New Instanton Effects in String Theory*

Chris Beasley

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Introduction

An important non-perturbative feature of string theory or M-theory compactifications which preserve only $\mathcal{N} = 1$ supersymmetry in four dimensions is that fundamental string worldsheet or brane instantons can sometimes generate an effective superpotential for the light chiral superfields that describe the classical moduli of the background. Such a superpotential drastically alters the low-energy behavior of the theory, since some or all of the branches of the classical moduli space can be lifted.

As a concrete example, I consider today perturbative heterotic compactification on a Calabi-Yau threefold X with a stable holomorphic gauge bundle V. In this case, the classical moduli to consider are those which describe the complexified Kähler class of X and the complex structure of X and V. As was first shown in this context by Dine, Seiberg, Wen, and Witten [1], a fundamental string worldsheet that wraps a smooth, isolated rational curve C in X can generate a superpotential for these moduli.

Schematically, the superpotential W_C generated by a worldsheet instanton wrapping C takes the form

$$W_C = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \times (\text{1-loop determinants}) \,. \tag{1}$$

Here $A(C)$ is the area of C in the Calabi-Yau metric on X, and B is the heterotic B-field. Thus, the argument of the exponential factor in W_C is simply the classical action of a string worldsheet wrapped once about C. Beyond tree-level, W_C is also weighted by a product of one-loop determinants that arise from integrating out the fluctuating modes of worldvolume bosons and fermions on C. (For now, I leave the one-loop factors implicit in W_C.) Because C is a holomorphic curve,

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its area $A(C)$ is given by the integral of the Kähler form over C, so that W_C depends explicitly on the complexified Kähler class of X. In general, the one-loop determinants in W_C also depend on the complex structure moduli of X and V.

The schematic formula (1) exhibits the superpotential contribution from only a single curve C in X. To compute the full superpotential, we must sum the contributions from all such curves in X. So again schematically,

$$W = \sum_{C \subset X} W_C \,. \tag{2}$$

Because X may contain thousands of rational curves, the direct evaluation of this instanton sum is generally a herculean task. If we are less ambitious, we can ask simply whether W is non-zero. Yet even this question is non-trivial!

To answer this question, one might naively argue that the superpotential contributions from individual curves in X are generic, so their sum in (2) is generically non-zero. However, such reasoning neglects the fact that the summand W_C depends holomorphically on the moduli of X and V, and arguments based on the "generiticity" of holomorphic quantities are dangerous. In fact, for the case in which X and V have a simple linear sigma model worldsheet description, a non-trivial cancellation occurs in the sum in (2), and the superpotential actually vanishes [2, 3, 4].

Nonetheless, one heuristic reason to think that string worldsheet and brane instantons often generate a superpotential in $\mathcal{N} = 1$ compactifications is that an analogous phenomenon already occurs in the much simpler context of four-dimensional supersymmetric gauge theory. As shown by Affleck, Dine, and Seiberg [5], instantons in supersymmetric QCD (or SQCD) with gauge group $SU(N_c)$ and with $N_f = N_c - 1$ flavors[1] generate a superpotential that completely lifts the classical moduli space of supersymmetric vacua of that theory.

Besides providing a sterling example of an instanton-generated superpotential, SQCD also provides an example of a class of more subtle instanton effects whose stringy analogues have not been much considered. The most prominent such effect occurs in SQCD with $N_f = N_c$ flavors. As shown by Seiberg [6], instantons in this theory do not generate a superpotential, but they nonetheless deform the complex structure of the classical moduli space. This quantum deformation is not so drastic as to lift any branches of the classical moduli space, but it instead smooths away a classical singularity at the origin of moduli space.

The exotic instanton effect in SQCD with $N_f = N_c$ flavors raises an immediate question about analogous phenomena in string theory. Namely, can worldsheet or brane instantons which do *not* generate a superpotential nonetheless generate a quantum deformation of the moduli space? If so, what form can this deformation take? These questions are the ones I will discuss in the remainder of my talk.

