# Geometric Quantum Computation and Errors

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# Team

- Jiannis Pachos (post-doc)
- Angelo Carollo (finishing PhD)
- Marcelo Santos (post-doc)
- Ivette Fuentes-Guridi (ex student, Perimeter, Oxford).
- Collaboration: A. Ekert, J. A. Jones, J. Anandan, E. Sjoqvist, M. Ericsson, M. Palma, R. Fazio, G. Falci, J. Siewert…

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- Berry phase of a spin-1/2 particle in a magnetic field.
- Any quantum computation can be executed with geometric phases only.
- We analyze the effect of simple errors and show that there is some natural inbuilt resistance.
- How far can this be generalized?

# Adiabatic theorem

If H changes slowly through some parameter, the adiabatic theorem assures that the system remains in the eigenstate of the Hamiltonian.

$$H(R(t))\left|\Psi(R(t))\right\rangle = E\left|\Psi(R(t))\right\rangle$$

And if we change the Hamiltonian so that in a time $\tau$ it returns to its initial form…

**Cyclic evolution**

$$\boldsymbol{H(t) = H(t=0)}$$

# Berry phase

M. Berry, Proc. Roy. Soc. A **392**, 45 (1984)

Then the state returns to its initial form but since eigenstates are defined up to a phase factor, the state could acquire a phase due to the adiabatic and cyclic evolution that took place.

$$\left|\Psi(t)\right\rangle = e^{ib_n(t)} e^{ig_n(t)} \left|\Psi(0)\right\rangle$$

$$b_n(T) = -\int_0^T E_n(t)\,dt$$

Usual dynamical phase

$$g_n(T) = i\oint \left\langle \Psi(t) \right| \frac{d}{dt} \left| \Psi(t) \right\rangle dt$$

Geometrical phase: It depends only on the path taken in the configuration space of parameter R.

# Spin 1/2 particle:

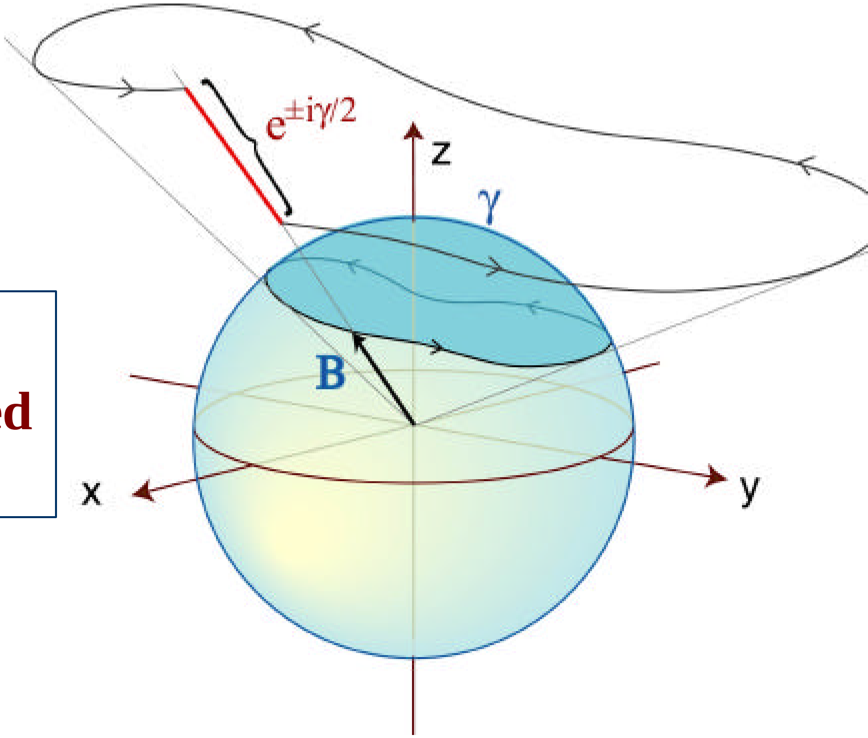
## Canonical example of Berry phase

A typical example is the phase acquired by a spin-1/2 particle interacting with a slowly varying magnetic field:

