Scaling and Renormalization in Fault-Tolerant Quantum Computers

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Fault-Tolerant Quantum Computation

Big Question: is it possible to run an arbitrarily long quantum computation in the presence of errors?

First Crack: Shor (1996) - yes, provided that *error rate is polylogarithmically small in circuit size*; a physically unreasonable assumption!

State-of-the-Art: Kitaev (1997), Knill, Laflamme, and Zurek (1998), Preskill (1998), Aharonov and Ben-Or (1999) - yes, provided that *error rate is smaller than some threshold value* η_c; requires polylogarithmic complexity overhead

The Scoop on Aharonov and Ben-Or

Rigorous proofs of the "threshold theorem" for:

- local stochastic noise
- noise with exponentially decaying correlations
- general noise

  (including computers with qubits arranged on the hypercubic lattice $\mathbf{Z}^d$, $d=1,2,\ldots$, with gates applied to nearest neighbors only)

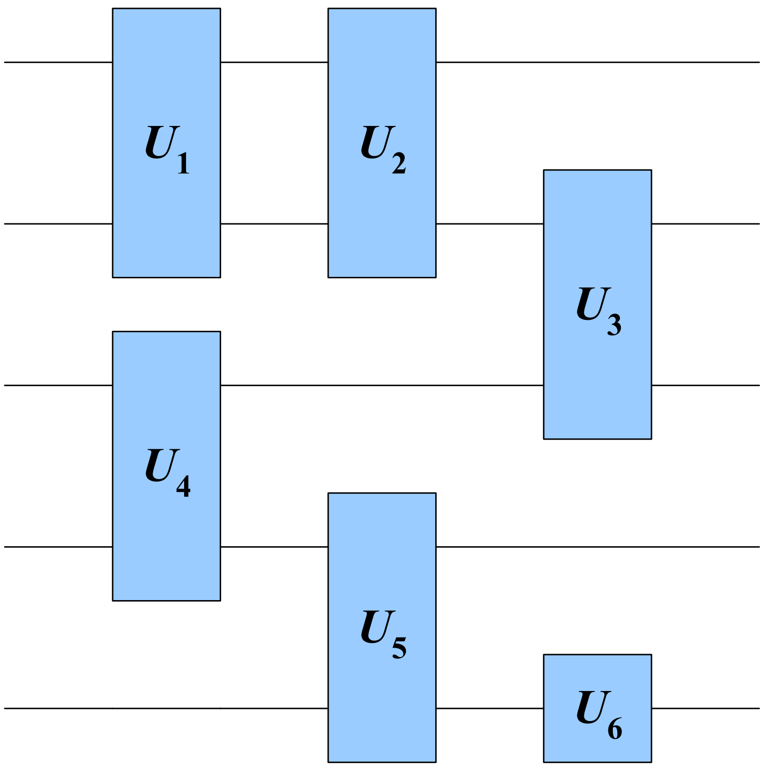
Key technique: hierarchical simulations using concatenated quantum error-correcting codes

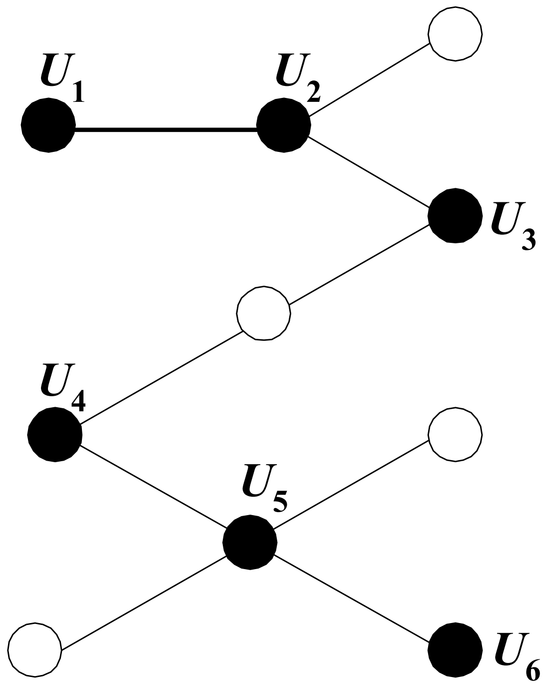
How to Interpret the Proof of Aharonov and Ben-Or

