# Operator-Schmidt Decompositions And the Fourier Transform:

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The following transparancies (except for this one) are just as given in my talk to facilitate web-surfers listening to the talk audio. My paper of the same title, will appear on the Archive within one day from the time these transparancies are posted. The construction of a two-qutrit unitary with Schmidt number 4 has since been carried out. (It was trivial, but I left it as an open problem.)

Note however, that if I were giving this talk again today, I would include material from the following paper, (as is explained in the arXiv version):

W. Dür, G. Vidal, J. I. Cirac, "Optimal conversion of nonlocal unitary operations," Phys. Rev. Lett. **89** (2002), 057901.

Their paper connects the study of operator-Schmidt numbers with probabilistic interconversion of unitary operators aided by (S)LOCC. Furthermore, they first discovered the fact that two-qubit unitaries have Schmidt numbers 1, 2, or 4. (I attributed this to Nielsen et. al, who rediscovered this subsequently, but didn't explain the significance of this fact in terms of (S)LOCC.)

1.Motivation of the operator-Schmidt decomposition: Nielsen's coherent communication complexity bound (and a slight generalization).

Nielsen et. al, "Quantum Dynamics as a Physical Resource," (To appear in Phys Rev A; quant-ph/0208077 ) & other work of Nielsen

2. Apply (1) to show the communication complexity of the quantum Fourier transform is maximal, generalizing previous special cases of Nielsen and Nielsen et al. (See quant-ph/0210100)

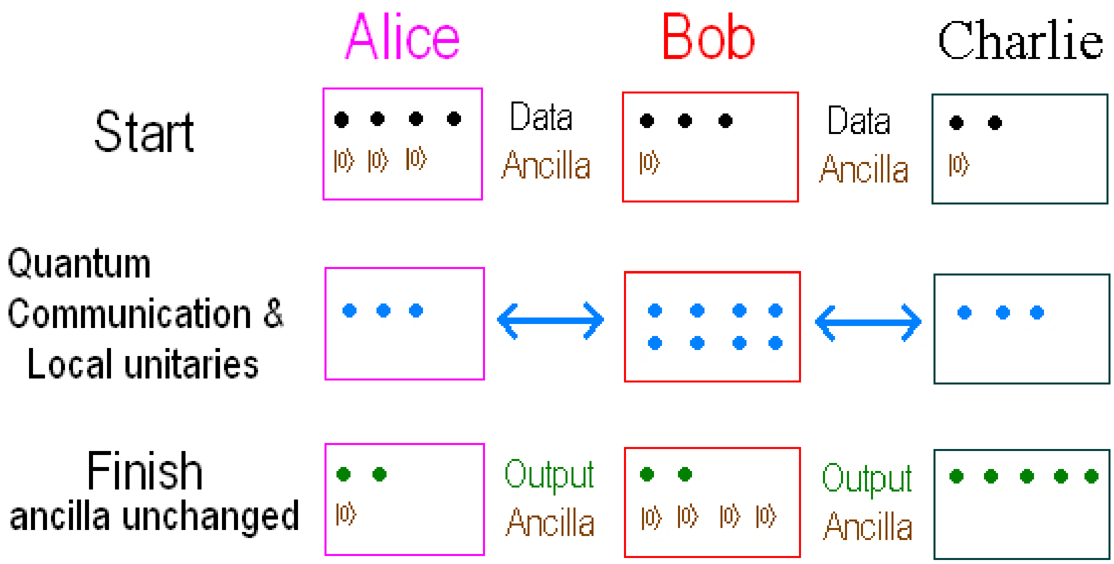
3. Operator-Schmidt decompositions computed using the Fourier transform:

   A. Application to operator-Schmidt number of unitaries.

   B. Construction of maximally-entangled unitaries on $\mathbb{C}^n \otimes \mathbb{C}^n$.

4. Remarks on the "magic basis" of Hill and Wootters. (Time permitting)

## Communication Complexity




alice, bob, eve.jpg

**Strength measures** $S(V)$ of linear operations $V$:

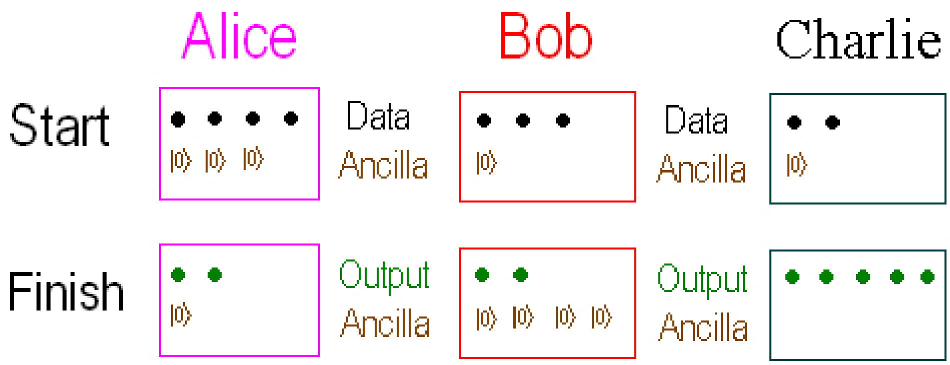
Desirable properties:

| | | |
|---|---|---|
| Nonnegativity: | $S(V) \geq 0$, | |
| Chaining: | $S(VW) \leq S(V) + S(W)$, | $\mathcal{H} \xrightarrow{W} \mathcal{K} \xrightarrow{V} \mathcal{L}$ |
| Locality: | $S(U_1 \otimes U_2 \otimes \ldots \otimes U_M) = 0$, | ($U_\alpha$ local unitaries) |

Nielsen et. al, "Quantum Dynamics as a Physical Resource,"

(To appear in Phys Rev A; quant-ph/0208077 )

# Ancilla Reduction

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$$U \ |\psi\rangle_{\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C}} = \langle 0,0,0|_{\mathcal{A}_1\mathcal{B}_1\mathcal{C}_1} \times V \times |\psi\rangle_{\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C}} \, |0,0,0\rangle_{\mathcal{A}_0\mathcal{B}_0\mathcal{C}_0}$$

$$\Longrightarrow S(U) \leq S(V)$$

Nielsen et al also consider other properties...

