Geometric measure of entanglement

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Summary:

I. GME can deal analytically with

(1) arbitrary two-qubit states
(2) generalized Werner states
(3) isotropic states

} bipartite, EF also known

(4) mixture of permutation-invariant states
(5) Smolin's bound entangled state
(6) Dür's bound entangled states

} multipartite, EF not known

II. GME is related to <u>entanglement witnesses</u> and <u>relative entropy of entanglement</u>

Outline

I. Introduction

II. Review of two entanglement measures

1. Entanglement of distillation
2. Entanglement of formation/cost

III. Geometric measure of entanglement

1. Pure states
2. Mixed states by convex hull
3. Examples

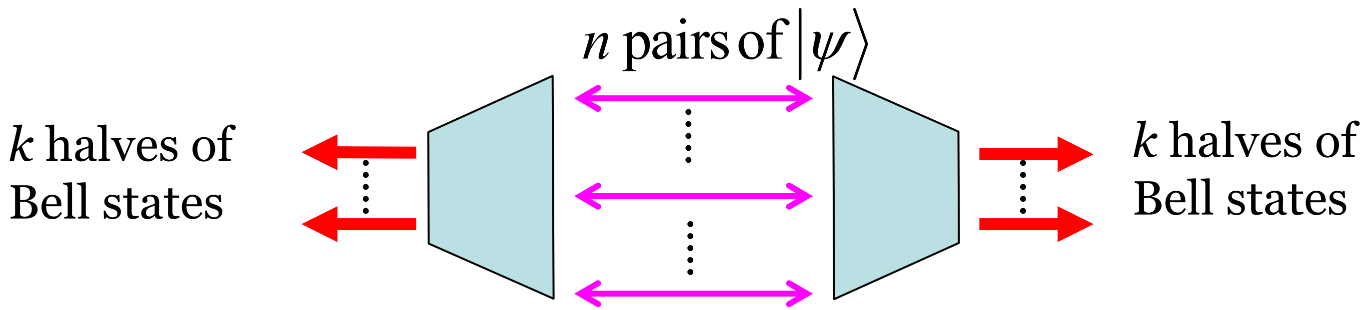
IV. GME and entanglement witness

V. GME and relative entropy of entanglement

VI. GME and quantum phase transitions?

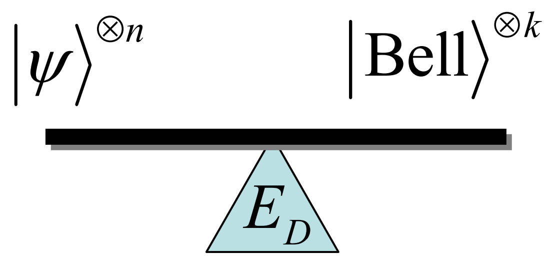
II. Quantifying entanglement

1. Entanglement of distillation [Bennett et al. '96]



$$E_D(\psi) \equiv \lim_{n\to\infty}(k/n)$$

(also applies to mixed states)

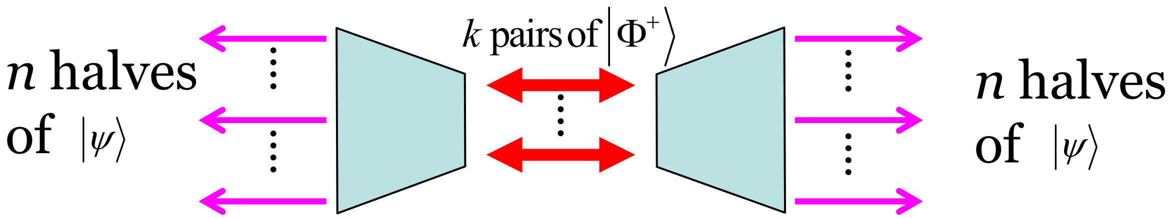


(after Nielsen)