$$\vec{B}(t) = |B|\,\vec{n}(t),$$

$$\hat{H}(t) = -m\,\vec{B}(t)\cdot\vec{s}$$

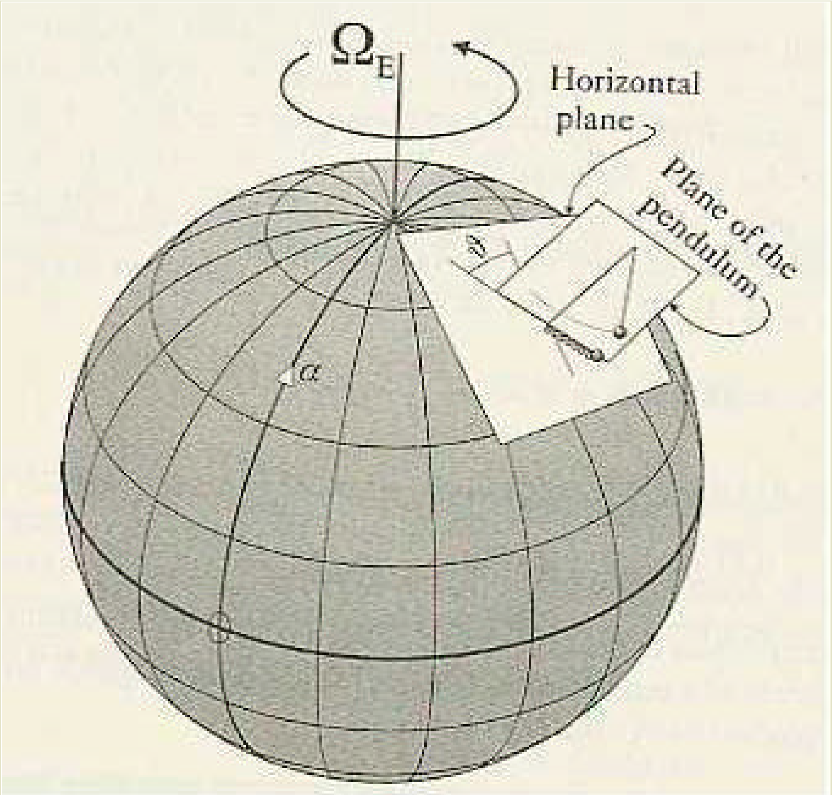
**The eigenstates acquire a geometric phase proportional to the area g enclosed in the path traversed by the field B.**




$$|\pm\rangle_{\vec{n}(0)} \to |\pm\rangle_{\vec{n}(T)} = e^{\pm\frac{i\gamma}{2}}|\pm\rangle_{\vec{n}(0)}$$

# Classical Berry Phase

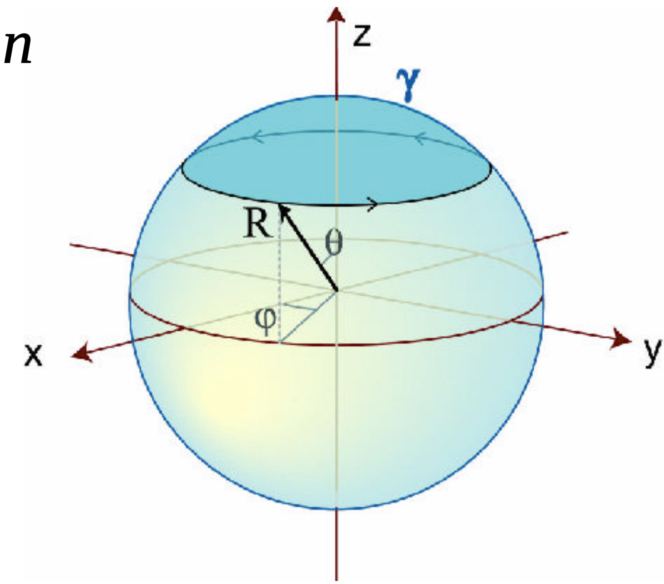
1. Cats and Astronauts;
2. The Earth is a sphere - Foucault's pendulum;



# Spin-1/2 interacting with an e-m field
## Semi-classical description

A 2-level system with Bohr frequency ω, interacting with a classical oscillating field with frequency ν and amplitude α, in the rotating frame is described:

$$\hat{H} = \frac{\Delta}{2} s_z + l\left(s_+ a e^{-ij} + s_- a e^{-ij}\right) \qquad \Delta = w - n$$




We can rewrite the Hamiltonian as $\hat{H} = \mathbf{R} \cdot \vec{s}$

Where: $\mathbf{R} = (l \cos j, l \sin j, \Delta/2)$

By rotating (adiabatically) the vector **R** as shown in the picture (the phase φ is rotated from 0 to 2π), the eigenstates acquire the geometric phase:

$$c_{\pm} = \pm g/2 = \pm p\,(1 - \cos q)$$

where $\cos q = \Delta \Big/ \sqrt{\Delta^2 + 4(a\,l)^2}$

# Spin-1/2 interacting with an e-m field
## Fully quantised description

In this case the interaction is described by the Jaynes Cumming Hamiltonian:

$$\hat{H} = \nu a^{\dagger} a + \frac{\omega}{2} \sigma_z + \lambda \left( \sigma_+ a + \sigma_- a^{\dagger} \right)$$

The change in the Hamiltonian is implemented through:

$$\hat{H}(\varphi) = U(\varphi) \hat{H} U^{\dagger}(\varphi) \qquad U(\varphi) = e^{-i \phi a^{\dagger} a}$$