## Complexity Bounds using Strength Measures

If $V$ is the implementation of $U$ using ancilla, and $V$ is decomposed into gates and local unitaries

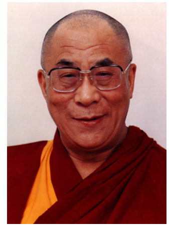
$$V = (C_0 \otimes D_0)\ G_1\ (C_1 \otimes D_1)\ G_2 ... G_n\ (C_n \otimes D_n)\,,$$

then

$$S(U) \leq S(V) \leq \sum_{k=1}^{n} S(G_k)$$

(Nielsen et al, 2002)

Strength from the Minimal



**Definition 1** *Let* $W : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{A}' \otimes \mathcal{B}' \otimes \mathcal{C}'$ *be nonzero. An expression*

$$W = \sum_{j=1}^{\ell} A_j \otimes B_j \otimes C_j, \quad A_j : \mathcal{A} \to \mathcal{A}', \text{ ect.}$$

*is called a* ***minimal decomposition*** *if there is no other such expression with a fewer than* $\ell$ *terms. The smallest such* $\ell$ *is called the* ***minimal length*** *of* $W$*, and denoted* $\mathrm{ML}(W)$*. The* ***minimal strength*** *is*

$$S_{\min}(W) = \log_2 \mathrm{ML}(W) .$$

# Properties of Minimal Strength: Chaining

Suppose $V$ and $W$ have minimal depositions

$$V = A_1 \otimes B_1 + A_2 \otimes B_2$$
$$W = C_1 \otimes D_1 + C_2 \otimes D_2$$

where

$$\mathcal{H} \otimes \mathcal{K} \xrightarrow{V} \mathcal{L} \otimes \mathcal{M} \xrightarrow{W} \mathcal{Q} \otimes \mathcal{R}$$

Then

$$VW = A_1C_1 \otimes B_1D_1 + A_1C_2 \otimes B_1D_2 + A_2C_1 \otimes B_2D_1 + A_2C_2 \otimes B_2D_2.$$

$$\mathrm{ML}\,(VW) \leq \mathrm{ML}\,(V)\,\mathrm{ML}\,(W)$$

Taking logs,

$$\boxed{S_{\min(VW) \leq S_{\min(V)+S_{\min(W)}}}}$$

Two tensor products

Let

$$W : \mathcal{A} \to \mathcal{A}'$$
$$X : \mathcal{B} \to \mathcal{B}'$$

$$W \otimes X : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}'$$
$$\phi \otimes \psi \mapsto (W\phi) \otimes (X\psi)$$

$$W \tilde{\otimes} X \in B\left(\mathcal{A} \to \mathcal{A}'\right) \otimes B\left(\mathcal{B} \to \mathcal{B}'\right)$$
$$\langle W_1, W_2 \rangle_{B(\mathcal{H} \to \mathcal{H}')} = \operatorname{Tr} W_1^\dagger W_2$$

Since

$$\left\langle \sum_i A_i \tilde{\otimes} B_i, \sum_j C_j \tilde{\otimes} D_j \right\rangle_{B(\mathcal{A}\to\mathcal{A}')\otimes B(\mathcal{B}\to\mathcal{B}')}$$

$$= \sum_{i,j} \langle A_i, C_j \rangle_{B(\mathcal{A}\to\mathcal{A}')} \langle B_i, D_j \rangle_{B(\mathcal{B}\to\mathcal{B}')}$$

$$= \left\langle \sum_i A_i \otimes B_i, \sum_j C_j \otimes D_j \right\rangle_{B(\mathcal{A}\otimes\mathcal{B}\to\mathcal{A}'\otimes\mathcal{B}')}$$

There exists a natural isomorphism

$$\Omega : B\left(\mathcal{A} \to \mathcal{A}'\right) \otimes B\left(\mathcal{B} \to \mathcal{B}'\right) \to B\left(\mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}'\right)$$
$$C \tilde{\otimes} D \to C \otimes D$$
$$\tilde{\otimes} \to \otimes$$

## Properties of Minimal Strength: General Reduction

$$V \overset{\min}{=} A_1 \otimes B_1 + A_2 \otimes B_2 + \ldots$$
$$(\mathbb{P}_1 \otimes \mathbb{P}_2)V = \mathbb{P}_1(A_1) \otimes \mathbb{P}_2(B_1) + \mathbb{P}_1(A_2) \otimes \mathbb{P}_2(B_2) + \ldots$$

$S_{\min}$ satisfies the **General reduction property:**

$$S_{\min}\left((\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_M)\, V\right) \leq S_{\min}(V)$$

$$\text{for } V : \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_M \to \mathcal{H}'_1 \otimes \ldots \otimes \mathcal{H}'_M$$

$$\text{and } \mathbb{P}_k : B(\mathcal{H}_k \to \mathcal{H}'_k) \to B(\mathcal{H}_k \to \mathcal{H}'_k)$$

General Reduction $\Longrightarrow$ Ancilla Reduction

Suppose $U$ is implemented using ancilla:

$$U \; |\psi\rangle_{\mathcal{A}\otimes\mathcal{B}} = \langle 0|_{\mathcal{A}_1\mathcal{B}_1} \times V \times |\psi\rangle_{\mathcal{A}\otimes\mathcal{B}} |0\rangle_{\mathcal{A}_0\mathcal{B}_0}$$

Define

$$\mathbb{P}_A : B\left(\mathcal{A}\otimes\mathcal{A}_0 \to \mathcal{A}\otimes\mathcal{A}_1\right) \to B\left(\mathcal{A}\to\mathcal{A}\right)$$

by

$$\mathbb{P}_A\left(W_A\right) \; |\psi\rangle_{\mathcal{A}} = \langle 0|_{\mathcal{A}_1} W_A \times |\psi\rangle_{\mathcal{A}} |0\rangle_{\mathcal{A}_0}$$

Then

$$U = \left(\mathbb{P}_A \otimes \mathbb{P}_B \otimes \mathbb{P}_C\right) V$$

and ancilla reduction follows:

$$S_{\min}\left(U\right) \leq S_{\min}\left(V\right)$$

We have proved the minimal strength satisfies
the desired properties, but

How can one compute it?

