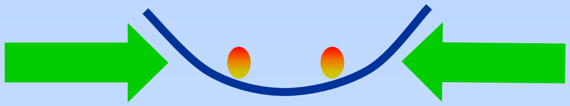
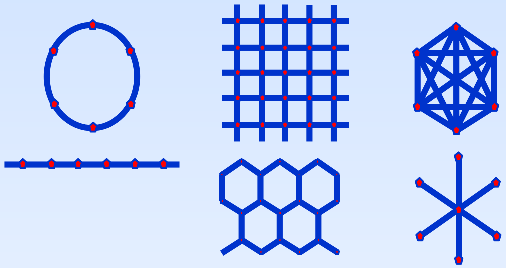
# Physical limits for quantum gates with ions



J. J. Garcia-Ripoll
P. Zoller (Innsbruck)

# Gaussian entanglement and symmetry graphs



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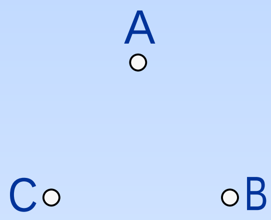
MPQ

Stony Brook, 29 May 2003

# Multiparticle entanglement

## 1. Shareability: (Wootters/Schumacher)




If A is very entangled with B, then it is weakly entangled with C.

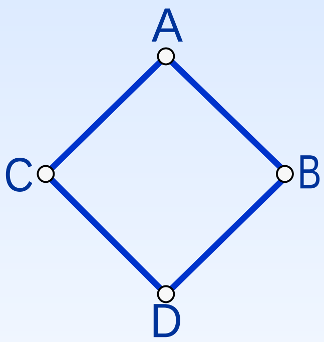
$$|\Psi\rangle_{ABC} = |\varphi^-\rangle_{AB}|0\rangle_C$$

To increase the entanglement AC one has to decreases that of AB.

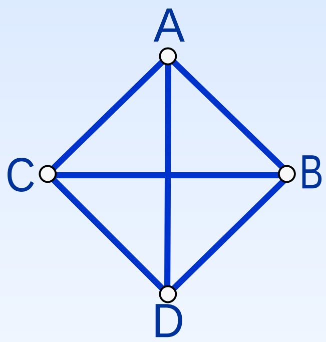
$$|\Psi\rangle_{ABC} = |001\rangle + |010\rangle + |100\rangle$$

(Dür, Vidal and Cirac)

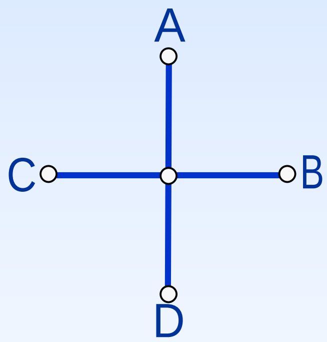
One may consider more particles and different configurations:




Entanglement
nearest neighbors

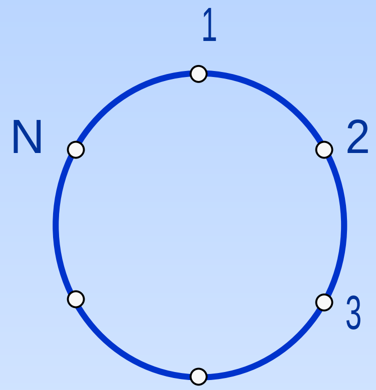



Entanglement
every partner




One and
the rest

# II. Entanglement in quantum statistical models: (Nielsen, Fazio, Vedral, Vidal, Korepin)



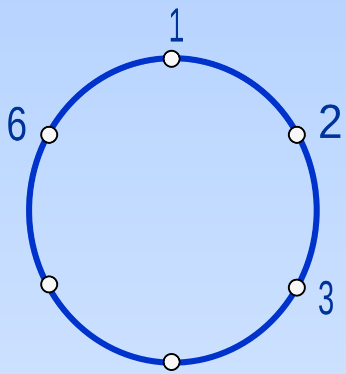

$$H = U\sum_n A_n \otimes A_{n+1} + t\sum B_n$$

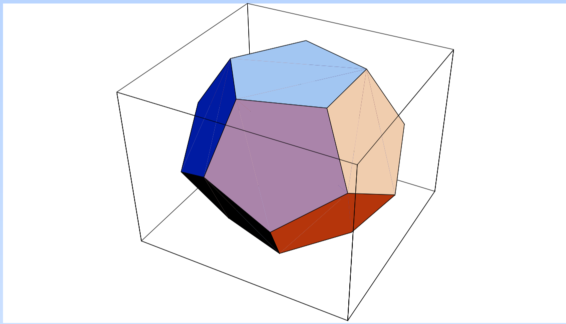
Entanglement between two particles:

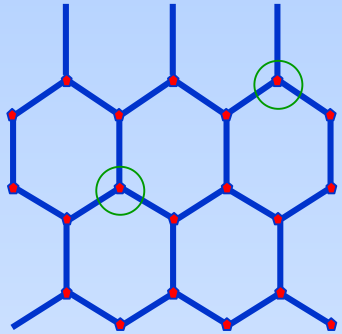
$$E_{i,j}(t,U,T) = E_F\left[Tr_{(i,j)}\left(e^{-H/T}\right)\right]$$

Which is the Hamiltonian for which this entanglement is maximal (at T=0)?

# This talk: Gaussian states





- Consider a undirected simple graph with N vertices.

Adjacency matrix A: $\begin{cases} A_{k,l} = 1 & \text{if k,l are connected} \\ A_{k,l} = 0 & \text{otherwise} \end{cases}$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Symmetry of the graph: automorphism group

$$G = \{ g \,/\, A = gAg^{-1} \} \subseteq S_N$$

Rotations
+ reflections

- Consider two points, k,l, such that for some

$$g \in G,\ g(k) = l \text{ and } g(l) = k$$

All pairs

We determine the Gaussian state, invariant under G, which maximizes EoF for (k,l)

$$\Psi \text{ gaussian with } T_g |\Psi\rangle = |\Psi\rangle, \text{ such that } E_F\left[\mathrm{Tr}_{(k,l)}\left(|\Psi\rangle\langle\Psi|\right)\right] \text{ is maximal}$$

- Gaussian states:

A B C

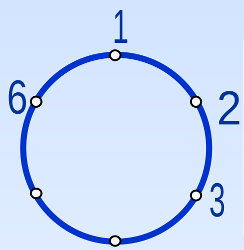
$$\dim(H_n) = \infty$$
$$[X_n, P_n] = i$$
$$H = \otimes_n H_n$$

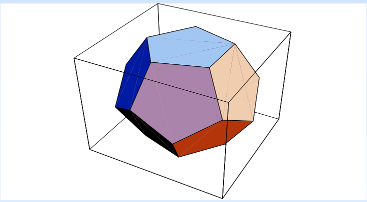
ρ is Gaussian if it can be written as:

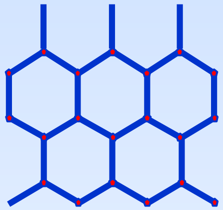
$$\rho = k e^{-Q(X_n, P_n)}$$

$Q \geq 0$ is a polynomial of degree 2

- Optimal state:





1. Build a Hamiltonian

$$H = \sum_{g \in G} T_g \left[ (x_k - x_l)^2 + (p_k + p_l)^2 \right] T_g^\dagger$$

2. Determine ground state ➡ $\Psi$
3. Determine ground state energy $E_0$.

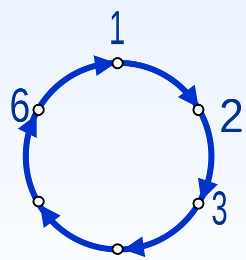
➡ $E_F = f(E_0)$

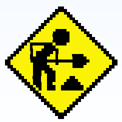
Everything can be easily determined starting from the Adjacency matrix A.

