Universal control of quantum subspaces and subsystems

Paolo Zanardi

Massachusetts Institute of Technology,
Institute for Scientific Interchange (**ISI**) Foundation

- *Arbitrary manipulation of information:: key requirement for classical and Q IP*

  When this goal is realized universality is achieved.

- *QIP case set of N qubits: State-space $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes N}$*

  1) Almost any two-qubit gate is universal [Lloyd, Barenco et al, Di Vincenzo (95)]

  2) single-qubit gates with one of arbitrary entangling two-qubit gate are universal [Brylinski's, Bremner et (2002)]

- *Operational constraints $\Rightarrow$ NO full state space Universality*

The Question

$\exists$? a subspace $\mathcal{C}$ of $\mathcal{H}$ over which the set of naturally available interactions are universal?

$\Rightarrow$ encoded universality *[Di Vincenzo, Bacon* et al (2000), Kempe *et al (2001), Lidar* et al , Viola (2002)]

The Control Group

- *Set of " naturally" available interactions* $\mathcal{I}_A := \{H(\lambda)\}_{\lambda\in\mathcal{M}} \subset$ *End*$(\mathcal{H})$

  $\mathcal{M} =$ *Set of control parameters.*

- *Allowed paths* $\mathcal{P}_A =$ *set physically realizable control processes* $\gamma : \mathbb{R} \to \mathcal{M}$

  NB $\mathcal{P}_A \subset \mathcal{F}(\mathbb{R}, \mathcal{M})$ *e.g., In* **HQC** $\mathcal{P}_A =$ adiabatic loops

- *Allowed evolutions*

$$\mathcal{U}_A := \{T\,\mathrm{exp}[-i\int dt H(\gamma(t))]\,/\,\gamma \in \mathcal{P}_A\} \qquad (1)$$

  Tech Assumption: $\mathcal{U}_A$ *is a* sub-GROUP *of* $\mathcal{U}(\mathcal{H})$

- *Definition: Set of* $\mathcal{U}_A$ *invariant subspaces* (codes) $\mathcal{C}_i \subset \mathcal{H}\, (i = 1.\ldots, M)$ of $\mathcal{U}_A$, such that

$$\mathcal{U}_A|_{\mathcal{C}_i} = \mathcal{U}(\mathcal{C}_i), \quad (i = 1.\ldots, M). \qquad (2)$$

  $\mathcal{U}_A$ is $\mathcal{C}_i$-universal.

Quantum control theory

(i) $\mathcal{P}_A = \mathcal{F}(\mathbb{R}, \mathcal{M})$ *(ii)* $\mathcal{I}_A = \{\sum_i \lambda_i H_i\}$

$$(i)\&(ii) \Rightarrow \mathcal{U}_A = e^{\mathcal{L}_A} \qquad (3)$$

where by $\mathcal{L}_A =$ *Lie algebra generated by the set of operators* $\mathcal{I}_A$

Restricted set of paths $\mathcal{P}_A \Rightarrow \mathcal{U}_A \subset e^{\mathcal{L}_A}$.

- *Example* **HQC** $\mathcal{I}_A =$*set of iso-degenerate Hamiltonian*

  $\mathcal{P}_A$ *is given by adiabatic* loops *around a* $\lambda_0 \in \mathcal{M}$.

- Adiabatic theorem: state space splits according to the eigenprojectors of $H(\lambda_0)$. $\Rightarrow$ NO universality

- *Irreducible connection*$\Rightarrow$ $\mathcal{U}_A = \oplus_r \mathcal{U}(\mathcal{H}_r)$, $\mathcal{H}_r$ *the* $r-th$ n_r*dimensional eigenspace of* $H(\lambda_0)$

- $\sum_r n_r^2 < (\sum_r n_r)^2$ – $\mathcal{U}_A$ *is strictly contained in* $\mathcal{U}(\mathcal{H})$. *$\mathcal{U}_A$ allows only for $\mathcal{H}_r$-universality*

- *XY-interactions (Lidar* et al *(2002)*

  $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ =*two-qubit space*

  $$\mathcal{I}_A = \{\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y, \sigma^x \otimes \sigma^y - \sigma^y \otimes \sigma^x, \sigma^z \otimes \mathbf{Id} - \mathbf{Id} \otimes \sigma^z\} \tag{4}$$

  is $\mathcal{H}_1$*-universal* $\mathcal{H}_1 =$ *linear span of* $|01\rangle$ *and* $|10\rangle$ *(* $\mathcal{H}_0 =$ *linear span of* $|00\rangle$ *and* $|11\rangle$*)*

  $(\mathcal{L}_A) \cong su(2) \Rightarrow \mathcal{H}$ *splits in a triplet (*$\mathcal{H}_1$*) and two singlets (*$\mathcal{H}_0$*).*

  $$\mathcal{I}'_A = \{\sigma^x \otimes \sigma^x - \sigma^y \otimes \sigma^y, \sigma^x \otimes \sigma^y + \sigma^y \otimes \sigma^x, \sigma^z \otimes \mathbf{Id} + \mathbf{Id} \otimes \sigma^z\} \tag{5}$$

  $\mathcal{H}_0$*-universal*

Remark The codes do not *have to be* $\mathcal{I}_A$*-invariant subspaces: auxiliary intermediate states ARE allowed*

- *Topological quantum computation [Kitaev, Freedman* et al*] Code= degenerate ground-state*

  protected by a broken topological symmetry Manipulations creating anyon-like excitations, braiding in global, fashion returning into the ground-state.