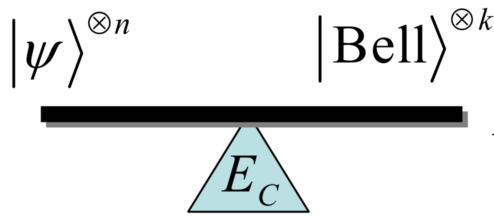
II. Quantifying entanglement

2. <u>Entanglement cost</u> and <u>entanglement of formation</u>

[Bennett et al. ’97]



$$E_C\left(|\psi\rangle\right) \equiv \lim_{k\to\infty}(k/n)$$ (also applies to mixed states)



$E_C = E_D$ for bipartite pure states

A variation for mixed states: Entanglement of formation

$$E_F(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_C\left(|\psi_i\rangle\right), \quad \text{with } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Some remarks…

1. Entanglement of distillation and entanglement cost are difficult to calculate for mixed states

2. Some progress with entanglement of formation:

   a) Wootters' formula for arbitrary two-qubit mixed states [Wootters, PRL'98]

   b) Formulas for symmetric states in higher dimensions

      ① Generalized Werner states [Vollbrecht & Werner, PRA'01]

      ② Isotropic states [Terhal & Vollbrecht PRL '00]

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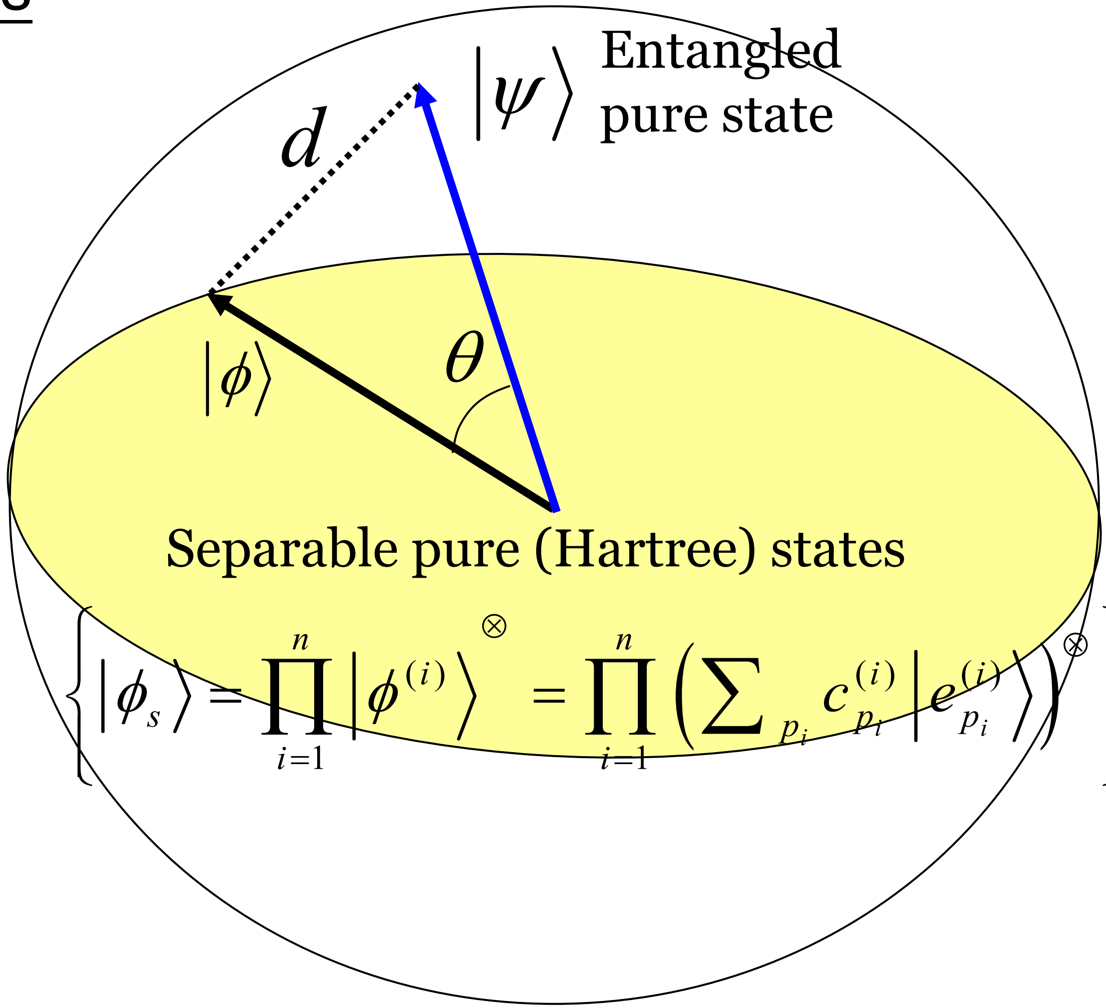
III. A geometric measure for multipartite entanglement