The Generalized operator-Schmidt decomposition

$$W : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}'$$

$$W = \sum_{i=1}^{\mathrm{Sch}(W)} \lambda_i \, A_i \otimes B_i, \quad \lambda_i > 0$$

where $A_i : \mathcal{A} \to \mathcal{A}'$ and $B_i : \mathcal{B} \to \mathcal{B}'$ are orthonormal:

$$\mathrm{Tr}\, A_i^\dagger A_j = \mathrm{Tr}\, B_i^\dagger B_j = \delta_{ij}.$$

and define the **Generalized Hartley Strength**

$$S_{\mathrm{har}}\,(W) \equiv \log_2 \mathrm{Sch}\,(W)$$

# Operator-Schmidt decomposition is just (vector)-Schmidt decomposition

$$W \in B\left(\mathcal{H} \otimes \mathcal{K} \to \mathcal{H}' \otimes \mathcal{K}'\right)$$

$$\Omega^\dagger W \in B\left(\mathcal{H} \to \mathcal{H}'\right) \otimes B\left(\mathcal{K} \to \mathcal{K}'\right)$$

$$\Omega^\dagger W = \sum \lambda_i \, A_i \tilde{\otimes} B_i$$

$$W = \sum \lambda_i A_i \otimes B_i$$

$U, W \in B(\mathcal{H} \otimes \mathcal{K} \to \mathcal{H}' \otimes \mathcal{K}')$ have the same Schmidt coefficients

$$\Longleftrightarrow$$

$U = (\mathbb{A} \otimes \mathbb{B})\, V$ for **unitary** "operator-operators"

$\mathbb{A} : B(\mathcal{H} \to \mathcal{H}') \to B(\mathcal{H} \to \mathcal{H}'),\, B : B(\mathcal{K} \to \mathcal{K}') \to B(\mathcal{K} \to \mathcal{K}')$

One would like invariants which determine much more:

Open Problem (except for qubits):

When does $U = (C \otimes D)\ V\ (E \otimes F)$ for local unitaries $C, D, E, F$ ??

**Theorem (Nielsen):** $S_{\text{har}}(U) = S_{\min}(U)$ for bipartite $U$.

**Proof.** $U : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}'$

$$U = \sum_{k=1}^{\text{Sch}(U)} \lambda_k C_k \otimes D_k \Longrightarrow S_{\min}(U) \leq S_{\text{har}}(U).$$

Note that

$$\text{Span}(\{D_k\}) = \left\{ \langle E|_{B(\mathcal{H})} |U\rangle_{B(\mathcal{H}\otimes\mathcal{K})} \,\middle|\, E : \mathcal{H} \to \mathcal{H}' \right\}$$

Taking a minimal decomposition

$$U = \sum_{k=1}^{\text{ML}(U)} G_k \otimes H_k$$

then

$$\left\{ \langle E|_{B(\mathcal{H}\to\mathcal{H}')} |U\rangle_{B(\mathcal{H}\otimes\mathcal{K}\to\mathcal{H}'\otimes\mathcal{K}')} \middle| E : \mathcal{H} \to \mathcal{H}' \right\} \subseteq \text{Span}(\{H_k\})$$

$$S_{\text{har}}(U) \leq S_{\min}$$

■

## The Communication Operator

The **communication operator** :

$$C : (\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}) \otimes \mathbb{C}^{n_3} \longrightarrow \mathbb{C}^{n_1} \otimes (\mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3})$$
$$(\, a \otimes b\,) \otimes c \mapsto a \otimes (\, b \otimes c) \,.$$

$C$ has generalized Schmidt decomposition

$$C = \sum_{j=1}^{n_2} A_j \otimes B_j, \quad \text{(Normalization ignored)}$$

where

$$A_j = \sum_{i=1}^{n_1} |i\rangle \langle ij| \quad \text{and} \quad B_j = \sum_{k=1}^{n_3} |jk\rangle \langle k| \,.$$

$$A_j \otimes B_j \ |\ell m\rangle \otimes |n\rangle = \delta_{jm} \ |\ell\rangle \otimes |mn\rangle$$

In particular $\mathrm{Sch}\,(C) = n_2$.

Nielsen's Coherent Communication Bound
(slight generalization to qudits and to allow net data transfer)

**Theorem** If $U : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}'$ is implemented using ancilla, local operations, and quantum communication (NO CC!) then

$$\sum_{d=2}^{\infty} N_d \log_2(d) \geq S_{\text{har}}(U),$$

where

$$N_d = \text{The number of qudits of dimension } d \text{ communicated}$$

Nielsen's complexity bound is:

$$\begin{aligned} S_{\text{har}}\left(U_{\mathcal{A}\otimes\mathcal{B}\to\mathcal{A}'\otimes\mathcal{B}'}\right) &\leq S_{\text{har}}\left(V_{\mathcal{A}_0\otimes\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{B}_0\to\mathcal{A}_1\otimes\mathcal{A}'\otimes\mathcal{B}'\otimes\mathcal{B}_1}\right) \\ &\leq \sum_{k=1}^{n} S_{\text{har}}(G_k) \end{aligned}$$

where $V$ is the implementation of $U$ using ancilla.

# References for Nielsen's Communication Bound (all take $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ and consider qubits.)

1. M. A. Nielsen, Ph.D. Thesis, University of New Mexico, 1998, Chapter 6; quant-ph/0011036.

$$\sum_{d=2}^{2} N_d \log_2(d) \geq \frac{1}{2} S_{\text{har}}(U),$$

2. M. A. Nielsen, "Entanglement and distributed quantum computation," Talk at the Benasque Center for Physics, July 19, 2000; URL: http://www.qinfo.org/talks/index.html. Proved

$$\sum_{d=2}^{2} N_d \log_2(d) \geq S_{\text{har}}(U),$$

using a clever argument using reference states, working on the level of vectors.

