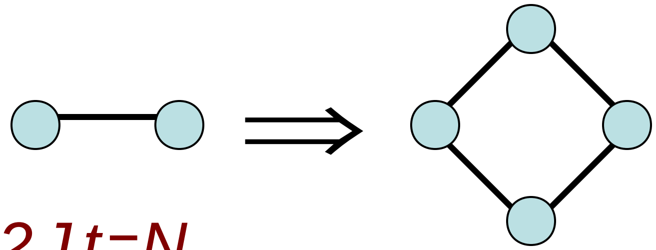
$$\boxed{\approx 2\left|J_N(2Jt)\right|}$$ at the maximum near $2Jt=N$

Closed chain:

$$\left|f_{N/2,1}^{N}\right| = 2\left|\sum_{k=0}^{\infty}(-1)^{(N/2)k} J_{(N/2)(2k+1)}(2Jt)\right|$$

$$\boxed{\approx 2\left|J_{N/2}(2Jt)\right|}$$ at the maximum near $2Jt=N/2$

1. Closed chain 2N $\leq$ Open chain N



2. Can find high fidelity transfer at $2Jt=N$

$$\boxed{\begin{array}{l} N = 10^3 \Rightarrow E_C(2Jt = 1005) = 0.13 \\ N = 10^4 \Rightarrow E_C(2Jt = 10017) = 0.06 \end{array}}$$

Distillable

Possible Future Work

Q.Comm part:

1. Sending higher D systems (Ex: 4 state systems by using up to 2 spin excitations --- with Korepin).

2. Study Graphs which improve comm. fidelity (suggested by Preskill)

3. Direct qubit comm. (without distillation) over arbitrary distances by using spin-1 chain (--- with Thapliyal).

4. Can measurements on the chain improve transfer? (suggested by Verstraete).

Q.Compu part:

2D extensions, extensions to clusters replacing single qubits etc.
--- for greater fault tolerance and robustness.