[1]We recall that a flavor is a massless chiral multiplet transforming in the sum of the fundamental and the anti-fundamental representations of the gauge group.

A Brief Sketch of the Main Idea

Before I discuss details, let me sketch the main idea. As I recall in moment, a quantum deformation of the classical moduli space $\mathcal{M}_{cl}$ is intrinsically described by an F-term correction to the effective action of the general form

$$\delta S = \int d^4x \, d^2\theta \, \omega_{\bar{i}\bar{j}} \, \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}} \, \overline{D}^{\dot{\alpha}} \overline{\Phi}^{\bar{j}} \, . \tag{3}$$

Here $\overline{D}_{\dot{\alpha}}$ is the usual spinor covariant derivative on superspace, and ω is a tensor on $\mathcal{M}_{cl}$ that represents the deformation. Among other terms, the interaction (3) leads in components to an effective four-fermion vertex, and for this reason I refer to (3) as a "multi-fermion F-term".

If we seek worldsheet instantons that can generate a deformation of the classical moduli space $\mathcal{M}_{cl}$, then we must consider worldsheet instantons that can generate the multi-fermion F-term in (3). Now in any instanton computation, the form of the effective operator generated by the instanton is largely determined by the structure of fermion zero-modes in the instanton background. For example, to generate a superpotential, the instanton must carry at least (and in the simplest case, precisely) two fermion zero-modes to generate the fermionic part of the chiral superspace measure $d^4x \, d^2\theta$. In contrast, to generate the correction (3), the instanton must carry at least four fermion zero-modes: two zero-modes to generate the measure $d^2\theta$ and two zero-modes to generate the fermionic superfields $\overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}} \, \overline{D}^{\dot{\alpha}} \overline{\Phi}^{\bar{j}}$ that appear in the operator itself.

In the case that C is an isolated rational curve in X, a worldsheet instanton wrapping C carries precisely two physical fermion zero-modes and hence naturally contributes to a superpotential. To discuss instantons which carry additional zero-modes, and which can thus generate multi-fermion F-terms as in (3), we simply relax our assumptions that C is isolated as a holomorphic curve in X and that C has genus zero. In either case, C carries extra fermion zero-modes and naturally contributes to higher F-terms than the superpotential.

Today I will focus on the particular case of worldsheet instantons that generate the multi-fermion F-term in (3) which describes a quantum deformation of the moduli space. As I explain, this effect arises naturally from a one-parameter holomorphic family of rational curves in X. For a complementary discussion of multi-fermion F-terms in the context of heterotic Calabi-Yau compactification, see [7].

General Remarks on Multi-Fermion F-terms

To start, I recall some essential properties of multi-fermion F-terms and how they are associated to deformations of the complex structure of the moduli space. This review follows Section 2 of [8], and for simplicity I consider only theories with global (as opposed to local) supersymmetry.

To motivate the study of multi-fermion F-terms in the effective action, let us begin by considering how to describe physically the infinitesimal deformation of some classical moduli space $\mathcal{M}_{cl}$ to a quantum moduli space $\mathcal{M}$. Abstractly, the classical effective action associated to motion on $\mathcal{M}_{cl}$ is a four-dimensional, $\mathcal{N}=1$ supersymmetric nonlinear sigma model which describes maps $\Phi : \mathbb{R}^4 \longrightarrow \mathcal{M}_{cl}$. This sigma model is governed by the usual action,

$$S = \int d^4x \, d^4\theta \; K\Big(\Phi^i, \overline{\Phi}^{\bar{i}}\Big) \, . \tag{4}$$

Here Φ^i and $\overline{\Phi}^{\bar{i}}$ are respectively chiral and anti-chiral superfields whose lowest bosonic components describe local holomorphic and anti-holomorphic coordinates on $\mathcal{M}_{cl}$, and K is the Kähler potential associated to some Kähler metric $ds^2 = g_{i\bar{i}} \, d\phi^i d\overline{\phi}^{\bar{i}}$ on $\mathcal{M}_{cl}$.