3. M. A. Nielsen et al, "Quantum dynamics as a physical resource", (accepted for publication in Phys. Rev. A); quant-ph/0208077. Used (ungeneralized) Hartley strength + `SWAP` to prove

$$\sum_{d=2}^{2} N_d \log_2(d) \geq \frac{1}{2} S_{\text{har}}(U),$$

# Application: Communication complexity of the QFT

$$\mathcal{F} : \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \to \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}, \quad N = M_1 M_2 = N_1 N_2$$

Alice Bob

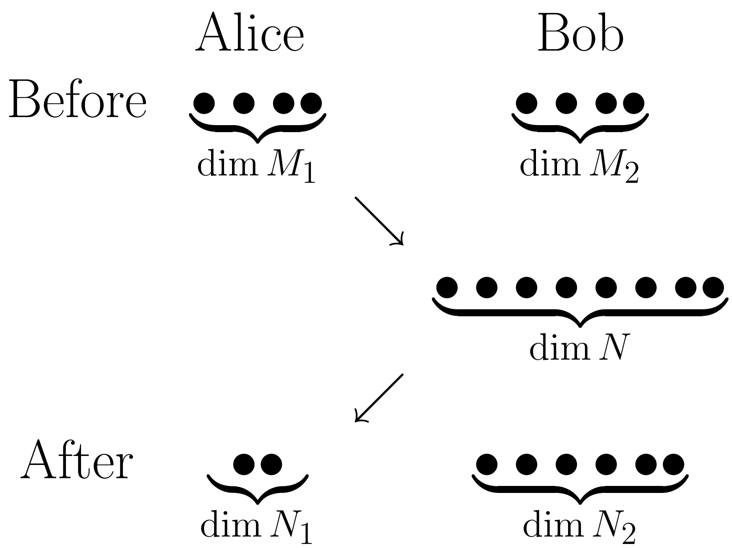
Before $\underbrace{\bullet\bullet\bullet\bullet}_{\dim M_1}$ $\underbrace{\bullet\bullet\bullet\bullet}_{\dim M_2}$

After $\underbrace{\bullet\bullet}_{\dim N_1}$ $\underbrace{\bullet\bullet\bullet\bullet\bullet\bullet}_{\dim N_2}$

What is maximal communication complexity?

# Maximal Communication Complexity

$$\mathcal{F} : \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \to \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}, \quad N = M_1 M_2 = N_1 N_2$$




Trivial upper bound:

$$\boxed{Q_0\left(\mathcal{F}_{\mathbb{C}^{M_1}\otimes\mathbb{C}^{M_2}\to\mathbb{C}^{N_1}\otimes\mathbb{C}^{N_2}}\right) \le \log_2 \min\left(N_1 M_1, N_2 M_2\right)}$$

# The QFT using Mixed-Decimals

$$N = M_1 M_2 = N_1 N_2.$$

$$\boxed{\mathcal{F}_{M1} M_2 \longrightarrow N_1 N_2 \colon \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2} \ \longrightarrow\ \mathbb{C}^N \ \stackrel{\mathcal{F}_N}{\longrightarrow}\ \mathbb{C}^N \ \longrightarrow\ \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}}$$

$\left|k_1\right\rangle_{M_1} \left|k_2\right\rangle_{M_2}$

$\longrightarrow \left|k_1 M_2 + k_2\right\rangle_N$

$\longrightarrow \frac{1}{N} \sum_{s=0}^{N-1} \exp\left(\frac{2\pi i}{N}\left(k_1 M_2 + k_2\right) s\right) \left|s\right\rangle_N$

$\longrightarrow \frac{1}{N} \sum_{p_1=0}^{N_1-1} \sum_{p_2=0}^{N_2-1} \exp\left(\frac{2\pi i}{N}\left(k_1 M_2 + k_2\right)\left(p_1 N_2 + p_2\right)\right) \left|p_1\right\rangle_{N_1} \left|p_2\right\rangle_{N_2}$

Warning

$\mathcal{F}_{N_1 N_2 \rightarrow M_1 M_2}$ does NOT in general have Schmidt coefficients related in any way to that of to $\mathcal{F}_{N_2 N_1 \rightarrow M_1 M_2}$, $\mathcal{F}_{N_1 N_2 \rightarrow M_2 M_1}$ or $\mathcal{F}_{N_2 N_1 \rightarrow M_2 M_1}$.

The Fourier transform knows which digit in the mixed-decimal expansion is the high-order qudit.

let $\omega = \exp(2\pi i/10)$

$$\sqrt{10}\mathcal{F}_{\mathbb{C}^2\otimes\mathbb{C}^5\to\mathbb{C}^2\otimes\mathbb{C}^5} = \begin{bmatrix} 1 & 1 \\ 1 & \omega^{25} \end{bmatrix} \otimes \begin{bmatrix} \omega^0 & \cdot & \omega^0 & \cdot & \omega^0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \omega^0 & \cdot & \omega^4 & \cdot & \omega^8 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \omega^0 & \cdot & \omega^8 & \cdot & \omega^{16} \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 1 \\ \omega^5 & \omega^{30} \end{bmatrix} \otimes \begin{bmatrix} \cdot & \omega^0 & \cdot & \omega^0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega^2 & \cdot & \omega^6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega^4 & \cdot & \omega^{12} & \cdot \end{bmatrix}$$

$$+ \text{one omitted term}$$

$$+ \begin{bmatrix} 1 & \omega^5 \\ \omega^5 & \omega^{35} \end{bmatrix} \otimes \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega^1 & \cdot & \omega^3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega^3 & \cdot & \omega^9 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Example: $\mathcal{F} : \mathbb{C}^2 \otimes \mathbb{C}^6 \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^3$
$\omega = \exp\left(\frac{2\pi i}{12}\right)$

$$\sqrt{12}\mathcal{F} = \begin{bmatrix} 1 & 1 \\ 1 & \omega^{15} \\ 1 & \omega^{30} \\ 1 & \omega^{45} \end{bmatrix} \otimes \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \omega^8 & \cdot \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 1 \\ \omega^5 & \omega^{20} \\ \omega^{10} & \omega^{40} \\ \omega^{15} & \omega^{60} \end{bmatrix} \otimes \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega^2 & \cdot & \cdot & \cdot & \omega^{10} \end{bmatrix}$$

$$+ \ldots$$

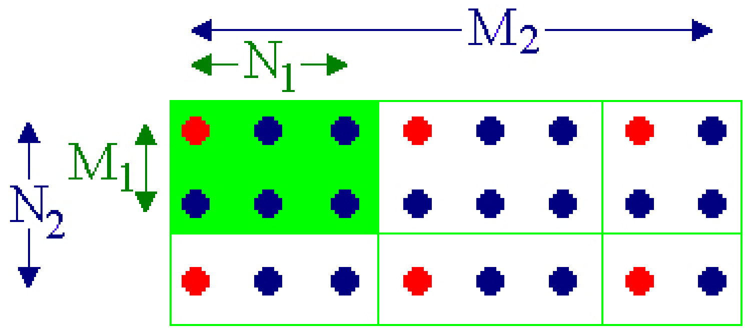
$$+ \begin{bmatrix} 1 & \omega^3 \\ 1 & \omega^{18} \\ 1 & \omega^{33} \\ 1 & \omega^{48} \end{bmatrix} \otimes \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \omega^4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$+ \ldots$$