[Shimony '95, Barnum & Linden '01 Wei & Goldbart '02]

Pure states

$$d = \| |\phi\rangle - |\psi\rangle \| = \sqrt{2(1-\cos\theta)}$$



$$\left\{ |\phi_s\rangle = \prod_{i=1}^{n} |\phi^{(i)}\rangle^{\otimes} = \prod_{i=1}^{n} \left(\sum_{p_i} c_{p_i}^{(i)} |e_{p_i}^{(i)}\rangle \right)^{\otimes} \right\}$$

Geometric measure of entanglement

Pure states

- A n-partite pure state described by

$$\left|\psi\right\rangle = \sum_{p_1 p_2 \cdots p_n} \chi_{p_1 p_2 \cdots p_n} \left|e_{p_1}^{(1)}\right\rangle \otimes \left|e_{p_2}^{(2)}\right\rangle \otimes \cdots \otimes \left|e_{p_n}^{(n)}\right\rangle$$

- Find the closest separable pure state, i.e. direct product of n single-particle states

$$\left|\phi_s\right\rangle = \prod_{i=1}^{n} \left|\phi^{(i)}\right\rangle^{\otimes} = \prod_{i=1}^{n} \left(\sum_{p_i} c_{p_i}^{(i)} \left|e_{p_i}^{(i)}\right\rangle\right)^{\otimes}$$

- The <u>shortest distance</u> or <u>smallest angle</u> from $\left|\psi\right\rangle$ to $\left|\phi_s\right\rangle$ is an indication of the degree of entanglement

GME : pure states

Minimizing $\|\phi_s - \psi\|^2 + (\lambda - 1)\langle \phi_s | \phi_s \rangle$,

➔ Equations satisfied by ϕ_s or $c's$:

$$\sum_{p_1 p_2 \cdots \hat{p}_i \cdots p_n} \chi^*_{p_1 p_2 \cdots p_n} c^{(1)}_{p_1} c^{(2)}_{p_2} \cdots \widehat{c^{(i)}_{p_i}} \cdots c^{(n)}_{p_n} = \lambda c^{(i)*}_{p_i}, \qquad \langle \psi | \left(\prod_{j \neq i}^{n} |\phi^{(j)}\rangle^{\otimes} \right) = \lambda \left(\langle \phi^{(i)} | \right)$$

$$\sum_{p_1 p_2 \cdots \hat{p}_i \cdots p_n} \chi_{p_1 p_2 \cdots p_n} c^{(1)*}_{p_1} c^{(2)*}_{p_2} \cdots \widehat{c^{(i)*}_{p_i}} \cdots c^{(n)*}_{p_n} = \lambda c^{(i)}_{p_i}, \qquad \left(\prod_{j \neq i}^{n} \langle \phi^{(j)} |^{\otimes} \right) |\psi\rangle = \lambda \left(| \phi^{(i)} \rangle \right)$$

1. Nonlinear eigenproblem; can be solved numerically
2. The largest eigenvalue $\Lambda = \max_{\phi\ sep.} |\langle \phi | \psi \rangle|$ (Ent. Eigenvalue)
3. *Geometric measure of entanglement* $E_{\sin^2} \equiv 1 - \Lambda^2$
4. For bipartite case, Λ equals the largest Schmidt coefficient