Similarly, the quantum effective action is also a nonlinear sigma model as above, but now with target space $\mathcal{M}$ instead of $\mathcal{M}_{cl}$. In principle, to pass from the sigma model with target $\mathcal{M}_{cl}$ to target $\mathcal{M}$, we must add a correction term δS to the classical effective action. So we ask — what form does δS take?

In general, a deformation of the complex structure on $\mathcal{M}_{cl}$ can be described intrinsically as a change in the $\overline{\partial}$ operator on $\mathcal{M}_{cl}$ of the form

$$\overline{\partial}_{\bar{j}} \longmapsto \overline{\partial}_{\bar{j}} + \omega_{\bar{j}}^{\;\,i} \, \partial_i \, . \tag{5}$$

Here $\omega_{\bar{j}}^{\;\,i}$ is a representative of a Dolbeault cohomology class in $H^1_{\overline{\partial}}(\mathcal{M}_{cl}, T\mathcal{M}_{cl})$, whose elements parametrize infinitesimal deformations of $\mathcal{M}_{cl}$. We use standard notation, with $T\mathcal{M}_{cl}$ and $\Omega^1_{\mathcal{M}_{cl}}$ denoting the holomorphic tangent and cotangent bundles of $\mathcal{M}_{cl}$.

We can equally well represent the change (5) in the $\overline{\partial}$ operator on $\mathcal{M}_{cl}$ as a change in the dual basis of holomorphic one-forms $d\phi^i$,

$$d\phi^i \longmapsto d\phi^i - \omega_{\bar{j}}^{\;\,i} \, d\overline{\phi}^{\bar{j}} \, . \tag{6}$$

As a result, under the deformation the metric on $\mathcal{M}_{cl}$ changes as

$$g_{i\bar{i}} \, d\phi^i d\overline{\phi}^{\bar{i}} \longmapsto g_{i\bar{i}} \left(d\phi^i - \omega_{\bar{j}}^{\;\,i} \, d\overline{\phi}^{\bar{j}} \right) d\overline{\phi}^{\bar{i}} \, , \tag{7}$$

so that the metric picks up a component of type $(0,2)$ when written in the original holomorphic and anti-holomorphic coordinates. (Of course, there is also a complex conjugate term of type (2,0) which we suppress.)

Since we know how the metric on $\mathcal{M}_{cl}$ changes when $\mathcal{M}_{cl}$ is deformed, we can immediately deduce the correction δS to the classical sigma model action. This correction takes the form of the F-term in (3),

$$\delta S = \int d^4x \, d^2\theta \; \omega_{\bar{i}\bar{j}} \, \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}} \, \overline{D}^{\dot{\alpha}} \overline{\Phi}^{\bar{j}} + c.c. = \int d^4x \; \omega_{\bar{i}\bar{j}} \, d\overline{\phi}^{\bar{i}} \, d\overline{\phi}^{\bar{j}} + \cdots \, , \tag{8}$$

with

$$\omega_{\bar{i}\bar{j}} = \frac{1}{2}\left(g_{i\bar{i}}\,\omega_{\bar{j}}^{\ i} + g_{i\bar{j}}\,\omega_{\bar{i}}^{\ i}\right). \tag{9}$$

Here $\overline{D}_{\dot\alpha}$ is the usual spinor covariant derivative on superspace. We have also performed the fermionic integral with respect to $d^2\theta$ in (8), from which we see that the leading bosonic term reproduces the correction to the metric in (7). As I mentioned earlier, other components of this F-term (indicated by the '$\cdots$' above) include a four-fermion, non-derivative interaction from which the multi-fermion F-term takes its name.

Chirality and Cohomology

The multi-fermion F-term in (8) that describes the complex structure deformation of the moduli space differs from the more familiar superpotential in two important ways.