Theorem: (Operator-Schmidt decomposition of the QFT)

$$\mathcal{F}_{M_1 M_2 \to N_1 N_2} = \sum_C \lambda_c \, A_C \otimes B_C,$$

where $N = N_1 N_2 = M_1 M_2$, and $C$ is a red equivalence class of $\mathbb{Z}_{N_2} \times \mathbb{Z}_{M_2}$ mod $M_1, N_1$ :




$$\lambda_C = \sqrt{\frac{N_1 M_1 \, \mathrm{Card}\,(C)}{N}}$$

$$(A_C)_{jk} = \frac{1}{\sqrt{N_1 M_1}} \exp\left[\frac{2\pi i}{N}\left(N_2 M_2 jk + M_2 k\hat{s} + N_2 j\hat{t}\right)\right], \ \left(\hat{s}, \hat{t}\right) \in C$$

$$(B_C)_{jk} = \frac{1}{\sqrt{\mathrm{Card}\,(C)}} \times \begin{cases} \exp\left(\frac{2\pi i}{N} jk\right) & \text{if } (j,k) \in C \\ 0 & \text{otherwise} \end{cases}$$

Corollary: The Communiciation Complexity of the QFT is Maximal in all cases.

Problems (Some open):

1. How much quantum communication is required to implement $U^{\otimes n}$, if one allows a small error which is negligible (in some appropriate sense) for large $n$? (open)

   (Warning: Nielsen et al show that the entanglement of $U : \mathcal{H} \times \mathcal{K} \to \mathcal{H}' \otimes \mathcal{K}'$ as an element of $B\left(\mathcal{H} \to \mathcal{H}'\right) \otimes B\left(\mathcal{K} \to \mathcal{K}'\right)$ does NOT satisfy chaining.)

2. Does the answer to (1) depend only on the operator-Schmidt coefficients of $U$?

3. Is there a nice canonical decomposition of the maximally-entangled unitaries? (Open) Can we at least construct infinitely many inequivalent maximal unitaries? (Maybe)

4. What operator-Schmidt numbers exist for unitaries on $\mathbb{C}^n \otimes \mathbb{C}^m$? (We'll consider $\mathbb{C}^3 \otimes \mathbb{C}^3$.)

## Schmidt number of two qubit unitaries

**Theorem (Makhlin 2000; Khaneja, Brockett, Glaser 2001):** Let $U$ be a two-qubit unitary. Then $\exists$ one-qubit unitaries $A, B, C, D$ and $\theta_\alpha \in (-\pi/4, \pi/4]$ such that

$$U = (A \otimes B)\, e^{i\left(\theta_x X\otimes X + \theta_y Y \otimes Y + \theta_z Z \otimes Z\right)}\, (C \otimes D)\,.$$

Furthermore, $V$ has the same invariants $\theta_\alpha$ iff

$$V = (E \otimes F)\, U\, (G \otimes H)$$

for some local unitaries $E$, $F$, $G$, and $H$.

**Corollary (Nielsen et al 2002):** A two-qubit unitary may have Schmidt-number 1, 2, or 4, but not 3.

**Conjecture (Nielsen et al 2002):** There exist unitary operators on $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ of Schmidt number $S$ iff $S$ divides $N_1 N_2$. An alternative conjecture is that unitary operators with Schmidt number $S$ exist iff $S$ and $N_1 N_2$ have a common factor.

## References for the Canonical Decomposition

1. Y. Makhlin, LANL e-print quant-ph/0002045 (2000).

   Gives the invariants $\theta_\alpha$.

2. N. Khaneja, R. Brockett, and S. J. Glaser, "Time optimal control of spin systems", *Phys. Rev. A* **63**, 032308 (2001); LANL e-print quant-ph quant-ph/0006114.

   Gives the decomposition.

3. B. Kraus and J. I. Cirac, "Optimal creation of entanglement using a two-qubit gate", *Phys. Rev. A* **63**, 062309 (2001); LANL e-print quant-ph/0011050.

   An elegant proof of the decomposition.

## Operator-Schmidt Decompositions given by the Discrete Fourier Transform

Theorem: There exists a basis $\phi_{\alpha\beta}$ of $\mathbb{C}^N \otimes \mathbb{C}^N$ such that the operator-Schmidt coefficients of the diagonal operator

$$D = \sum_{\alpha,\beta=0}^{N-1} \lambda(\alpha,\beta) |\phi_{\alpha\beta}\rangle \langle\phi_{\alpha\beta}|$$

are the nonzero values of $\left|\hat{\lambda}(\alpha,\beta)\right|$, where $\hat{}$ is the discrete Fourier transform

$$\hat{\lambda}(\alpha,\beta) = \frac{1}{N} \sum_{\alpha',\beta'=0}^{N-1} \exp\left(\frac{2\pi i}{N}(\alpha\alpha' + \beta\beta')\right) \lambda(\alpha',\beta').$$

**Technical Remark:** We will work on $\mathcal{H} \otimes \mathcal{H}^*$, with $\dim \mathcal{H} = N$, rather than on $\mathbb{C}^N \otimes \mathbb{C}^N$, to employ operator-theory techniques to this problem.