- Local stochastic noise in the computer C = defects (disorder) on a suitably constructed graph Γ_C
- Concatenation = "renormalization" of Γ_C; start with a subthreshold system, iterate the renormalization map to push the system away from the threshold and toward the trivial ($\eta \to 0$) behavior
- Threshold error rate η_c = nontrivial fixed point of the renormalization map

Quantum Circuits, Interaction Graphs, and Error Locations

Given a quantum circuit C, we construct the interaction graph Γ_C





Concatenated Codes and Hierarchical Simulation

- Concatenated code: encode each qubit in a block of m qubits, encode each of those in a block of m qubits, …
  r levels of concatenation $\rightarrow$ each qubit is encoded in m^r qubits
- Hierarchical simulation: compute directly on encoded states; start with circuit C_0, replace each gate with a fault-tolerant procedure (including error correction) to get $C_1=M(C_0)$; continue r times to get $C_r=M^r(C_0)$

Quantum Computation Codes

Ingredients:

- quantum error-correcting code 1-to-m, can correct up to d errors
- spread s of the code: one error anywhere during a procedure will result in at most s single-qubit errors at the end of the procedure; demand $s \leq d$
- all operations (encoding, decoding, computation) are performed using gates from some universal set

Local Stochastic Noise and Bernoulli Site Percolation on Γ_C

<u>Local stochastic noise</u>: each vertex of Γ_C is affected by an error with probability η and left intact with probability $1-\eta$, independently of all other vertices; thus all vertices are statistically equivalent

<u>Key feature:</u>

- percolation threshold η_*: no infinite connected cluster a.s. for $\eta < \eta_*$, infinite connected cluster a.s. for $\eta > \eta_*$
- mean cluster size is finite for $\eta < \eta_*$,

Estimating the Threshold: Renormalization Argument

- One level of concatenation: $C_0 \to C_1$, $\Gamma_0 \to \Gamma_1$
- *Renormalize* Γ_1 by replacing all of the vertices corresponding to a procedure with a single vertex, draw edges appropriately - the resulting graph $R(\Gamma_1)$ should be isomorphic to Γ_0 (from local self-similarity)
- *New site percolation process* on $R(\Gamma_1)$: a vertex is occupied if $k+1$ or more errors occurred in the corresponding procedure in Γ_1, where $k = [d/s]$.

Estimating the Threshold: Renormalization Argument (cont'd)

- The renormalized site occupation probability:

$$R(\eta) \le \sum_{l=k+1}^{A} \binom{A}{l} \eta^{l} (1-\eta)^{A-l} \le 2^{A} \eta^{k+1}$$

  A = maximum number of locations in a procedure

- The effective error rate goes down, i.e., $R(\eta) < \eta$, for η satisfying the threshold condition $\eta < \eta_{c} = 2^{-A/k}$ (expect it to be much smaller than the percolation threshold η_{*})
- Iterate r times to get $R^{r}(\eta) \le 2^{-A/k} (2^{A/k} \eta)^{(k+1)^{r}}$; for $\eta < \eta_{c}$ we have $R^{r}(\eta) < R^{r-1}(\eta) < \ldots < R(\eta) < \eta$