First, the superpotential arises from a holomorphic function $W(\Phi^i)$ on $\mathcal{M}_{cl}$ and hence is manifestly supersymmetric. In contrast, the multi-fermion F-term is not manifestly supersymmetric, since the corresponding operator $\mathcal{O}_\omega = \omega_{\bar{i}\bar{j}}\,\overline{D}_{\dot\alpha}\overline{\Phi}^{\bar{i}}\,\overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{j}}$ is not manifestly chiral. Instead, the chirality of $\mathcal{O}_\omega$ (in the on-shell supersymmetry algebra of the classical sigma model) follows from the fact that the tensor $\omega_{\bar{j}}^{\ i}$ is annihilated by $\overline{\partial}$.

Another important distinction between the multi-fermion F-term in (8) and the superpotential is that, unlike a holomorphic function, the cohomology class in $H^1_{\overline{\partial}}(\mathcal{M}_{cl}, T\mathcal{M}_{cl})$ that actually determines the deformation is locally trivial. This fact implies that locally on $\mathcal{M}_{cl}$, the multi-fermion F-term δS can be integrated to a D-term, having the form $\int d^4\theta\,(\cdots)$. However, because the cohomology class represented by ω is globally non-trivial, we cannot write the correction δS globally on $\mathcal{M}_{cl}$ as a D-term, and in this sense δS is an F-term.

Multi-Fermion F-terms of Higher Degree

The multi-fermion F-term in (8) that describes a deformation of the moduli space is only the first in a series of multi-fermion F-terms that we can consider. To exhibit the generalization, we begin with a section ω of $\overline{\Omega}^p_{\mathcal{M}_{cl}} \otimes \overline{\Omega}^p_{\mathcal{M}_{cl}}$. (Were it not for the requirement of Lorentz-invariance, we could more generally start with a section of $\overline{\Omega}^p_{\mathcal{M}_{cl}} \otimes \overline{\Omega}^q_{\mathcal{M}_{cl}}$ for $p \neq q$.) Explicitly, ω is given by a tensor $\omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p}$ that is antisymmetric in the indices $\bar{i}_k$ and $\bar{j}_k$. Given such a tensor, we construct a possible term in the effective action that generalizes what we found in (8):

$$\delta S = \int d^4x\, d^2\theta\; \omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p}\;\left(\overline{D}_{\dot\alpha_1}\overline{\Phi}^{\bar{i}_1}\,\overline{D}^{\dot\alpha_1}\overline{\Phi}^{\bar{j}_1}\right)\cdots\left(\overline{D}_{\dot\alpha_p}\overline{\Phi}^{\bar{i}_p}\,\overline{D}^{\dot\alpha_p}\overline{\Phi}^{\bar{j}_p}\right). \tag{10}$$

Given the form of this operator, we can assume that ω is symmetric under the overall exchange of $\bar{i}$'s and $\bar{j}$'s.

As explained in [8], the general multi-fermion F-terms of degree $p > 1$ have no effect on the classical algebraic geometry of the moduli space, and for this reason our primary interest lies in the F-term of degree $p = 1$ associated to a deformation of the moduli space. See Section 2 of [8] for remarks on the chirality and the cohomological interpretation of the operator in (10).

Worldsheet Instanton Computations

In the rest of my talk, I will sketch two worldsheet instanton computations. These computations can be done in various ways, corresponding to the various descriptions (RNS, Green-Schwarz, hybrid) of the fundamental string. I find it convenient to use a "physical gauge" formalism, which corresponds to the Green-Schwarz description of the fundamental string after its inherent κ-symmetry is fixed. Equivalently, the physical gauge formalism is the fundamental string analogue of the formalism introduced by Becker, Becker, and Strominger [9] for general p-brane instanton computations. My discussion follows [10], where the physical gauge formalism for worldsheet instanton computations is introduced.