- Extensions:

- General group (including undirected graphs, etc) if $\mathrm{Index}(k,k) = \mathrm{Index}(k,l)$

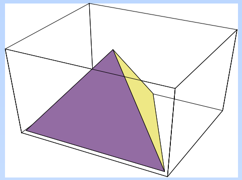
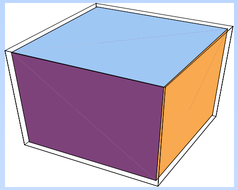
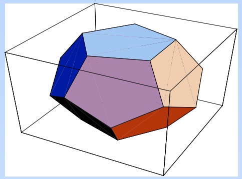
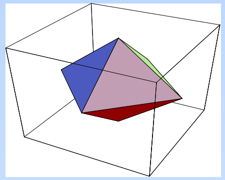
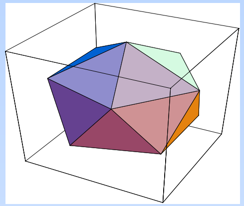
➡ add $g \in G,\ g(k) = l$ and $g(l) = k$




- Qubits 

# Examples

1. Platonic solids:

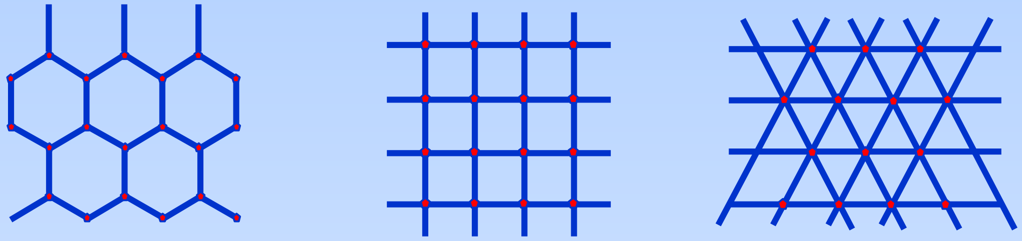






| Platonic Solid | EoF x 100 | N adjacent | N nodes |
|---|---|---|---|
| Tetrahedron | 19.74 | 3 | 4 |
| Cube | 19.74 | 3 | 8 |
| Dodecahedron | 11.12 | 3 | 20 |
| Octahedron | 10.75 | 4 | 6 |
| Icosahedron | 5.37 | 5 | 12 |

Tendencies:

- E decreases with the number of adjacent nodes.
- E decreases with the total number of nodes.
- E is suppressed in loops with an odd number of nodes.

## II. Infinite lattices:

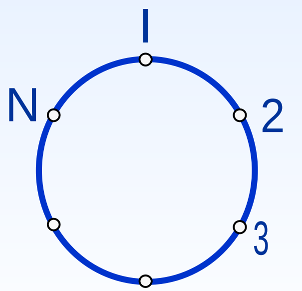


| Lattice | EoF x 100 | N adjacent |
|---|---|---|
| Hexagonal (2d) | 10.61 | 3 |
| Square (2d) | 6.31 | 4 |
| Trigonal (2d) | 2.69 | 6 |
| Cubic (3d) | 2.62 | 6 |

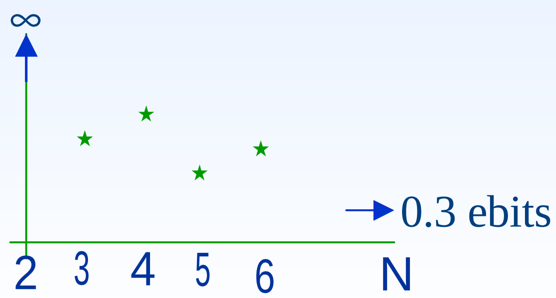
Tendencies:

- E decreases with the number of adjacent nodes.

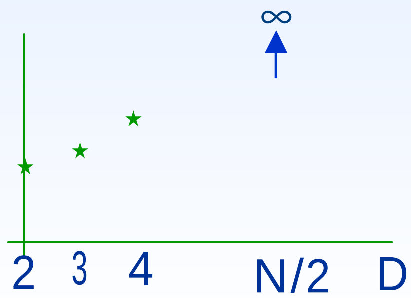
## III. Finite lattices:




Nearest neighbors



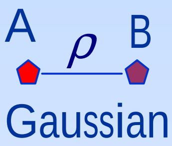

Separation



# Key points of the derivation

1. Entanglement of formation of symmetric Gaussian states:

(Giedke, Wolf, Krüger, Werner and Cirac)




Covariant matrix $\gamma = \begin{pmatrix} n & 0 & k_x & 0 \\ 0 & n & 0 & -k_p \\ k_x & 0 & m & 0 \\ 0 & -k_p & 0 & m \end{pmatrix}$

Symmetric Gaussian states: $m = n$

$$E_F(\rho) = f(\Delta)$$

with $\Delta = \sqrt{(n - k_x)(n - k_p)}$ and $f(\Delta)$ a monotonic function.