Again The **QUESTION**

Given the available set $\mathcal{U}_A$ of operations, can some encoded universality be achieved?

An **Answer: GO to IRREPS!**

- *Group representation theory* $\Rightarrow$ *decomposition of* $\mathcal{H}$ *according the* $\mathcal{U}_A$*-irreps*

$$\mathcal{H} = \oplus_J \mathbb{C}^{n_J} \otimes \mathcal{H}_J \qquad (6)$$

  the J-th irrep $\mathcal{H}_J$, with dimension d_J, and multiplicity n_J : symmetries *for* $\mathcal{U}_A$ *e.g., permutational in DFS theory, with n_J-dim irreps*

- *The group $\mathcal{U}_A$ acts irreducibly over the subspaces $\mathcal{C}_J = |\phi\rangle \otimes \mathcal{H}_J$.*

NOTICE: *irreducibility does not imply* $\mathcal{U}_A|_{\mathcal{C}} \cong \mathcal{U}(\mathcal{C})$

$\mathcal{U}_A$ *Lie-Group* $\Rightarrow$

If dim $\mathcal{U}_A|_{\mathcal{H}_J} = d_J^2 - 1$ **then** $\mathcal{U}_A$ **is** $\mathcal{C}_J$**-universal where** $\mathcal{C}_J$ **is any** d_J**-dimensional subspace of the form** $|\phi\rangle \otimes \mathcal{H}_J, (|\phi\rangle \in \mathbb{C}^{n_J})$.

- **RECIPE**

  (0) Determine the group $\mathcal{U}_A$

  (1) Split $\mathcal{H}$ *according the* $\mathcal{U}_A$ *irreps*

  (2) Check $\forall J$, $d_J^2 -$ *dim*$\mathcal{U}_A|_{\mathcal{H}_J} \geq 0$: *If YES* $\Rightarrow$ n_J-*parameters family of codes for* $\mathcal{I}_A$

- *Back to the Two-qubit Example* *Stabilizer Group* $\mathcal{G} = \{\mathbf{Id}, \sigma^{z\otimes 2}\} \cong \mathbf{Z}_2$. $\mathcal{G}|_{\mathcal{H}_\alpha} = (-1)^\alpha \mathbf{Id}$

  $\mathcal{U}_A =$ *Commutant (of the group algebra* $\mathbb{C}\mathcal{G}$*)*

  $$\text{span}\{\mathbf{Id}, \sigma^z \otimes \mathbf{Id}, \mathbf{Id} \otimes \sigma^z, \sigma^{z\otimes 2}, \sigma^\alpha \otimes \sigma^\beta\, (\alpha, \beta = x, y)\}. \quad (7)$$

  This algebra is universal simultaneously over $\mathcal{H}_0$ *and* $\mathcal{H}_1$.

Lattice Bosons

- L bosonic modes, $[b_i, b_j^\dagger] = \delta_{ij}$, $(i, j = 1, \dots, L)$.

- $\mathcal{I}_A = \{b_j^\dagger b_i \, / \, i, j = 1, \dots, L\}$. The bilinears $b_j^\dagger b_i$ span a algebra $\mathcal{L}_A$ isomorphic to $u(L)$.

- The Fock space $\mathcal{H}_F = h_\infty^{\otimes L}$ (h_∞ is the state-spae of a single quantum oscillator) splits in $su(L)$-invariant subspaces $\mathcal{H}_N$ with dimensions $d_{N,L} := \begin{pmatrix} N+L-1 \\ L-1 \end{pmatrix}$ corresponding to the eigenvalues N of the total number operator $\sum_{j=1} b_j^\dagger b_j$.

- Typically $d_{N,L}^2 > L^2 = \dim u(L) \Rightarrow \mathcal{L}_A$ is not $\mathcal{H}_N$-universal. When $N = 1$, with L-arbitrary, one obtains the fundamental irrep for which $d_{1,L} = L$.

Group algebra universality

The group of unitaries over $\mathcal{C}$ *is given by the restriction to* $\mathcal{C}$ *of the unitary part of the* group algebra *of* $\mathcal{U}_A$ *i.e.,* $\mathcal{U}(\mathcal{C}) = U\mathbb{C}\mathcal{U}_A|_{\mathcal{C}}$.

