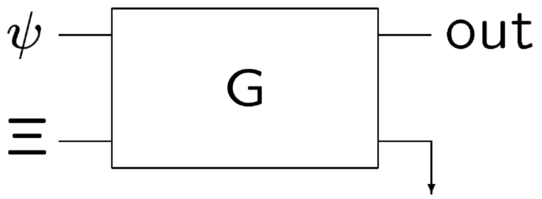
Programmable Quantum Circuits:
Controlling quantum states with quantum states

Two scenarios

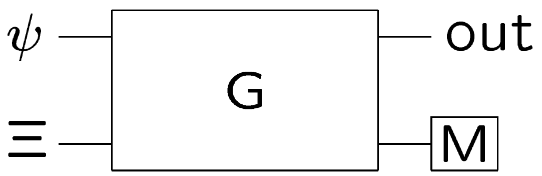
1. Deterministic



$$\rho_{in} = |\psi\rangle\langle\psi| \rightarrow T_{\Xi}(\rho_{in})$$

where T is a trace-preserving, completely positive map.

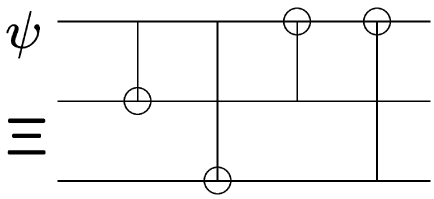
2. Probabilistic



$$\rho_{in} = |\psi\rangle\langle\psi| \rightarrow T_{(\Xi,M)}(\rho_{in})$$

where $T_{(\Xi,M)}$ is a completely positive map.

Example:



Deterministic

$$
\begin{aligned}
|\Phi_\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \\
|\Psi_\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)
\end{aligned}
$$

Then

$$
\begin{aligned}
|\Xi\rangle = |\Phi_+\rangle &\Rightarrow |\psi\rangle \to \sigma_x|\psi\rangle \\
|\Xi\rangle = |\Phi_-\rangle &\Rightarrow |\psi\rangle \to (-i\sigma_y)|\psi\rangle \\
|\Xi\rangle = |\Psi_+\rangle &\Rightarrow |\psi\rangle \to |\psi\rangle \\
|\Xi\rangle = |\Psi_-\rangle &\Rightarrow |\psi\rangle \to \sigma_z|\psi\rangle
\end{aligned}
$$

If program state is

$$|\Xi\rangle = c_0|\Psi_+\rangle + c_1|\Phi_+\rangle + c_2|\Phi_-\rangle + c_3|\Psi_-\rangle$$

then

$$\begin{aligned} T_\Xi(|\psi\rangle\langle\psi|) &= |c_0|^2\rho_{in} + |c_1|^2\sigma_x\rho_{in}\sigma_x \\ &\quad + |c_2|^2\sigma_y\rho_{in}\sigma_y + |c_3|^2\sigma_z\rho_{in}\sigma_z \end{aligned}$$

Can implement bit-flip, phase-flip, and depolarizing channels, and many other possibilities.

Probabilistic

$|\phi\rangle$ is a one-qubit state. Define

$$U_{init}|00\rangle = -|10\rangle \quad U_{init}|10\rangle = -|11\rangle$$
$$U_{init}|11\rangle = |01\rangle \quad U_{init}|01\rangle = |00\rangle$$

Program

$$|\Xi\rangle = \frac{1}{\sqrt{2}} U_{init}(|\phi\rangle|\phi_\perp\rangle + |\phi_\perp\rangle|\phi\rangle)$$

Measurement is projection onto

$$\frac{1}{\sqrt{3}}(|\Phi_+\rangle + |\Phi_-\rangle + |\Psi_-\rangle).$$

Then with probability 1/3

$$|\psi\rangle \to (I - 2|\phi\rangle\langle\phi|)|\psi\rangle$$

Also have that if

$$|\Xi\rangle = \frac{1}{\sqrt{2}} U_{init}(|\phi\rangle|\phi\rangle + |\phi_\perp\rangle|\phi_\perp\rangle)$$

then

$$|\psi\rangle \rightarrow (|\phi\rangle\langle\phi_\perp| + |\phi_\perp\rangle\langle\phi|)|\psi\rangle$$

Deterministic processors - general formalism

$$|\Psi_{in}\rangle = |\psi\rangle_d \otimes |\Xi\rangle_p \rightarrow \ \ |\Psi_{out}\rangle = G|\Psi_{in}\rangle$$

$$\rho_d^{(out)} = \text{Tr}_p(|\Psi_{out}\rangle\langle\Psi_{out}|)$$

where G is unitary. We can express G as

$$G = \sum_{j,k=1}^{N} A_{jk} \otimes |j\rangle_{p\,p}\langle k|$$

where

$$\sum_{j=1}^{N} A_{jk}^{\dagger} A_{jm} = I\delta_{k,m}$$

We then have that

$$\begin{aligned} \rho_d^{(out)} &= \sum_{j=1}^{N} A_j(\Xi)|\psi\rangle_{d\,d}\langle\psi|A_j^{\dagger}(\Xi) \\ A_j(\Xi) &= \sum_{k=1}^{N} {}_p\langle k|\Xi\rangle_p A_{jk} \end{aligned}$$

Resources necessary to implement maps deterministically

One-parameter group

$$T_\alpha(\rho_{in}) = U(\alpha)\rho_{in}U^\dagger(\alpha) \qquad U(\alpha) = e^{i\alpha\sigma_z}$$

where $0 \leq \alpha < 2\pi$.