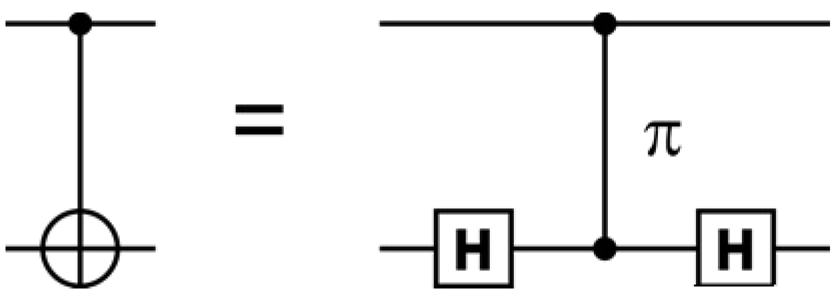
In analogy with the semi-classical case, we apply an adiabatic transformation, by varying φ from 0 to 2π. The eigenstates of the system acquire the geometric phases:

$$|\psi_n^+\rangle \rightarrow +\pi(1 - \cos\theta_n) + 2\pi n,$$
$$|\psi_n^-\rangle \rightarrow -\pi(1 - \cos\theta_n) + 2\pi(n+1).$$

$$\cos\theta_n = \Delta \Big/ \sqrt{\Delta^2 + 4\lambda^2 (n+1)}$$


Carollo, Fuentes-Guridi, Bose and Vedral, PRL (2002).
Carollo, Santos, Vedral, PRA (2003).