**Definition** Let $\mathcal{H}$ be an $N$-dimensional Hilbert space. Define the **Twist** $T : \mathcal{H} \to \mathcal{H}$ and **Discrete Rotation** $R : \mathcal{H} \to \mathcal{H}$ and by

$$T\,|j\rangle = \exp\left(\frac{2\pi i j}{N}\right)|j\rangle\,, \quad j \in \{0, ..., N-1\}$$
$$R\,|j\rangle = |j + 1 \text{ mod } N\rangle$$

Products of the form $R^\alpha T^\beta$ are called **Twisted Rotations.**

$$\boxed{\begin{array}{l} T^k \longleftrightarrow e^{itX} \\ R^m \longleftrightarrow e^{isP} \end{array}}$$

## The Discrete Fourier Transform and Twisted Rotations

**Theorem 2 (Weyl 1930)** *$R$ and $T$ satisfy the discrete Weyl relations*

$$R^\alpha = I \text{ iff } \alpha \in N\mathbb{Z} \tag{1}$$

$$T^\alpha = I \text{ iff } \alpha \in N\mathbb{Z} \tag{2}$$

$$RT = \exp(-2\pi i/N)\, TR \tag{3}$$

*and the associated Fourier relations*

$$\mathcal{F}R\mathcal{F}^\dagger = T \tag{4}$$

$$\mathcal{F}T\mathcal{F}^\dagger = R^\dagger. \tag{5}$$

*Furthermore, any irreducible representation of (1)−(3) is unitarily equivalent to the defined rotations and twists, and in such a representation an $F$ satisfying (4) − (5) is represented up to a phase as the discrete Fourier transform.*

**Observation (J. Schwinger 1960):** The set of twisted rotations

$\left\{ \frac{1}{\sqrt{N}} R^{\alpha} T^{\beta} \right\}_{\alpha, \beta \in \{0, \ldots, N-1\}}$ forms an orthonormal basis of $B\left(\mathcal{H}\right)$.

# The Space $\mathcal{H} \otimes \mathcal{H}^*$

$$\psi = \sum a_{ij} \left|i\right\rangle \otimes \left\langle j\right| \in \mathcal{H} \otimes \mathcal{H}^*$$

$$A_\psi = \sum a_{ij} \left|i\right\rangle \left\langle j\right| : \mathcal{H} \to \mathcal{H}$$

## The Hilbert Space of Bras

**Definition:** The **Dual space** $\mathcal{H}^*$ is the space of (continuous) linear functionals on $\mathcal{H}$. In Dirac notation, $\mathcal{H}^*$ is the space of bras.

What is the inner product on $\mathcal{H}^*$?

Define the map $\psi \to \bar{\psi} : \mathcal{H} \to \mathcal{H}^*$ by

$$\overline{|\psi\rangle} = \langle\psi| \,.$$

The inner product on $H^*$ is given by

$$\left\langle \bar{f}, \bar{g} \right\rangle_{\mathcal{H}^*} = \langle g, f \rangle_{\mathcal{H}} \,.$$

Unfortunately, Dirac notation would become confusing in what follows. Equivalently,

$$\Big\langle \langle\psi| \quad , \langle\phi| \Big\rangle_{\mathcal{H}^*} = \langle \phi, \psi \rangle_{\mathcal{H}}$$

# The Conjugate Operator

Let $A : \mathcal{H} \to \mathcal{H}$. The **conjugate operator** $\bar{A} : \mathcal{H}^* \to \mathcal{H}^*$ is given by

$$\bar{A}\bar{f} = \overline{Af}$$

# Why the Bar?

$$f = \sum_k a_k \, |k\rangle \Longrightarrow \bar{f} = \sum_k \bar{a}_k \; \langle k| = \sum_k \bar{a}_k \overline{|k\rangle}$$

$$A \; |j\rangle = \sum_k a_{jk} \, |k\rangle \Longrightarrow \bar{A} \; \overline{|j\rangle} = \sum_k \bar{a}_{jk} \; \overline{|k\rangle}$$

The Natural Isomorphism

**Theorem:** The linear map $\Xi : B(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}^*$satisfying

$$\Xi \left( |f\rangle \langle g| \right) = f \otimes \bar{g}$$

is a unitary equivalence. Furthermore, for

$$A, B, C \in B(\mathcal{H}).$$

one has

$$\left( A \otimes \bar{B} \right) \, \Xi C = \Xi \left( A C B^{\dagger} \right).$$

Furthermore, $\Xi C$ is maximally entangled iff $C$ is a nonzero scalar multiple of a unitary.

$$C \text{ is unitary } \Longleftrightarrow C = \sum |e_j\rangle \langle f_j| \Longleftrightarrow \Xi C = \sum e_j \tilde{\otimes} \bar{f}_j$$

Schmidt Decompositions given by the Fourier Transform

**Theorem:** Define

$$\Phi_{\alpha\beta} = \Xi\left(T^{\alpha} R^{-\beta}\right) \in \mathcal{H} \otimes \mathcal{H}^{*}, \quad \alpha, \beta \in \{0, ..., N-1\}$$

Then the $\{\Phi_{\alpha\beta}\}$ form a maximally entangled orthonormal basis. Furthermore, the diagonal operator $D$

$$D = \sum_{\alpha,\beta=0}^{N-1} \lambda(\alpha, \beta) \left|\Phi_{\alpha\beta}\right\rangle \left\langle\Phi_{\alpha\beta}\right| : \mathcal{H} \otimes \mathcal{H}^{*} \to \mathcal{H} \otimes \mathcal{H}^{*},$$

with arbitrary $\lambda(\alpha, \beta) \in \mathbb{C}$, satisfies

$$D = \sum_{a,b=0}^{N-1} \hat{\lambda}(a, b) \times \left(\frac{1}{\sqrt{N}} R^{a} T^{b}\right) \otimes \overline{\left(\frac{1}{\sqrt{N}} R^{a} T^{b}\right)}.$$

In particular, the Schmidt coefficients of $D$ are the nonzero values of $\left|\hat{\lambda}(a, b)\right|$.

**Proof.** $\Phi_{\alpha\beta}$ is an eigenvector of each $\left(R^aT^b\right)\otimes\overline{(R^aT^b)}$:

$$\begin{aligned}\left(R^aT^b\right)\otimes\overline{(R^aT^b)}\,\Phi_{\alpha\beta} &= \left(R^aT^b\right)\otimes\overline{(R^aT^b)}\,\Xi\left(T^\alpha R^{-\beta}\right)\\ &= \Xi\left(R^aT^bT^\alpha R^{-\beta}\left(R^aT^b\right)^\dagger\right)\\ &= \exp\left(-\frac{2\pi i}{N}\left(a\alpha+b\beta\right)\right)\Xi\left(T^\alpha R^{-\beta}\right)\\ &= \exp\left(-\frac{2\pi i}{N}\left(a\alpha+b\beta\right)\right)\;\Phi_{\alpha\beta}\end{aligned}\tag{6}$$