2. Express it in a linear form:

$$\Delta(\rho) = \inf_{s>0} \mathrm{Tr}\left[\gamma\left(sX + \frac{1}{s}P\right)\right]$$

3. Optimization problem:

$$\Delta_{i,j} = \sup_{\Psi\to\gamma_{i,j}} \inf_{s>0} \mathrm{Tr}\left[\gamma_{i,j}\left(sX + \frac{1}{s}P\right)\right]$$ with $\Psi$ invariant under G.

4. Use symmetry:

$$\Delta_{i,j} = \sup_{\Gamma} \inf_{s>0} \mathrm{Tr}\left[\Gamma\left(sH_X + \frac{1}{s}H_P\right)\right]$$

where $$H_X = \sum_{g\in G} T_g \left(X \oplus \grave{}0\right) T_g^{\dagger}$$

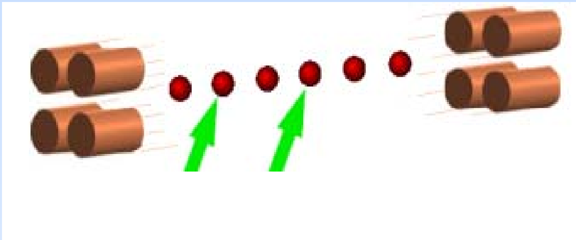
5. Algebra + properties of covariant matrices:

$$\Delta_{i,j} = \inf_{\Psi} \langle\Psi\,|\,H\,|\,\Psi\rangle$$

where $H$ is a Hamiltonian of harmonic osc. which are coupled according to the graph.

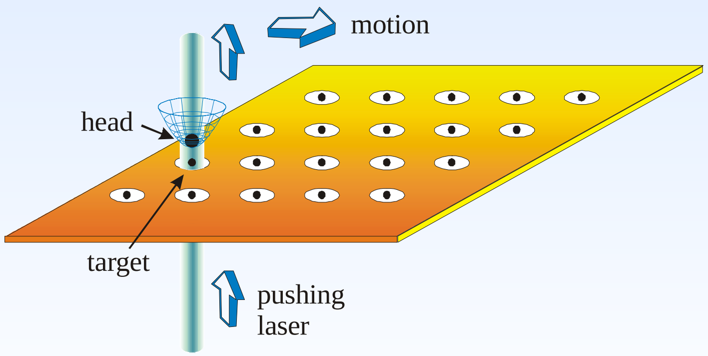
# 2. Limits on gates for trapped ions

## Quantum information processing with trapped ions:

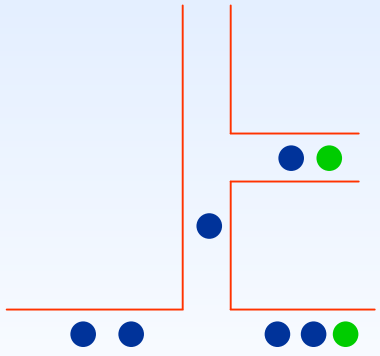


- Internal states manipulated with lasers.
- Two-qubit gates by exciting motional states.
- Difficult to scale-up.

## Scalable proposals



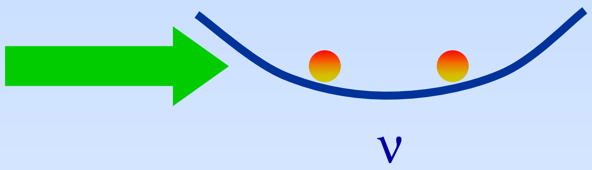

(J.I. Cirac and P. Zoller, Nature 2000)



(Wineland and col, Nature 2001)

# Two-qubit gates

- Most of the errors occur during two-qubit quantum gates.
- We would like to design the most efficient quantum gates.
- One just has consider two ions in a trap.