GME: Mixed states via convex hull

$$E_{\sin^2}(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i \sin^2\theta\left(|\psi_i\rangle\right), \quad \text{with } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Criteria for good entanglement measures

[Vedral et al. '97, Vidal '00, Horodecki et al. '00]

1. (a) $E(\rho) \geq 0$; (b) $E(\rho) = 0$, if ρ is not entangled
2. Local unitary transformations should not change the amount of entanglement
3. Local operations and classical communication should not increase the expectation value of entanglement
4. Entanglement cannot increase under discarding information

$$\sum_i p_i E(\rho_i) \geq E\left(\sum_i p_i \rho_i\right)$$

GME: Examples of bipartite states

① Two-qubit pure states $|\psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$

$\lambda_{\max} = \max(\sqrt{p}, \sqrt{1-p})$ cf $C = 2\sqrt{p}\sqrt{1-p}$ concurrence

$E_{\sin^2} = 1 - \lambda_{\max}^2 = \dfrac{1-\sqrt{1-C^2}}{2}$ (valid for all 2-qubit *mixed* states [Vidal '00])

② Generalized Werner states ($\int dU\, U \otimes U \rho U^\dagger \otimes U^\dagger = \rho$)

$$\rho_{Werner}(f) \equiv \frac{d^2 - fd}{d^4 - d^2} I \otimes I + \frac{fd^2 - d}{d^4 - d^2} F, \qquad \text{where } F \equiv \sum_{ij} |ij\rangle\langle ji|$$

$$E_{\sin^2}(f) = \frac{1-\sqrt{1-f^2}}{2}, \quad \text{for } f \le 0; \quad 0 \text{ otherwise}$$

GME: Examples of bipartite states

③ Isotropic states ($\int dU\, U \otimes U^* \rho\, U^\dagger \otimes \left(U^\dagger\right)^* = \rho$)

$$\rho_{iso}(F) \equiv \frac{1-F}{d^2-1} I \otimes I + \frac{Fd^2-1}{d^2-1} \left|\Phi^+\right\rangle\left\langle\Phi^+\right|, \qquad \text{where } \left|\Phi^+\right\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \left|ii\right\rangle$$

$$E_{\sin^2}(F) = 1 - \frac{1}{d}\left(\sqrt{F} + \sqrt{(1-F)(d-1)}\right)^2, \quad \text{for } F \geq \frac{1}{d}; \quad 0 \text{ otherwise}$$

Works for bipartite states; what about multipartite states?

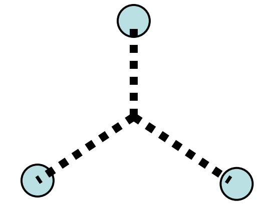
GME: Examples of tri- and multi-partite pure states

④ Tripartite pure states

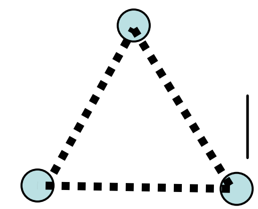
$$\left|GHZ\right\rangle = \frac{1}{\sqrt{2}}\left(\left|000\right\rangle + \left|111\right\rangle\right),\ \Lambda = \frac{1}{\sqrt{2}},\ E_{\sin^2} = \frac{1}{2}$$

$$\left|W\right\rangle = \frac{1}{\sqrt{3}}\left(\left|001\right\rangle + \left|010\right\rangle + \left|100\right\rangle\right),\quad \Lambda = \frac{2}{3}, E_{\sin^2} = \frac{5}{9}$$

$$\left|\tilde{W}\right\rangle = \frac{1}{\sqrt{3}}\left(\left|110\right\rangle + \left|101\right\rangle + \left|011\right\rangle\right),\quad \Lambda = \frac{2}{3}, E_{\sin^2} = \frac{5}{9}$$