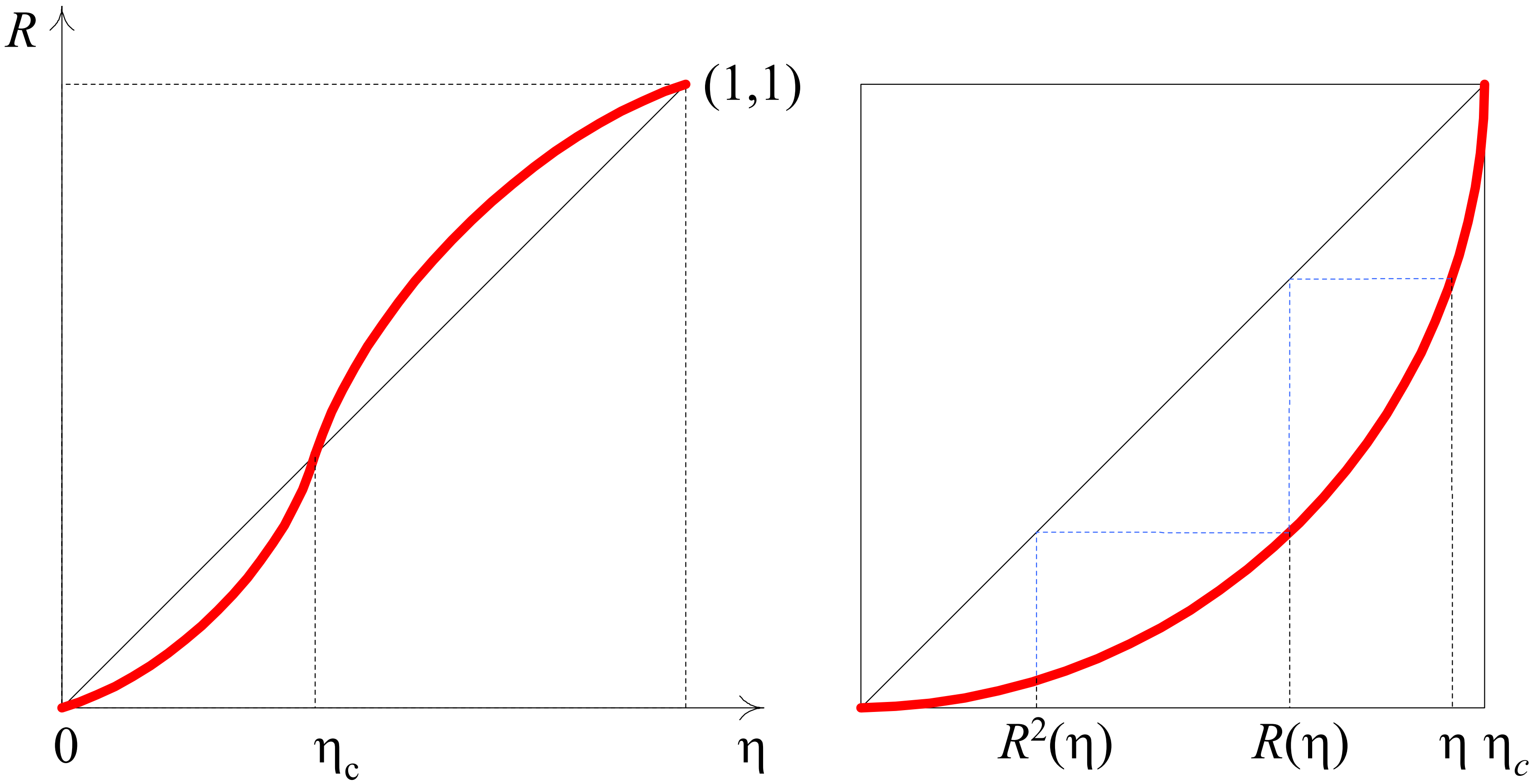
Estimating the Threshold: Renormalization Argument (cont'd)

What is the required number of concatenation levels?

- N = number of locations in the original circuit C_0 (including identity gates)
- Given some $\varepsilon > 0$, we want the effective error rate per procedure in C_r to be smaller than ε/N, so that the total error is less than ε

The required number of concatenation levels is polylogarithmic in N: $r = O(\mathrm{polylog}(N/\varepsilon))$

Exact Value of the Threshold



Trade-Off between Complexity Overhead and Threshold Error Rate

- From general arguments in percolation theory [Chayes, Chayes, Fisher, and Spencer (1986)]:

$$\frac{R(\eta)(1-R(\eta))}{\eta(1-\eta)} \leq R'(\eta) \leq \sqrt{\frac{A}{\eta(1-\eta)}}$$

- The rate at which the effective error rate goes down with concatenation in a subthreshold computer is controlled by $\lambda = R'(\eta_c)$

Key inequality: $1 < \lambda \leq \sqrt{\frac{A}{\eta_c(1-\eta_c)}}$

Trade-Off … (cont'd)

For r concatenation levels to suffice, we must have $\lambda = 2^{K/r}$, where K is some constant

Then

$$2^{K/r} \le \sqrt{\frac{A}{\eta_c(1-\eta_c)}}$$

We get the *trade-off inequality*:

$$\boxed{\eta_c(1-\eta_c) \le A2^{-2K/r}}$$

Summary

- The idea behind the proof of the “threshold theorem” for fault-tolerant quantum computation can be seen as a renormalization argument for a suitably defined site percolation process
- In general, subthreshold quantum computers will be well into the subcritical phase of the percolation process
- There is a trade-off between the threshold error rate and the complexity overhead needed for the fault-tolerant circuit