No-go theorem (Nielsen and Chuang):
For each unitary operator implemented by a deterministic processor, need an extra dimension in the program space.

What if maps are not unitary?

A. Phase-damping channel

$$T_\theta(\rho_{in}) = \theta\rho_{in} + (1-\theta)\sigma_z\rho_{in}\sigma_z$$

where $0 \leq \theta \leq 1$. Can be programmed with 2-D program space.

B. Amplitude-damping channel

$$T_\theta(\rho_{in}) = \sum_{j=1}^{2} B_j(\theta)\rho_{in}B^\dagger(\theta)$$

where

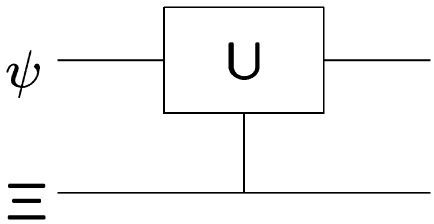
$$B_1(\theta) = |0\rangle\langle 0| + \sqrt{1-\theta}|1\rangle\langle 1| \qquad B_2 = \sqrt{\theta}|0\rangle\langle 1|$$

Requires infinite program space

Suppose we only want to approximate the one-parameter group $U(\alpha)$?

Processor: Controlled-U gate with single qubit date state as target and program state as control.

Possible operations $\{U_1, \ldots U_n\}$.



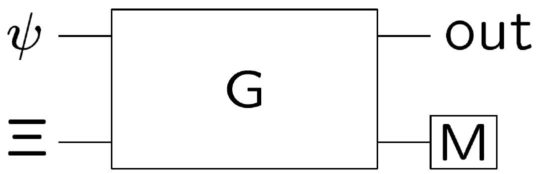
Maximize average fidelity

$$\overline{F} = \frac{1}{4\pi} \int d\Omega \langle\psi|U^\dagger(\alpha) T_\Xi(|\psi\rangle\langle\psi|) U(\alpha)|\psi\rangle.$$

Optimal program state is $\Xi_m \leftrightarrow U_m$ where U_m maximizes $\mathrm{Tr}(U_m^\dagger U(\alpha))$.

Classical behavior - superpositions do not help.

Probabilistic programmable circuits



Suppose we want

$$\text{out} = \frac{1}{\|A(\Xi)\|} A(\Xi)|\psi\rangle$$

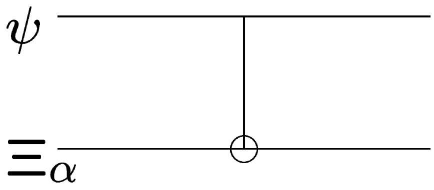
where $A(\Xi)$ is any linear operator. Can this be done?

Yes, if $\mathcal{H}_d$ has dimension D, then a program space of dimension D^2 will work.

Increasing the probability

Can we systematically increase the success probability of a probabilistic quantum processor?

Example (Vidal and Cirac, and Preskill):



If

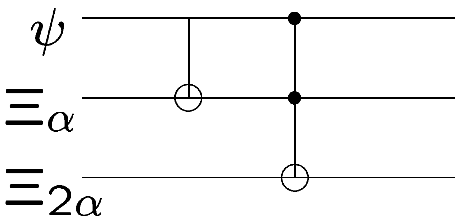
$$|\Xi_\alpha\rangle = \frac{1}{\sqrt{2}}(e^{i\alpha}|0\rangle + |e^{-i\alpha}|1\rangle)$$

then

$$|\Psi_{out}\rangle = \frac{1}{\sqrt{2}}[U(\alpha)|\psi\rangle|0\rangle + U^\dagger(\alpha)|\psi\rangle|1\rangle]$$

Measure program output in computational basis. With probability 1/2 the output of the data register will be $U(\alpha)|\psi\rangle$.

To increase the success probability use the circuit



where the second gate is a Toffoli gate. Measure program output in computational basis. Get

$$00, 01, 10 \;\Rightarrow\; |\psi\rangle \to U(\alpha)|\psi\rangle$$

$$11 \;\Rightarrow\; |\psi\rangle \to U^\dagger(3\alpha)|\psi\rangle$$

Success probability is increased to 3/4.

Same procedure works to increase success probability of nonunitary operation

$$A = \cos\theta|0\rangle\langle 0| + e^{i\phi}\sin\theta|1\rangle\langle 1|.$$

Conclusions

- Programmable circuits can be either deterministic or probabilistic.

- Not all sets of maps can be performed deterministically.

- Probabilistic circuits can perform a much wider class of maps.

- There are methods to increase the success probability of probabilistic circuits.

Future directions

- Further explore approximate deterministic circuits

- Find methods for increasing the success probability for more complicated operations

- Find useful examples of programmable circuits with simple program states