$\left|GHZ\right\rangle$



$\left|W\right\rangle, \left|\tilde{W}\right\rangle$

⑤ N-qubit pure states

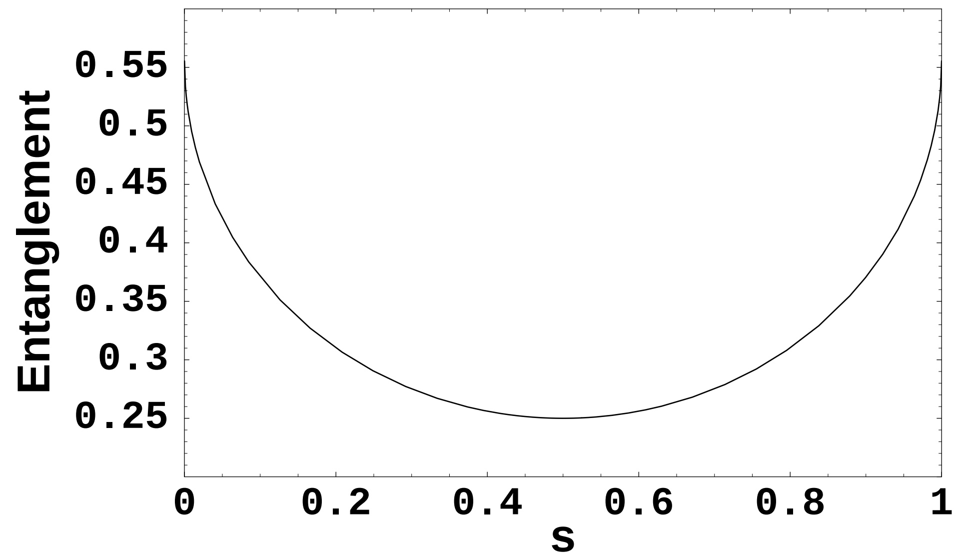
For $\left|S(n,k)\right\rangle \equiv \sqrt{\frac{k!(n-k)!}{n!}} \sum_{permutations} \left|\underbrace{0\cdots0}_{k}\underbrace{1\cdots1}_{n-k}\right\rangle$ permutationally invariant under parties

$$\Lambda(n,k) = \sqrt{\frac{n!}{k!(n-k)!}\left(\frac{k}{n}\right)^{k}\left(\frac{n-k}{n}\right)^{n-k}}$$

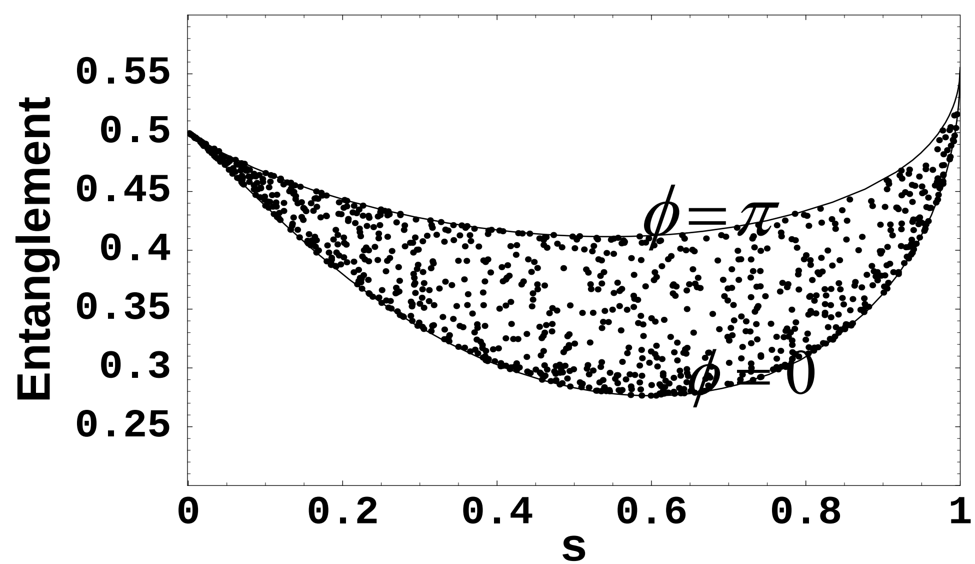
GME: Examples of tri- and multi-partite pure states