Since the $\{\Phi_{\alpha\beta}\}$ form an orthonormal basis,

$$\left(R^aT^b\right)\otimes\overline{(R^aT^b)}=\sum_{\alpha\beta=0}^{N-1}\exp\left(-\frac{2\pi i}{N}\left(a\alpha+b\beta\right)\right)\;\left|\Phi_{\alpha\beta}\right\rangle\left\langle\Phi_{\alpha\beta}\right|.$$

By the Fourier inversion theorem,

$$\left|\Phi_{\alpha\beta}\right\rangle\left\langle\Phi_{\alpha\beta}\right| = \frac{1}{N^2}\sum_{a,b=0}^{N-1}\exp\left(\frac{2\pi i}{N}\left(a\alpha+b\beta\right)\right)\left(R^aT^b\right)\otimes\overline{\left(R^aT^b\right)}$$

Hence

$$\begin{aligned}
D &= \sum_{\alpha,\beta=0}^{N-1}\lambda\left(\alpha,\beta\right)\ \left|\Phi_{\alpha\beta}\right\rangle\left\langle\Phi_{\alpha\beta}\right| \\
&= \sum_{\alpha,\beta=0}^{N-1}\lambda\left(\alpha,\beta\right)\frac{1}{N^2}\sum_{a,b=0}^{N-1}\exp\left(\frac{2\pi i}{N}\left(a\alpha+b\beta\right)\right)\left(R^aT^b\right)\otimes\overline{\left(R^aT^b\right)} \\
&= \frac{1}{N}\sum_{a,b=0}^{N-1}\hat{\lambda}\left(a,b\right)\ \left(R^aT^b\right)\otimes\overline{\left(R^aT^b\right)}.
\end{aligned}$$

■

Application: Operator-Schmidt numbers on $\mathbb{C}^3 \otimes \mathbb{C}^3$

**Question:** How many zeros may the discrete Fourier transform of a phase-valued function have?

**Theorem:** There exists a phase-valued function $\lambda$ on $\mathbb{Z}_3^2$ such that the support of $\hat{\lambda}$ has cardinality $S$ iff $S \in \{1,3,5,6,7,8,9\}$.

**Proof.**

| $S$ | $f$ | $\hat{f}$ |
|---|---|---|
| 1 | $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |

| $S$ | $f$ | $\hat{f}$ |
|---|---|---|
| 3 | $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \omega & \omega & \omega \end{bmatrix}$ | $\begin{bmatrix} \omega+2 & 0 & 0 \\ \omega+2 & 0 & 0 \\ i\sqrt{3} & 0 & 0 \end{bmatrix}$ |
| 5 | $\begin{bmatrix} 1 & -1 & 1 \\ \omega & -\omega & \omega^2 \\ \omega^2 & -\omega^2 & \omega \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 1 & \omega^2 & \omega \\ 0 & 1-\omega & 1-\omega^2 \end{bmatrix}$ |
| 6 | $\begin{bmatrix} 1 & 1 & 1 \\ \omega & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 1 & \omega^2 & \omega \\ 2 & 1+\omega & 1+\omega^2 \end{bmatrix}$ |

| $S$ | $f$ | $\hat{f}$ |
| --- | --- | --- |
| 7 | $\begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & -\omega^2 & \omega \\ -1 & \omega^2 & -\omega \end{bmatrix}$ | $\frac{1}{3}\begin{bmatrix} 0 & 0 & 3 \\ -2+2\omega & 1+2\omega & 7+2\omega \\ 2-2\omega & -1-2\omega & -1-2\omega \end{bmatrix}$ |
| 8 | $\begin{bmatrix} \omega & \omega & \omega^2 \\ 1 & -\omega^2 & \omega^2 \\ -1 & 1 & -\omega \end{bmatrix}$ | $\frac{1}{3}\begin{bmatrix} 0 & -3+3\omega & 3 \\ -2+2\omega & 1+2\omega & 4+5\omega \\ -1+\omega & -1-2\omega & -1-2\omega \end{bmatrix}$ |
| 9 | $S=9$ is generic | |

Now take $S = 2$ or $4$ and assume there exists a phase-valued function $f$ on $\mathbb{Z}_3^2$ so that the cardinality of the support of $\hat{f}$ is $S$. Then since $\bar{f}f = 1$ identically, then

$$\begin{aligned}\delta_{\vec{v},0} &= \widehat{\left(\bar{f}f\right)}(-\vec{v}) \\ &= \frac{1}{3} \sum_{w \in \{0,\ldots,2\}^2} \overline{\hat{f}} \left(\vec{v} + \vec{w} \bmod 3\mathbb{Z}^2\right) \hat{f}(\vec{w})\end{aligned}$$

But it is not hard to see that for any subset of $\mathbb{Z}_3^2$ of cardinality 2 or 4 there exists a nonzero $\vec{v} \in \mathbb{Z}_3^2$ such that $f(\vec{w}) \neq 0$ and $f\left(\vec{v} + \vec{w} \bmod 3\mathbb{Z}^2\right) \neq 0$ for exactly one value of $\vec{w}$, yielding a contradiction.

■

The Fourier transform doesn't give everything for $\mathbb{C}^3 \otimes \mathbb{C}^3$

**Theorem (the commutative case):** There exists a unitary $U$ on $\mathbb{C}^N \otimes \mathbb{C}^N$ with $\mathrm{Sch}\,(U) = S$ for any $S \in \{1, ..., N\}$..

**Proof.** Fix $S$ and set $\psi^S = \sum_{ij=1}^N \psi_{ij}^S \, |ij\rangle$ with

$$\psi_{ij}^S = \begin{cases} -1 & \text{if } i = j \text{ and } i < S \\ 1 & \text{otherwise} \end{cases}.$$

Then $\mathrm{Sch}\left(\psi^S\right) = S$, since the matrix $\left[\psi_{ij}^S\right]$ is rank $S$. Schmidt-decompose

$$\psi = \sum_{i=1}^S \lambda_i v_i \otimes w_i, \;\; \lambda_i > 0.$$

A diagonal unitary operator $U_\psi$ with $\mathrm{Sch}\,(U_\psi) = S$ is constructed by changing the vectors $v_i$ and $w_i$ in this decomposition into diagonal matrices.