To illustrate this formalism, let us consider a worldsheet instanton which wraps a holomorphic curve C smoothly embedded in $\mathbb{C}^2 \times X$, where for convenience we choose a complex structure on the four Euclidean directions transverse to X. In physical gauge, C carries a set of four complex bosons x^μ and y^m, for $\mu, m = 1, 2$, which describe fluctuations normal to C in $\mathbb{C}^2 \times X$. As such, the bosons x^μ and y^m are valued respectively in the bundles $\mathcal{O}^2$ and N, where $\mathcal{O}^2$ is the rank two trivial bundle on C and where N is the holomorphic normal bundle to C in X.

We now consider the worldvolume fermions on C in physical gauge. If we work for simplicity with the $SO(32)$ heterotic string, then the left-moving current algebra on C is described by thirty-two left-moving fermions λ which transform as sections of the bundle $S_-(TC) \otimes V|_C \equiv V_-$. Here $S_-(TC)$ is a left-moving spin bundle on C, and we recall that V is the holomorphic gauge bundle on X. By convention, the kinetic operator for these fermions is the $\overline{\partial}$ operator on C coupled to the bundle V_-.

As for the right-moving worldvolume fermions on C, these fermions are naturally twisted and transform as sections of the bundles below,

$$\begin{aligned} \theta^\alpha \quad & \text{in} \quad S_-(\mathcal{O}^2) \otimes \overline{\mathcal{O}}\,, \\ \theta^\alpha_{\overline{z}} \quad & \text{in} \quad S_-(\mathcal{O}^2) \otimes \overline{\Omega}^1_C\,, \\ \overline{\chi}^{\overline{m}}_{\dot\alpha} \quad & \text{in} \quad S_+(\mathcal{O}^2) \otimes \overline{N}\,. \end{aligned} \tag{11}$$

Here $S_\pm(\mathcal{O}^2)$ denotes the respective positive- or negative-chirality spin bundle associated to the rank two trivial bundle $\mathcal{O}^2$ on C. We indicate the corresponding spinor indices by α and $\dot\alpha$. By convention, the kinetic operator for a right-moving fermion on C is the ∂ operator coupled to the appropriate bundle, so we naturally regard the fermions in (11) as transforming in *anti-holomorphic* bundles on C. Thus, $\overline{\mathcal{O}}$ is still the trivial line-bundle on C. Also, $\overline{\Omega}^1_C$ is the bundle of $(0,1)$ forms on C, indicated by the subscript $\overline{z}$ on $\theta^\alpha_{\overline{z}}$. Finally, $\overline{N}$ is the anti-holomorphic normal bundle to C

in X, indicated by the superscript $\overline{m}$ on $\overline{\chi}^{\overline{m}}_{\dot\alpha}$.

Let us count right-moving fermion zero-modes on C. These zero-modes arise from anti-holomorphic sections of the anti-holomorphic bundles $\overline{\mathcal{O}}$, $\overline{N}$, and $\overline{\Omega}^1_C$. By conjugation, such anti-holomorphic sections are related to holomorphic sections of the holomorphic bundles $\mathcal{O}$, N, and Ω^1_C. So for instance, we see that C always carries two fermion zero-modes arising from θ^α. These zero-modes are Goldstone modes for the two supersymmetries broken by C, and they generate the chiral measure $d^2\theta$ in any instanton computation. If C has genus g, then the fermion $\theta^\alpha_{\overline{z}}$ similarly has $2g$ zero-modes. Finally, if we let $p = \dim_{\mathbb{C}} H^0(C,N)$, so that C has p infinitesimal holomorphic deformations inside X, then the fermion $\overline{\chi}^{\overline{m}}_{\dot\alpha}$ has $2p$ zero-modes.

We now perform the worldsheet instanton computation in two special cases. First, we assume that C is a smooth, isolated rational curve in X. In this case $N = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$, so that $g = p = 0$ and C carries precisely two fermion zero-modes. Hence C naturally contributes to the superpotential.

In the second case, we assume that C is a rational curve that moves in a one-parameter holomorphic family in X, with normal bundle $N = \mathcal{O} \oplus \mathcal{O}(-2)$. Hence $g = 0$ but $p = 1$, so that C carries four fermion zero-modes and naturally generates a multi-fermion F-term as in (3).