⑥ Superposition of pure states

$$\sqrt{s}\left|W\right\rangle + e^{i\phi}\sqrt{1-s}\left|\tilde{W}\right\rangle$$



$$\sqrt{s}\left|W\right\rangle + e^{i\phi}\sqrt{1-s}\left|GHZ\right\rangle$$



GME: Examples of multipartite mixed states

⑦ Mixture of S(n,k)

What is GME for the mixed state?

$$\rho(p_0, p_1, \cdots, p_n) \equiv \sum_k p_k \left| S(n,k) \right\rangle \left\langle S(n,k) \right|$$

Ans. 1. Compute entanglement eigenvalue $\Lambda(p_0, p_1, \cdots, p_n)$ for the pure state

$$\sum_k \sqrt{p_k} \left| S(n,k) \right\rangle$$

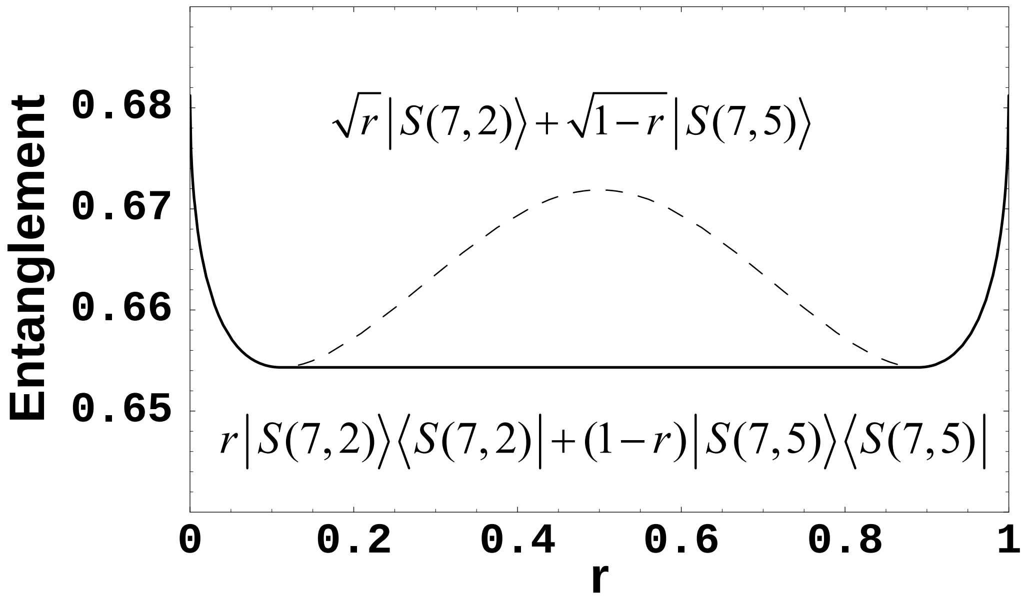
2. GME for the mixed state $\rho(p_0, p_1, \cdots, p_n)$ is then the convex hull of the function

$$f(p_0, p_1, \cdots, p_n) \equiv 1 - \Lambda^2 (p_0, p_1, \cdots, p_n)$$

$$\mathrm{E}(\rho) = (\text{convex hull of } f)(p_0, p_1, \cdots, p_n)$$

GME: Examples of multipartite mixed states

⑧ Seven-qubit example $r|S(7,2)\rangle\langle S(7,2)|+(1-r)|S(7,5)\rangle\langle S(7,5)|$