## Current proposals: limitations

(Molmer and Sorensen, Plenio, Monroe, Milburn and James, Leibfried and Wineland, Cirac and Zoller,...)

Time: $\nu T \gg 1$ (tipically, of the order of 100 to achieve F=0.99).

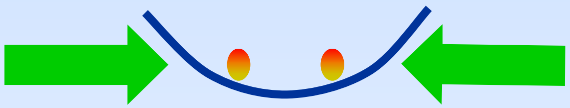
Addressability: Different lasers acting on different ions.

Low temperatures: $\eta\sqrt{N+1} \ll 1$

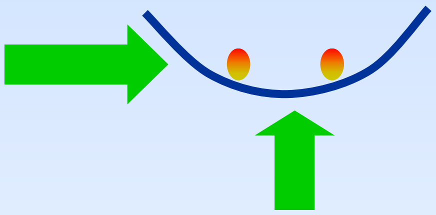
# New scheme based on a different concept

- No time limitation by the trap frequency.
- Insensitive to temperture:
- No addressability:

Idea:



(as in Poyatos, Cirac, and Zoller, 97)

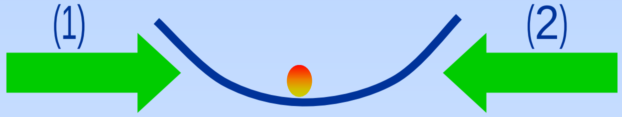


- Use on-resonant lasers to kick the atoms ➡ No spectroscopic limitations.
- Laser pulses must be short, and come from two different directions:

**Limitation is laser control**

## Basic principle (with one ion):

(i) Use lasers to kick both ions.

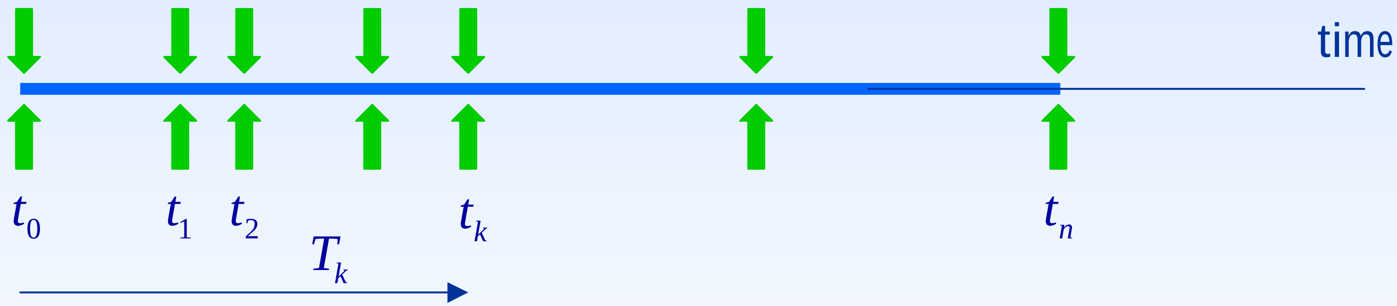



Kick depends on the internal state of the ion and laser direction.

$$|g\rangle|\mathrm{mot}\rangle \xrightarrow{(1,2)} |g\rangle e^{2ikx_1}|\mathrm{mot}\rangle$$

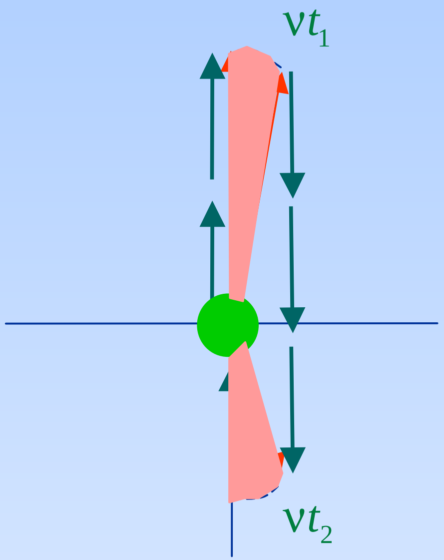
$$|e\rangle|\mathrm{mot}\rangle \xrightarrow{(1,2)} |e\rangle e^{-2ikx_1}|\mathrm{mot}\rangle$$