Example: C is an isolated rational curve

We first evaluate the superpotential contribution from C in the case that C is an isolated, rational curve. Evaluating the contribution from C to the superpotential is now admirably direct in the physical gauge formalism. To determine the contribution from C to the low-energy effective action — which amounts to integrating out the physical degrees of freedom on C — we merely evaluate the worldvolume partition function. By standard reasons of holomorphy, any superpotential contribution from C cannot have a non-trivial perturbative dependence on the string tension α', so we need only evaluate this partition function to one-loop order. Hence the superpotential contribution from C can be computed as an elementary Gaussian integral over the fluctuating, physical degrees of freedom on the worldvolume.

With our previous description of the physical degrees of freedom on C, we can perform this Gaussian integral immediately. We find that the F-term contribution to the effective action from C is given formally by

$$\delta S \,=\, \int d^4x\, d^2\theta\, W_C\,, \tag{12}$$

with

$$W_C \,=\, \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \frac{\mathrm{Pfaff}\left(\overline{\partial}_{V_-}\right)}{\left(\det{}'\,\overline{\partial}_{\mathcal{O}}\right)^2 \left(\det \overline{\partial}_{\mathcal{O}(-1)}\right)^2}\,. \tag{13}$$

In this expression, the chiral measure $d^4x\, d^2\theta$ on superspace arises as an integral over the collective coordinates of the instanton. As in the introduction, we recognize the argument of the exponential factor in W_C as the classical action of a worldsheet wrapped on C.

Beyond tree-level, the partition function receives contributions from the one-loop determinants of the kinetic operators for the fluctuating modes on C. Because of the twisting of the right-moving worldvolume fermions, the determinants associated to these fermions cancel the corresponding determinants associated to the non-zero, right-moving modes of the bosons. Equivalently, this cancellation is a consequence of the two residual supercharges preserved by C. So the non-trivial determinantal factors appearing in W_C arise only from the left-moving sector of the worldvolume theory.

In the left-moving sector, the path integral over the $SO(32)$ current algebra fermions is represented by the Pfaffian factor in the numerator of W_C, and the path integral over the non-zero, left-moving modes of the worldvolume bosons is represented by the product of determinants in the denominator of W_C. In these expressions, $\overline{\partial}_{V_-}$, $\overline{\partial}_{\mathcal{O}}$, and $\overline{\partial}_{\mathcal{O}(-1)}$ denote the respective $\overline{\partial}$ operators on C coupled to the corresponding holomorphic bundles. Because the bosons x^μ valued in the trivial bundle $\mathcal{O}^2$ have zero-modes, we include a "prime" on the determinant of $\overline{\partial}_{\mathcal{O}}$ to indicate that this determinant is to be computed with the zero-mode omitted; it otherwise vanishes.

Example: C is a rational curve that moves in a one-parameter holomorphic family

To discuss the case that C moves in a one-parameter holomorphic family, we consider the relatively trivial case that the Calabi-Yau threefold X factorizes as $X = E \times Y$, where E is an elliptic curve and Y is a $K3$ surface. We also assume that the bundle V over X factorizes as the tensor product of a flat bundle V_E over E and a holomorphic bundle V_Y over Y. In this case, heterotic compactification on X preserves $\mathcal{N} = 2$ supersymmetry in four dimensions, and the moduli space is now locally a product of a hypermultiplet moduli space $\mathcal{M}_H$ associated to Y and a vector multiplet moduli space associated to E.

We assume that the worldsheet instanton wraps a genus zero surface C inside Y. Supersymmetry implies that C is actually a rational curve in some complex structure on Y, and in this complex structure we have a trivial family of rational curves in X parametrized by E.