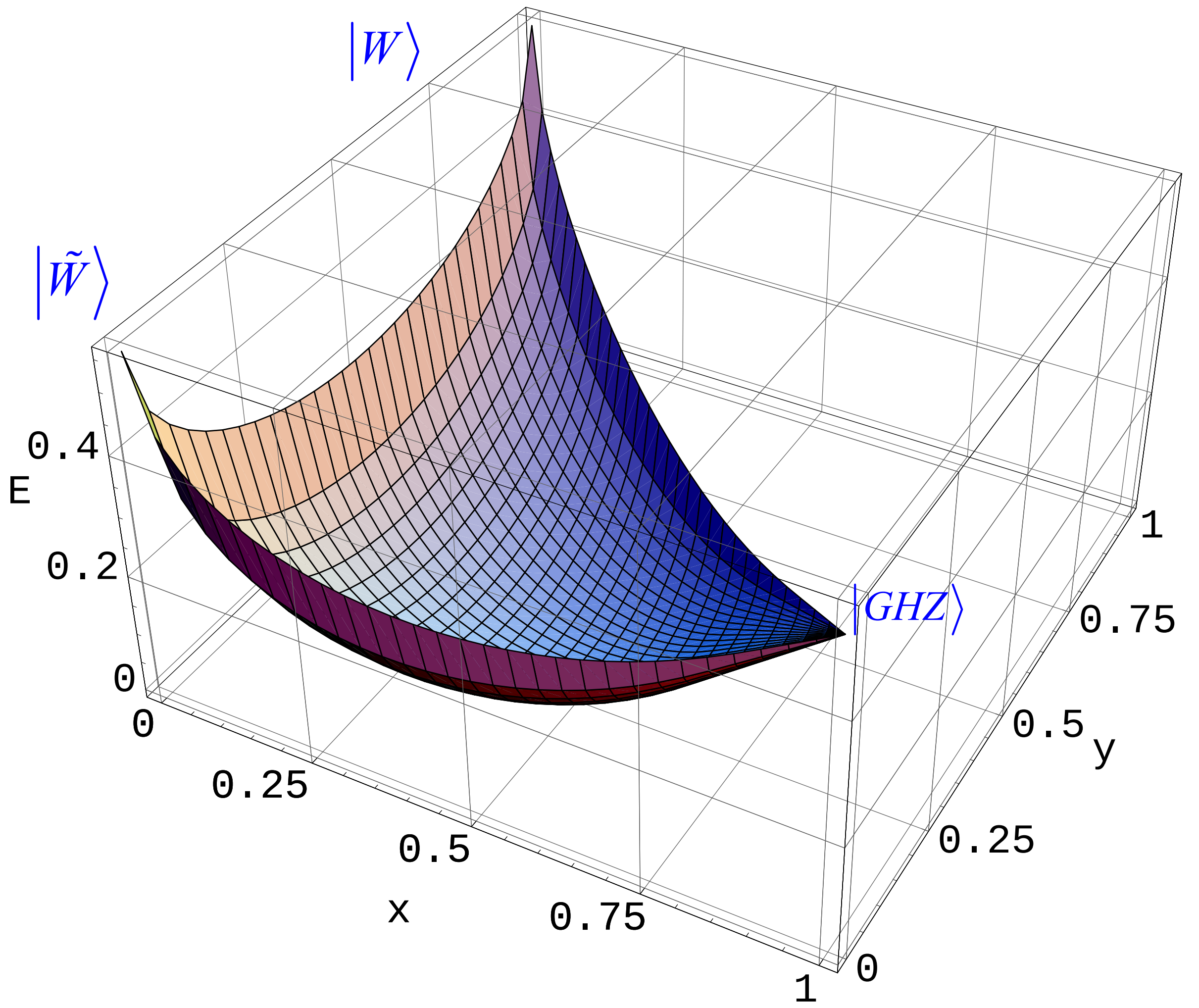
Recall S(n,k) $|S(n,k)\rangle \equiv \sqrt{\frac{k!(n-k)!}{n!}} \sum_{permutations} |\underbrace{0\cdots0}_{k}\underbrace{1\cdots1}_{n-k}\rangle$

$$\Lambda(n,k)=\sqrt{\frac{n!}{k!(n-k)!}\left(\frac{k}{n}\right)^{k}\left(\frac{n-k}{n}\right)^{n-k}}$$