# Controlled Not = Phase shift

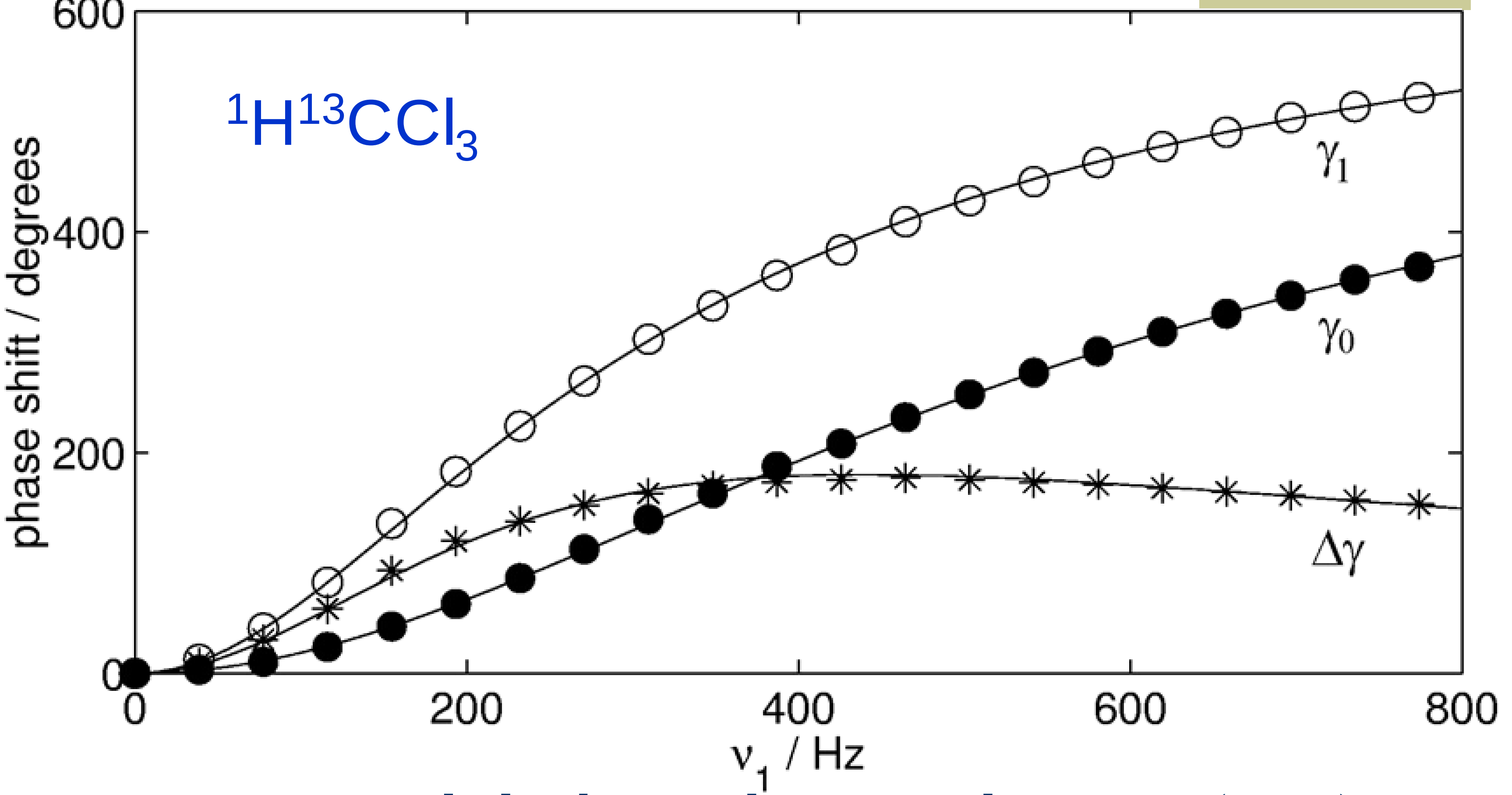



$$|11\rangle \xrightarrow{p} -|11\rangle$$

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{if} \end{pmatrix}$$

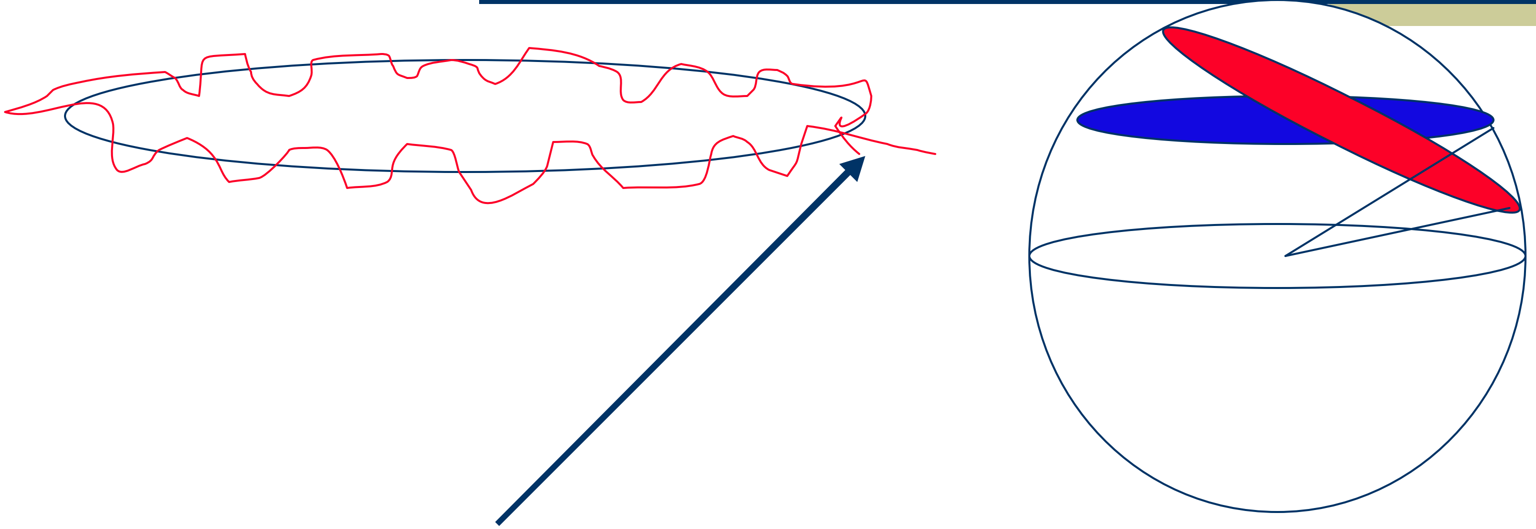
f operates symmetrically on the two qubits: it does not distinguish between control and target bits

# Conditional Geometry




Jones, Vedral, Ekert and Castagnoli, Nature (2000).

# Fault-tolerance



Random motion

Systematic displacement

Geometric phase is invariant under the above.

G. De Chiara and G. M. Palma, quant-ph (2003).

(but see also Blais and Tremblay, PRA (2003))

# Master Equation

- Master Equation

$$\dot{r} = \frac{1}{i}[H, r] - \frac{1}{2}\sum_{k=1}^{n}\{\Gamma^{\dagger}{}_{k}\Gamma_{k} r + r\Gamma^{\dagger}{}_{k}\Gamma_{k} - 2\Gamma_{k} r \Gamma^{\dagger}{}_{k}\}$$

- For small time intervals, $r(t+\Delta t) \approx \sum_{k=0}^{n} W_k r(t) W_k{}^{\dagger}$

where $W_0 = \hat{1} - i\tilde{H}\Delta t$ $\quad W_k = \Gamma_k \sqrt{\Delta t}$

$$\tilde{H} = H - \frac{i}{2}\sum_{k=1}^{n} \Gamma^{\dagger}{}_{k}\Gamma_{k}$$

Carollo, Fuentes-Guridi, Santos and Vedral, PRL (2003).

# Quantum Jumps

- For a given jump "k", $r(t_{m+1}) \approx W_k\, r(t_m) W_k^{\dagger}$ with probability $p_k = \mathrm{Tr}\left\{ W_k\, r(t_m) W_k^{\dagger} \right\}$
- Different trajectories = different set of W's

$$\left|\Psi^{i}{}_{m}\right\rangle = \prod_{l=1}^{m} \mathrm{W}_{\mathrm{i(l)}} \left|\Psi_0\right\rangle$$

= different set of pure states

$$\left\{ \left|\Psi_0\right\rangle, \left|\Psi_0{}^{i}\right\rangle, \ldots, \left|\Psi_N{}^{i}\right\rangle \right\}$$

# Geometric phase

- Pantcharatnam formula

$$\gamma_g = -\arg\{ \langle \Psi_0 | \Psi_1 \rangle \langle \Psi_1 | \Psi_2 \rangle \ldots \langle \Psi_{N-1} | \Psi_N \rangle \langle \Psi_N | \Psi_0 \rangle \}$$

- Continuous limit

$$\gamma_g = -\operatorname{Im} \int_0^T \frac{\langle \Psi(t) | \frac{\mathrm{d}}{\mathrm{d}t} | \Psi(t) \rangle}{\langle \Psi(t) | \Psi(t) \rangle} \mathrm{d}t - \arg \langle \Psi(T) | \Psi(0) \rangle$$

Carollo, Santos, Fuentes-Guridi, Vedral, Phys. Rev. Lett. (2003).