Because this example preserves $\mathcal{N} = 2$ supersymmetry, a worldsheet instanton wrapping C cannot generate a superpotential. However, it can generate a correction to the metric on hypermultiplet moduli space $\mathcal{M}_H$. We again compute the instanton correction by evaluating the one-loop partition function of the worldvolume theory on C. In complete analogy to our result for the superpotential, we find

$$\delta S = \int d^4x\, d^2y\, d^2\theta\, d^2\overline{\chi}\ \Psi_C\,, \tag{14}$$

where

$$\Psi_C = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \frac{\mathrm{Pfaff}\left(\overline{\partial}_{V_-}\right)}{\left(\det{}'\,\overline{\partial}_{\mathcal{O}}\right)^3 \left(\det{}'\,\overline{\partial}_{\mathcal{O}(-2)}\right)}\,. \tag{15}$$

Here d^2y is the measure on the elliptic curve E induced from the background metric on E, and the fermionic measure $d^2\theta\, d^2\overline{\chi}$ is the four-dimensional reduction of the six-dimensional chiral measure. Finally, $V_- \equiv S_- \otimes V_Y|_C$, and because the normal bundle to C in X is now $N = \mathcal{O} \oplus \mathcal{O}(-2)$, the bosonic denominators in Ψ_C differ in the obvious way from the corresponding denominators in W_C.

We now wish to exhibit δS in (14) as a multi-fermion F-term of the form (3). To this end, we perform the bosonic integral over E and the fermionic integral with respect to $d^2\overline{\chi}$.

The bosonic integral over E in (14) is trivial, since nothing in the integrand depends on E. This integral produces a factor of the area of E, which is then reabsorbed when we rescale the four-dimensional metric to Einstein frame. We note in passing that the modulus associated to the area of E transforms in a vector multiplet, and any correction to the metric on the hypermultiplet moduli space cannot depend on the vector multiplets.

In contrast to the bosonic integral over E, the fermionic integral with respect to $d^2\overline{\chi}$ is quite interesting. From the perspective of a perturbative worldvolume computation, the latter integral is the integral over the zero-modes of the worldvolume fermions $\overline{\chi}^{\overline{m}}_{\dot\alpha}$ tangent to E, so performing this integral implicitly reveals how the fermionic zero-modes associated to the family are "soaked up" in worldvolume perturbation theory. Of course, at this point we could perform such an analysis directly by considering the various interaction terms involving $\overline{\chi}^{\overline{m}}_{\dot\alpha}$ in the worldvolume Green-Schwarz action. However, a much more elegant approach is to use the structure of $\mathcal{N}=2$ supersymmetry present in this example.

We now make the simplifying assumption that the $K3$ surface Y is non-compact, so that gravity is effectively decoupled and our discussion of multi-fermion F-terms in the context of global supersymmetry is valid. In this case, the low-energy effective action for the moduli associated to Y is a hyperkahler sigma model whose target space is the hypermultiplet moduli space $\mathcal{M}_H$. Here we use the non-compactness of Y, since in the context of $\mathcal{N}=2$ supergravity, $\mathcal{M}_H$ is only quaternionic Kahler.

Although a general quaternionic Kahler manifold need not be Kahler, a hyperkahler manifold certainly is Kahler. Our implicit choice of a distinguished $\mathcal{N}=1$ subalgebra in (14) — the subalgebra associated to the superspace coordinate θ — then corresponds geometrically to the choice of a distinguished complex structure on $\mathcal{M}_H$, in which we regard $\mathcal{M}_H$ as an ordinary Kahler manifold appropriate for a sigma model with only $\mathcal{N}=1$ supersymmetry.

Because the fermionic measure $d^2\overline{\chi}$ in (14) can be interpreted as part of the chiral measure on $\mathcal{N}=2$ superspace, the corresponding fermionic integral can be equivalently evaluated by acting on the integrand Ψ_C with the operator $\{\overline{Q}^{(2)}_{\dot\alpha}, [\overline{Q}^{(2)\,\dot\alpha}, \cdot\,]\}$, where $\overline{Q}^{(2)}_{\dot\alpha}$ is the anti-chiral supercharge generating translations along $\overline{\chi}$ in superspace.