GME: Examples of multipartite mixed states

⑨ GME for the mixture

$$\rho(x,y) \equiv x\left|GHZ\right\rangle\left\langle GHZ\right| + y\left|W\right\rangle\left\langle W\right| + (1-x-y)\left|\tilde{W}\right\rangle\left\langle \tilde{W}\right|$$



GME: Bound entangled states

I. Smolin's 4-partite bound entangled state [quant-ph/0001001]

$$\rho_{\text{Smolin}} = \frac{1}{4}\sum_{i=1}^{4} \left|\Psi_i\right\rangle_{AB}\left\langle\Psi_i\right| \otimes \left|\Psi_i\right\rangle_{CD}\left\langle\Psi_i\right|$$

II. Dür's N-partite bound entangled states [PRL '01]

$$\rho_{Dur} = \frac{1}{N+1}\left|N-GHZ\right\rangle\left\langle N-GHZ\right| + \frac{1}{2(N+1)}\sum_{k=1}^{N}\left(P_k + Q_k\right)$$

$$P_k = \left|0\cdots 1_k \cdots 0\right\rangle\left\langle 0\cdots 1_k \cdots 0\right|, \quad Q_k = \left|1\cdots 0_k \cdots 1\right\rangle\left\langle 1\cdots 0_k \cdots 1\right|$$

III. In fact, the following state is bound entangled if $x \le \frac{1}{N+1}$

$$\rho(x) = x\left|N-GHZ\right\rangle\left\langle N-GHZ\right| + \frac{1-x}{2N}\sum_{k=1}^{N}\left(P_k + Q_k\right)$$

➔GME can be found analytically for all these cases

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GME and entanglement witnesses

- An entanglement witness $\mathbb{W}$ for an entangled state ρ is a Hermitian operator (observable) that

  I. $Tr(\mathbb{W}\sigma) \geq 0$ for all separable states σ

  II. $Tr(\mathbb{W}\rho) < 0$

- For a pure state $|\psi\rangle$

$$\mathbb{W} = \lambda^2 I - |\psi\rangle\langle\psi|$$

  is a good entanglement witness if $\Lambda^2(|\psi\rangle) \leq \lambda^2 < 1$

- Equivalently

$$\min_{\mathbb{W}} Tr\left(\mathbb{W}|\psi\rangle\langle\psi|\right) = -1 + \Lambda^2(|\psi\rangle) = -E_{\sin^2}(\psi)$$

GME and relative entropy of entanglement

I. For any pure state $|\psi\rangle$, $E_{\log}\left(|\psi\rangle\right) \equiv -2\log\Lambda\left(|\psi\rangle\right)$

is a lower bound of its relative entropy of entanglement

$$E_R\left(|\psi\rangle\right) \geq -2\log\Lambda\left(|\psi\rangle\right)$$

II. For the n-qubit pure state $|S(n,k)\rangle \equiv \sqrt{\frac{k!(n-k)!}{n!}} \sum_{permutations} \left|\underbrace{0\cdots0}_{k}\underbrace{1\cdots1}_{n-k}\right\rangle$

$$E_R\left(|S(n,k)\rangle\right) = -2\log\Lambda(n,k)$$

where $\Lambda(n,k) = \sqrt{\frac{n!}{k!(n-k)!}\left(\frac{k}{n}\right)^k\left(\frac{n-k}{n}\right)^{n-k}}$

GME and quantum phase transitions?

$$H = -J\sum_i \left(\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x\right)$$ has a QPT at g=1

- If one knows the exact ground state, one can compute its geometric measure of entanglement and study scaling behavior of entanglement

- What would be the advantage of GME over other, say, EF?

  - EF can, up to now, only deal with pairwise entanglement
  - GME can deal with n-partite entanglement

- However, the exact form of ground state in the spin (up, down) basis is difficult to find explicitly, even though it is known in the fermionic basis.

  (work in progress…)

Conclusions:

Quant-ph/0212030, 0303079, 0303158,
& two others coming up soon…

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} bipartite, EF also known

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