# No Jump

$$\left|\Psi^0{}_m\right\rangle = (\mathrm{W}_0)^m \left|\Psi_0\right\rangle = \left(\hat{1} - i\frac{T}{N}\tilde{H}\right)^{\frac{N}{T}t} \left|\Psi_0\right\rangle$$

- In the limit N >> 1, $i\frac{\mathrm{d}}{\mathrm{dt}}\left|\Psi^0(t)\right\rangle = \tilde{H}\left|\Psi^0(t)\right\rangle$

$$\tilde{H} = H - \frac{i}{2}\sum_{k=1}^{n} \Gamma^{\dagger}{}_k \Gamma_k \qquad W_0 = \hat{1} - i\tilde{H}\Delta t$$

$$g^0{}_g = -\int_0^T \frac{\left\langle \Psi^0(t) \middle| H \middle| \Psi^0(t) \right\rangle}{\left\langle \Psi^0(t) \middle| \Psi^0(t) \right\rangle} - \arg\left\langle \Psi^0(T) \middle| \Psi^0(0) \right\rangle$$

# No Jump

- In particular,

$$\text{If } \sum_{k=1}^{n} \Gamma^{\dagger}{}_{k}\Gamma_{k} \propto \hat{1} \quad \text{then} \quad W_0 = (1-a)\,\hat{1} + i\hat{H}\Delta t$$

and the geometric phase for the no-jump trajectory coincides with the one for the decoherence free evolution.

# 1 jump

- For 1 jump at time $t_1$

$$g^{1}{}_{j} = \int_0^{t_1} \frac{\left\langle \Psi'(t) \middle| \frac{\mathrm{d}}{\mathrm{d}t} \middle| \Psi'(t) \right\rangle}{\left\langle \Psi'(t) \middle| \Psi'(t) \right\rangle} \mathrm{d}t - \arg\left\{ \left\langle \Psi'(t_1) \middle| \Gamma_j \middle| \Psi'(t_1) \right\rangle \right\} +$$

$$\int_{t_1}^{T} \frac{\left\langle \Psi''(t) \middle| \frac{\mathrm{d}}{\mathrm{d}t} \middle| \Psi''(t) \right\rangle}{\left\langle \Psi''(t) \middle| \Psi''(t) \right\rangle} \mathrm{d}t - \arg\left\{ \left\langle \Psi''(T) \middle| \Psi'(0) \right\rangle \right\}$$

$$\left| \Psi'(0) \right\rangle = \left| \Psi_0 \right\rangle, \ \left| \Psi''(t_1) \right\rangle = W_j \left| \Psi'(t_1) \right\rangle$$

# Example: Spin 1/2

- Spin 1/2 coupled to a magnetic field.

$$H = \frac{w}{2} \hat{s}_z$$

- Dephasing, no-jump - same phase!

$$\Gamma = l\, \hat{s}_z ,\quad \Gamma^{\dagger}\Gamma \propto \hat{1}$$

$$g = p\left(1 - \langle \Psi_0 | \hat{s}_z | \Psi_0 \rangle\right) = p\left(1 - \cos q\right)$$

# Dephasing: 1 jump - same phase!!

$$g^{1}_{\hat{s}_z} = -\int_0^{t_1} \frac{w}{2} \langle \Psi(0) | \hat{s}_z | \Psi(0) \rangle dt - \arg\left\{ \langle \Psi(0) | \hat{s}_z | \Psi(0) \rangle \right\}$$

$$- \int_{t_1}^{2p/w} \frac{w}{2} \langle \Psi(0) | \hat{s}_z | \Psi(0) \rangle dt - \arg\left\{ \langle \Psi(0) | e^{i\frac{\hat{s}_z}{2}(2p - wt_1)} \hat{s}_z e^{i\frac{\hat{s}_z}{2} wt_1} | \Psi(0) \rangle \right\}$$