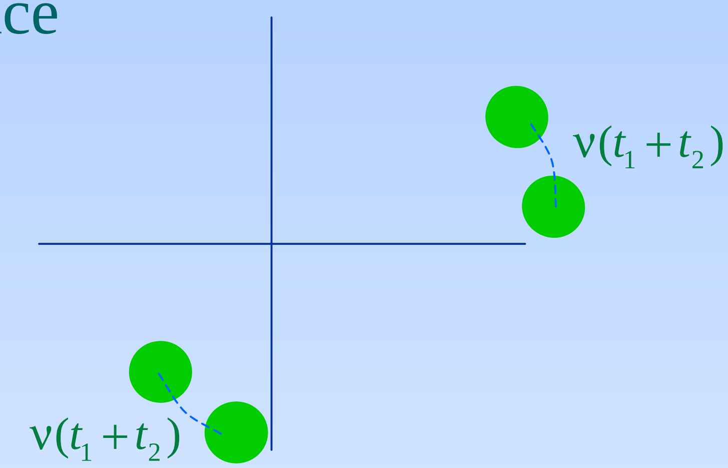
(ii) Free evolution in the harmonic trap.




If $\sum_{k=1}^{N} n_k e^{i\upsilon T_k} = 0$ then $U = e^{i\phi\sigma_z} \otimes 1$

# Phase Space




- We return to the initial state.
- The wavefunction acquires a phase.

$$\phi = A$$

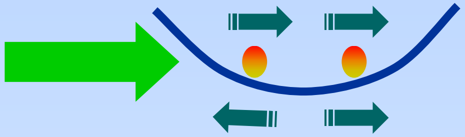
- Can be made independent of the initial coherent state.
- The phase is also independent.

$$\text{if} \quad \sum_{k=1}^{N} n_k e^{i\theta_k} = 0 \quad \text{then} \ |\alpha\rangle \rightarrow e^{i\phi} |\alpha e^{-i\nu t}\rangle$$

The angle $\phi$ depends on the path.

The path depends on the initial state. ➡ Quantum gate.

# With two ions



Two modes

Conditions

$$\begin{cases} \sum_{k=1}^{N} n_k e^{i\sqrt{3}\upsilon T_k} = 0 \\ \sum_{k=1}^{N} n_k e^{i\upsilon T_k} = 0 \end{cases}$$

$$U = e^{i\phi\sigma_z^1\sigma_z^2} \otimes 1$$

Control-phase gate

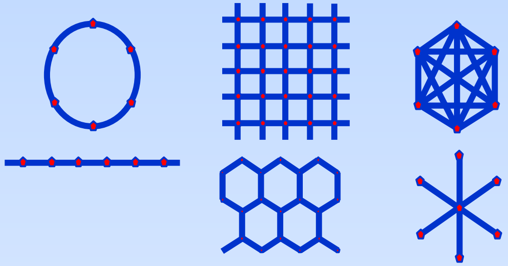
- For any chosen time T, it is always possible to find a sequence of pulses.

**There is not limitation by the trap frequency!**

- Completely independent of the motional state (no LDL restriction).
- No addressability.
- The number of required pulses increases if the time decreases. $N_p \sim \dfrac{8}{(\nu T)^{3/2}}$
- Limitations:
  - Length of the laser pulses.
  - Laser stability.

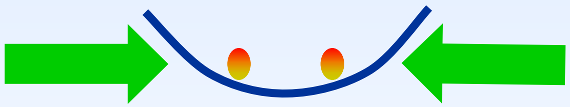
# Conclusions

## Entanglement of Gaussian states



- Solved the problem of shareability for arbitrary graphs.
- Connection with quantum statistics: ground state of a symmetric Hamiltonian.

## New concepts for quantum gates with ions



-No limitation by trap frequency.

- New scheme:
  - New concept (resonant interaction).
  - Fast (not limited by trap frequency).
  - Robust (arbitrary temperatures).
  - Simple (no addressability).