To describe geometrically the action of $\overline{Q}^{(2)}_{\dot\alpha}$ in the hyperkahler sigma model, we again introduce $\mathcal{N}=1$ chiral and anti-chiral superfields Φ^i and $\overline{\Phi}^{\bar{i}}$ to describe local holomorphic and anti-holomorphic coordinates on $\mathcal{M}_H$ in the distinguished complex structure. This complex structure corresponds to a covariantly constant endomorphism $\mathbf{I}$ of $T\mathcal{M}_H$ satisfying $\mathbf{I}^2=-1$, and the action of the associated supercharge $\overline{Q}^{(1)}_{\dot{\alpha}}$ is identified with the action of the Dolbeault $\overline{\partial}$ operator on $\mathcal{M}_H$.

Because $\mathcal{M}_H$ is hyperkahler, we also have covariantly constant tensors $\mathbf{J}$ and $\mathbf{K}$ which define additional complex structures on $\mathcal{M}_H$ and which satisfy the quaternion algebra with $\mathbf{I}$. Of course, the tensor $\mathbf{J}$ is used to define the extra supercharges of the $\mathcal{N}=2$ supersymmetry algebra, and the action of $\overline{Q}^{(2)}_{\dot{\alpha}}$ can be identified geometrically with the action of the differential operator $\mathbf{J}(\partial)$, or in components $\mathbf{J}^{i}_{\bar{i}}\partial_i$, on $\mathcal{M}_H$.

With this geometric description of $\overline{Q}^{(2)}_{\dot{\alpha}}$, we immediately deduce that

$$\int d^2\overline{\chi}\,\Psi_C \,=\, \left\{\overline{Q}^{(2)}_{\dot{\alpha}},\,\left[\overline{Q}^{(2)\,\dot{\alpha}},\,\Psi_C\right]\right\} \,=\, \mathbf{J}^{i}_{\bar{i}}\mathbf{J}^{j}_{\bar{j}}\,(\nabla_i\nabla_j\Psi_C)\;\overline{D}_{\dot{\alpha}}\overline{\Phi}^{\bar{i}}\,\overline{D}^{\dot{\alpha}}\overline{\Phi}^{\bar{j}}\,. \tag{16}$$

Here ∇ denotes the covariant derivative associated to the background hyperkahler metric on $\mathcal{M}_H$. In writing (16), we note that Ψ_C transforms globally as a function on $\mathcal{M}_H$, and we use the fact that $\mathbf{J}$ is covariantly constant and so annihilated by ∇.

In the notation of a four-dimensional effective action with only $\mathcal{N}=1$ supersymmetry, the instanton correction δS in (14) is then given by

$$\delta S \,=\, \int d^4x\,d^2\theta\;\mathbf{J}^{i}_{\bar{i}}\mathbf{J}^{j}_{\bar{j}}\,(\nabla_i\nabla_j\Psi_C)\;\overline{D}_{\dot{\alpha}}\overline{\Phi}^{\bar{i}}\,\overline{D}^{\dot{\alpha}}\overline{\Phi}^{\bar{j}}\,. \tag{17}$$

Hence the trivial family of instantons parametrized by E generates the multi-fermion F-term of degree $p=1$ that is represented geometrically by the tensor

$$\omega_{\bar{i}\bar{j}} \,=\, \mathbf{J}^{i}_{\bar{i}}\mathbf{J}^{j}_{\bar{j}}\,(\nabla_i\nabla_j\Psi_C)\,. \tag{18}$$

We see that, even in the simple example $X = E\times Y$, a one-parameter holomorphic family of rational curves generates the multi-fermion F-term describing a deformation of the moduli space. This simple example preserving $\mathcal{N}=2$ supersymmetry can then be extended fiberwise to a similar instanton computation in Calabi-Yau backgrounds preserving only $\mathcal{N}=1$ supersymmetry.

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