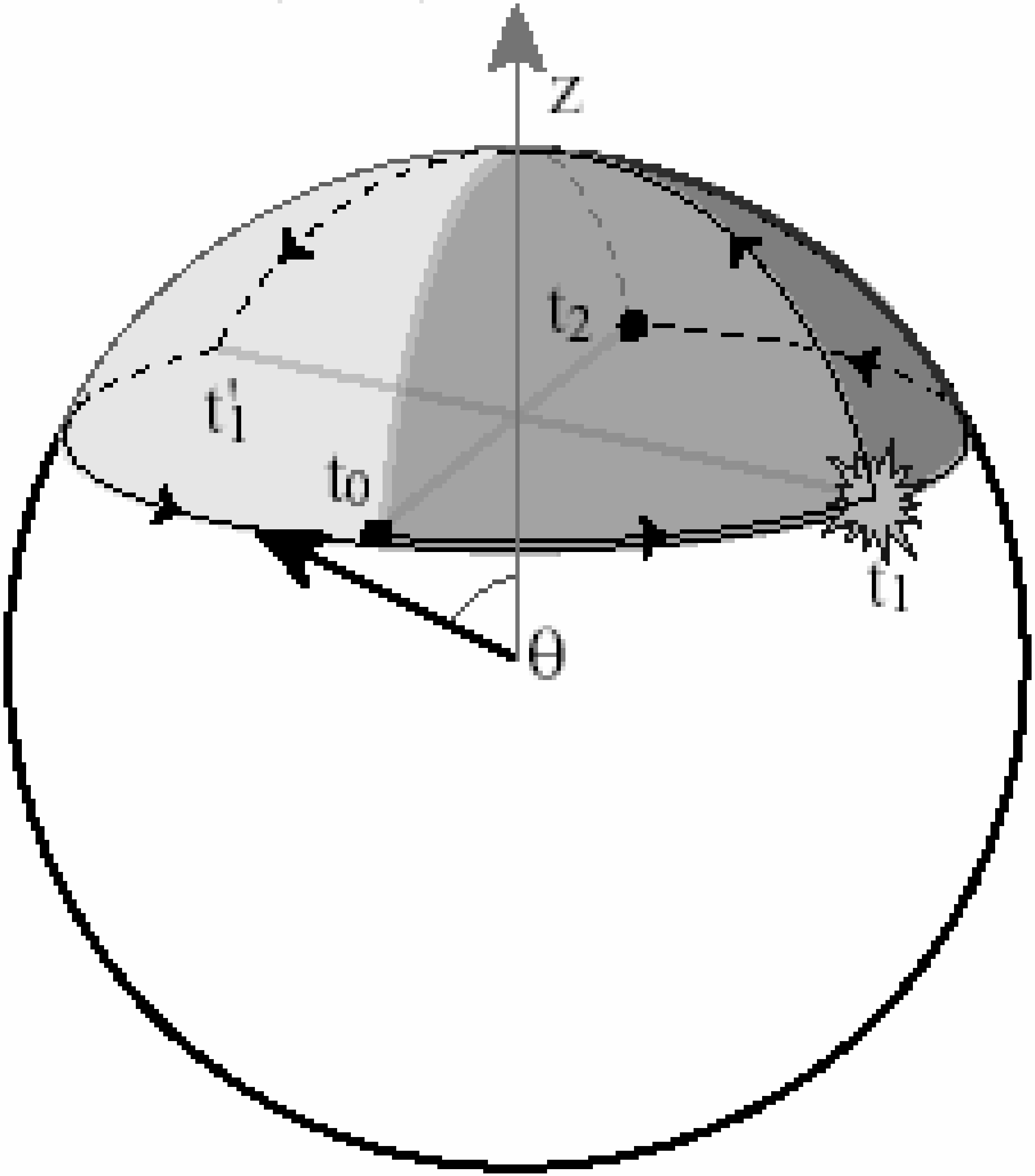
$$= p\left(1 - \langle \Psi(0) | \hat{s}_z | \Psi(0) \rangle\right) = p\left(1 - \cos q\right)$$

- Dephasing, k jumps - same phase!!!

$$g^{k}_{\hat{s}_z} = -\int_0^{2p/w} \frac{w}{2} \langle \Psi(0) | \hat{s}_z | \Psi(0) \rangle dt - \arg\left\{ \langle \Psi(0) | \hat{s}_z | \Psi(0) \rangle^k \right\}$$

$$- \arg\left\{ \langle \Psi(0) | e^{ip\hat{s}_z} \hat{s}_z^{\,k} | \Psi(0) \rangle \right\}$$

$$= p\left(1 - \langle \Psi(0) | \hat{s}_z | \Psi(0) \rangle\right) = p\left(1 - \cos q\right)$$