Allowed interactions are completely controllable and belong to the group algebra of a non-abelian group $\mathcal{K}$ *i.e.,* $\mathcal{I}_A \subset \mathbb{C}\mathcal{K}$.

Then the group $\mathcal{U}_A$ is *generically* $\mathcal{C}$-universal for all $\mathcal{C} = |\phi\rangle \otimes \mathcal{H}_J$, where $\mathcal{H}_J$ is a $\mathcal{K}$-irrep space and $|\phi\rangle \in \mathbb{C}^{n_J}$ (n_J is the multiplicity of the J-th irrep)

NOTICE

Tracing out the $|\phi\rangle \Rightarrow$ *universal control over the* $\mathcal{H}_J$
[Virtual Subystems, Zanardi (2001)]

- *One-qubit universality: (one irrep with multiplicity one). A generic Hamiltonian in* $\mathbb{C}SU(2)$: $H = \sum_{\alpha=x,yz} \lambda_\alpha \sigma^\alpha$. *This latter is universal over* $\mathcal{H}$.

- N *spin* 1/2 *systems coupled by exchange interactions:* $\mathcal{I}_A \subset \mathcal{S}_N$ generic universality in any $\mathcal{S}_N$-irrep.

- $N = 3 \Rightarrow (\mathbb{C}^2)^{\otimes 3} = 4$ totally symmetric irrep ($J = 3/2$) $\bigoplus$ 2 two-dimensional $\mathcal{S}_3$ irrep (two $J = 1/2$ $SU(2)$-irreps).

  $\Rightarrow$ *Two-parameter family of* $\mathcal{S}_3$*-codes* [Di Vincenzo *et al* (2000)]

Tensor product structure

State space $\mathcal{H}_N = \mathcal{H}^{\otimes N}$

We assume that $\mathcal{U}_A \subset \mathcal{U}(\mathcal{H}_N) \supset \mathcal{U}(\mathcal{H})^{\otimes N}$ is locally universal i.e., is $\mathcal{C}$-universal for some $\mathcal{C} \subset \mathcal{H}^{\otimes M}$ ($n := N/M \in \mathbf{N}$).

- Let $\mathcal{U}_A$ be locally universal and $\exists X \in \mathcal{U}_A$ such that $\forall i, j = 1, \ldots, M$: i) X acts trivially in all the clusters but the i-th and the j-th; ii) $\mathcal{C}^{(i)} \otimes \mathcal{C}^{(j)}$ is an X-invariant subspace and X is entangling $\Rightarrow$ $\mathcal{U}_A$ is $\mathcal{C}^{\otimes N}$-universal.

Singlet Coding

- $\rho: \mathcal{K} \to \mathcal{U}(\mathcal{H})$ *Group Representation*

  SINGLET SECTOR (Trivial Irrep)

  $\mathcal{C}_\rho := \{|\psi\rangle \in \mathcal{H} \Rightarrow \rho(\mathcal{K})|\psi\rangle = |\psi\rangle\}$

- *Transformations over $\mathcal{C}_\rho$ are elements of the commutant $\rho(\mathcal{K})'$ i.e., the singlet is an irrep of that algebra.*

- *Crucial Fact:* $\mathcal{C}_{\rho^{\otimes N}} \supset \mathcal{C}_\rho^{\otimes N}$

DFS and Exchange Interaction

- $\mathcal{H} \cong \mathbb{C}^d$ *and* $\mathcal{I}_A =$ *exchange Hamiltonians between the different factors in* $\mathcal{H}^{\otimes M}$. $\mathcal{I}_A \subset \mathbb{C}\mathcal{S}_M$.

- $(\mathbb{C}\mathcal{S}_M)' = M$*-fold tensor representation of* $SU(d)$.

- *QI is encoded in the* $SU(d)$*-singlets:* *decoherence-free (for collective errors)* *[Zanardi & Rasetti (1997)]*

- *For* $M = 2\,d$ *the state-space contains a two-dimensional* $SU(2)$-singlet *sector* $\mathcal{C}$ *(*$= S_N - irrep$*)*

- $n = N/M$ *clusters coupled together by Hamiltonians in* $\mathbb{C}S_N$ *the* $SU(d)$*-singlet sector of* $\mathcal{H}^{\otimes N}$ *strictly* includes $\mathcal{C}^{\otimes n} \Rightarrow$

  exchange Hamiltonians allow generically for universality on the former $\Rightarrow \mathcal{C}^{\otimes n}$*-universality.* *[Lidar* et al]

Conclusions

- General framework for universal quantum control and **QIP** on subspaces/subsystems
- All physical examples known so far in **QIP** fit in
- General conditions for obtaining encoded-universality.
- Tool: Algebraic formalism describing error correction/avoidance schemes

DUALITY: between the task of "not allowing many bad things to happen" in error correction and " making as many as good things happen as possible" in quantum control.

DO YOU WANNA KNOW MORE?

P. Zanardi, S. Lloyd, `quant-ph/0305013` !