■

## Construction of Maximally Entangled Unitaries on $\mathbb{C}^N \otimes \mathbb{C}^N$

Recall that

$$D = \sum_{\alpha,\beta=0}^{N-1} \lambda(\alpha,\beta) \, |\Phi_{\alpha\beta}\rangle \langle\Phi_{\alpha\beta}| : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^*,$$

satisfies

$$D = \sum_{a,b=0}^{N-1} \hat{\lambda}(a,b) \times \left(\frac{1}{\sqrt{N}} R^a T^b\right) \otimes \overline{\left(\frac{1}{\sqrt{N}} R^a T^b\right)}.$$

Hence

$D$ is maximally entangled $\iff$ $\lambda$ and $\hat{\lambda}$ are phase-valued functions.

## Biunimodular Functions and Maximally Entangled Unitaries

**Definition:** A function $f$ is **biunimodular** iff both $f$ and $\hat{f}$ are phase-valued functions.

The problem of characterizing biunimodular functions on $\mathbb{Z}_N^2$ proper has not been studied. Note however, that if $f$ and $g$ are biunimodular on $\mathbb{Z}_N$ then $f \otimes g$ is biunimodular on $\mathbb{Z}_N^2 = \mathbb{Z}_n \times \mathbb{Z}_n$.

## Known results on biunimodular functions on $\mathbb{Z}_N$

- Guassian biunimodulars were known to Guass:

$$g_{N,a,b}(k) = \exp\left(\frac{2\pi i}{N}\left(ak^2 + bk\right)\right), \quad N \text{ odd}, a \text{ coprime to } N$$
$$g_N(k) = \exp\left(\frac{2\pi i}{N}k^2\right), \quad N \text{ even}$$

- For $N$ divisible by a square, Björck and Saffari (1995) have constructed an uncountably infinite family of biunimodulars on $\mathbb{Z}_N$. Furthermore, they conjecture that there exist an infinite number (up to phases) iff $N$ is divisible by a square. Interestingly, their examples use mixed-decimals.

- The conjecture of Björck and Saffari was already known to be valid for $N \leq 9$, as all such biunimodulars had already been found. (G Björck, 1990, J. Backelin and R. Fröberg 1991, G Björck and R Fröberg 1991 and 1994)

Remarks on Hill & Wootters's Magic Basis
and the Gradient of the Determinant

The gradient of the determinant is defined by

$$\frac{d}{dt}\det(A) = \left\langle \nabla \det(A), \frac{dA}{dt} \right\rangle_{B(\mathcal{H})}$$

In particular

$$\boxed{\nabla \det(A) = \begin{cases} \left(\det(A)\, A^{-1}\right)^{\dagger} & \text{if } \det(A) \neq 0 \\ \text{the continuous extension, otherwise} \end{cases}}$$

What is the meaning of $\nabla \det$ in the $B(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^*$ formalism?

**Theorem** Let $\mathfrak{C} : B(\mathcal{H}) \to B(\mathcal{H})$ be the continuous extension of

$$\mathfrak{C}(A) = \left((\det A)\, A^{-1}\right)^{*}$$

Define the corresponding map $\mathfrak{D} : \mathcal{H} \otimes \mathcal{H}^{*} \to \mathcal{H} \otimes \mathcal{H}^{*}$ by

$$\mathfrak{D}(\psi) = \Xi \mathfrak{C}\left(\Xi^{*} \psi\right).$$

Then

**Theorem 3** *1.* $\mathfrak{C}(AB) = \mathfrak{C}(A)\,\mathfrak{C}(B)$ *and* $\mathfrak{C}(A^{*}) = (\mathfrak{C}(A))^{*}$. *In particular,* $\mathfrak{C}$ *acts independently on the factors of the polar decomposition.*

*2.* $\mathfrak{D}\left(\left(A \otimes \bar{B}\right)\psi\right) = \left(\mathfrak{C}(A) \otimes \overline{\mathfrak{C}(B)}\right)\mathfrak{D}(\psi)$. *In particular, if A and B are unitary then* $\mathfrak{D}\left(\left(A \otimes \bar{B}\right)\psi\right) = (\det A^{*}B)\left(A \otimes \bar{B}\right)\mathfrak{D}(\psi$

*3.* $\mathfrak{D}(\psi) = \alpha\psi$ *for some* $\alpha \in \mathbb{C}$ *iff* $\psi$ *is maximally entangled or zero.*

*4. Temporarily allowing Schmidt coefficients to vanish, the product of the Schmidt coefficients of* $\psi$ *is given by* $N^{-1}\left|\langle \psi, \mathfrak{D}\psi \rangle_{\mathcal{H}\otimes\mathcal{H}^{*}}\right|$

5. *Furthermore, if* $N=2$ *then*

(a) $\mathfrak{C}$ *and* $\mathfrak{D}$ *are conjugations, i.e. antiunitary maps squaring to the identity, and*

$$\mathfrak{C}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix}$$

(b) $\psi$ *is separable iff* $\langle \psi, \mathfrak{D}\psi \rangle = 0$.

(c) *Denote* $|i\bar{j}\rangle \equiv |i\rangle \otimes \overline{|j\rangle} \in \mathcal{H} \otimes \mathcal{H}^*$. *Then each of the following basis vectors are invariant under* $\mathfrak{D}$*:*

$$\left\{ |0\bar{0}\rangle + |1\bar{1}\rangle\,,\ i\,|0\bar{0}\rangle - i\,|1\bar{1}\rangle\,,\ i\,|0\bar{1}\rangle + i\,|1\bar{0}\rangle\,,\ |0\bar{1}\rangle - |1\bar{0}\rangle \right\}.$$

(d) *If* $A = e^{i\theta} U P$, *where* $U \in SU(2)$ *and* $P = \operatorname{diag}(\lambda_1, \lambda_2)$ *is positive, then* $\mathfrak{C}(A) = e^{-i\theta} U \operatorname{diag}(\lambda_2, \lambda_1)$. *In particular,* $\mathfrak{D}$ *preserves Schmidt coefficients.*

Vollbrecht and Werner make the following point in “Why two qubits are special”:

> The remarkable properties of the [magic] basis...are in some sense not so much a property of that basis, but of the antiunitary operation of *complex conjugation* in [that] basis.

In particular, one may canonically translate the results and the magic-basis or magic-conjugation proofs of Wooters and Makhlin on $\mathbb{C}^2 \otimes \mathbb{C}^2$ into basis-free results and proofs on $\mathcal{H} \otimes \mathcal{H}^*$.