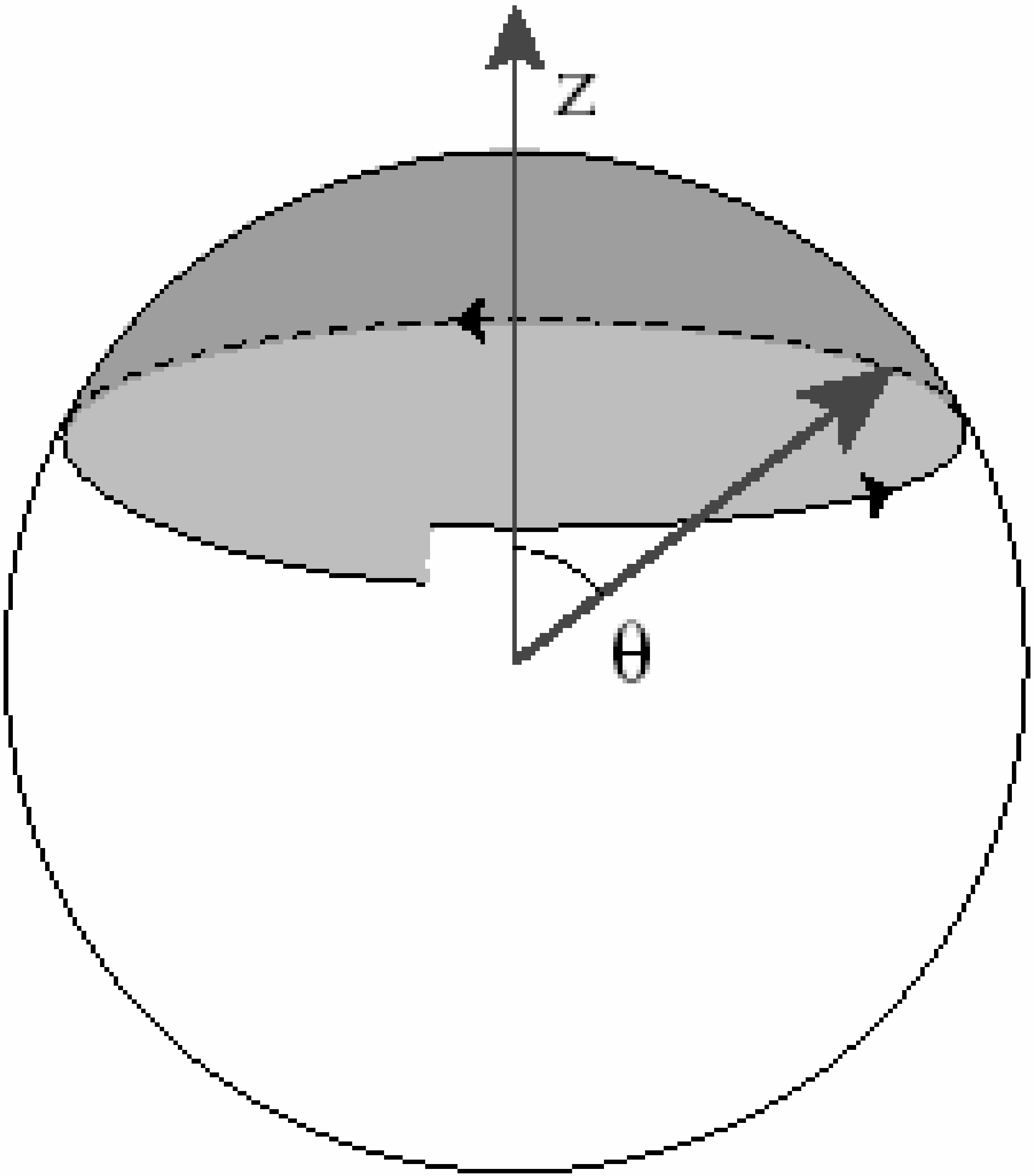

z
t2
t'1
t0
t1
θ

# Spontaneous emission

$$\Gamma = a\,\hat{s}_{-}$$

$$g^{0}_{\hat{s}_{-}} = p + \frac{w}{2a}\ln\left(\left\langle \Psi(0)\right| e^{-2p\frac{w}{a}\hat{s}_{z}} \left|\Psi(0)\right\rangle\right)$$

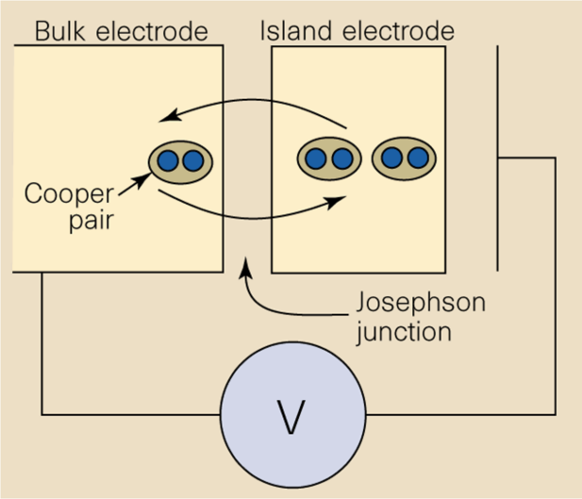
$$g^{0}_{\hat{s}_{-}} \approx p\,(1-\cos q) + (2p)\frac{a}{w}\sin^{2} q + O(\frac{a}{w})^{2},\ \text{for}\ \ w >> a$$


Z
θ

# Implementations

1. NMR - confirmed experimentally - Jones et al, Nature (2000)

2. Josephson Junctions - Fazio et al, Nature (2000).



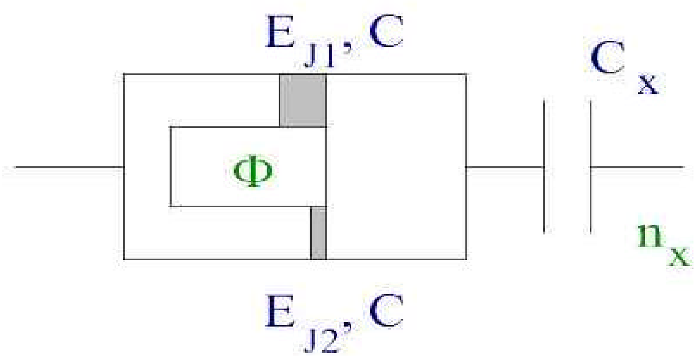

$|0\rangle$ 0 Cooper pairs

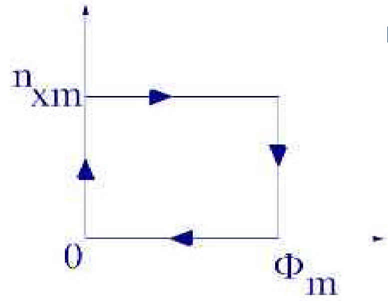
$|1\rangle$ 1 Cooper pair

Y.Nakamura, Y.A.Pashkin, J.S.Tsai, Nature 398, 786 (1999)

(for error analysis see, Whitney and Gefen, PRL (2003))

# Geometry in Josephson




Regime:

$E_{J_1}, E_{J_2} << E_{ch}$

$$H = E_{ch}(n - n_x)^2 - E_J(\Phi)\cos(q - a)$$

$$E_J(\Phi) = \sqrt{(E_{J_1} - E_{J_2})^2 + 4E_{J_1}E_{J_2}\cos^2\left(p\frac{\Phi}{\Phi_0}\right)}$$

$$\tan(a) = \frac{(E_{J_1} - E_{J_2})}{(E_{J_1} + E_{J_2})}\tan\left(p\frac{\Phi}{\Phi_0}\right)$$

$\Phi_0 = h/2e$

# Josephson = Spin 1/2

Only n=0 and n=1 are important:

Effective Hamiltonian:

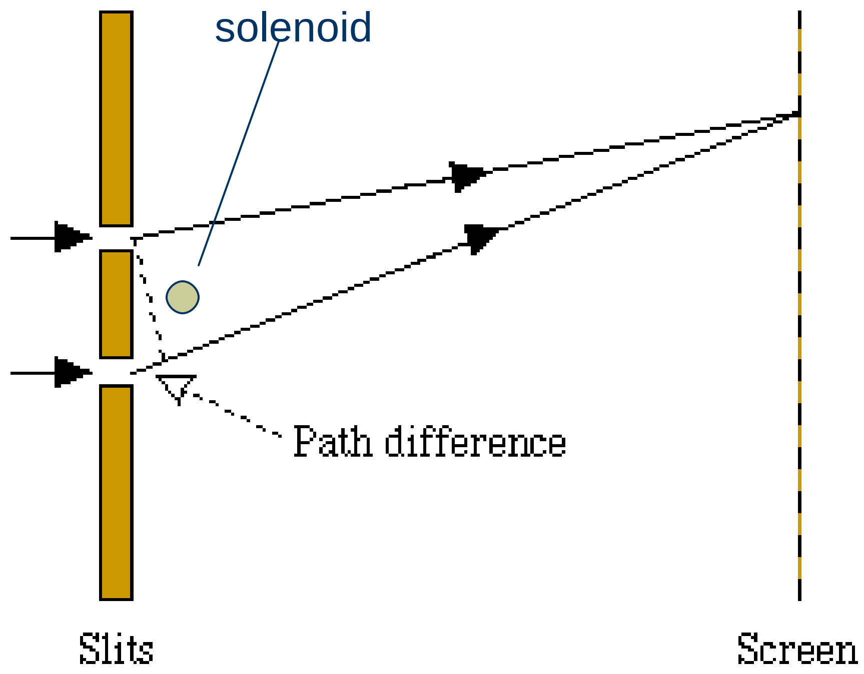
$$H = -\frac{1}{2}\vec{B}\vec{s}$$

where

$$\vec{B} = (E_J(\Phi)\cos(a), -E_J(\Phi)\sin(a), E_{ch}(1-2n_x))$$

# Aharonov-Bohm Effect

Electrons in Phase




$$f = p\,(d_1 - d_2)\,/\,l + \oint_c A\,ds$$

# Degeneracy

$$\begin{pmatrix} |\psi_0\rangle \\ |\psi_1\rangle \end{pmatrix} \xrightarrow{adiab} \exp\left( P\int_0^t A(t)dt \right) \begin{pmatrix} |\psi_0\rangle \\ |\psi_1\rangle \end{pmatrix}$$

where

$$A = \begin{pmatrix} \langle\psi_0|\frac{d}{dt}|\psi_0\rangle & \langle\psi_0|\frac{d}{dt}|\psi_1\rangle \\ \langle\psi_1|\frac{d}{dt}|\psi_0\rangle & \langle\psi_1|\frac{d}{dt}|\psi_1\rangle \end{pmatrix}$$

Wilczek and Zee, PRL 1984.

# Summary and Future

- Geometry offers some protection.
- Topology – how far?
- Implementations?
- Combining other mechanisms of protection.

V. Vedral, Int. J. Q. Info. (2003).
J. Pachos and V. Vedral, quant-ph (2003)