N=1 Thermodynamics, Black-holes and Bose-Einstein Condensation

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August 14th, 2006

Talk at the Fourth Simons Workshop in Mathematics and Physics
Stony Brook University, July 24 - August 25, 2006

The AdS/CFT is a new tool for QFT because it opens up the possibility for analyzing gauge theories at strong coupling. On the other hand, certain questions regarding quantum gravity can be related to questions about ordinary QFTs. It therefore becomes important to understand the way that the weak and strong coupling regimes match.

In this talk, we consider the extrapolation between weak and strong coupling for the thermodynamic properties of $\mathcal{N}=4$ supersymmetric Yang-Mills in the grand canonical ensemble. The gauge theory has an $SU(4)$ R-symmetry and so one can introduce at most 3 chemical potentials corresponding to the maximal abelian subgroup $U(1)^3 \subset SU(4)$. We will choose for simplicity to consider just a single chemical potential corresponding to rotations in the (ϕ_1, ϕ_2) plane of the six scalars. The grand canonical partition function of the theory on a three sphere of radius R is then the Euclidean functional integral on $S^3 \times S^1$, where the thermal circle has circumference $\beta = 1/T$, with the $U(1) \subset SU(4)$ gauged with a constant background field $\mathcal{A}_0 = i\mu t_{U(1)}$ (the component around the thermal circle), where $t_{U(1)}$ is the generator of the $U(1)$. This auxiliary gauge connection is, of course, not to be confused with the dynamical $SU(N)$ gauge field. The partition function is then

$$\mathcal{Z}(T,\mu) = \int_{S^3\times S^1} [dA_\mu][d\phi_a][d\psi_A]\, e^{-S_E} \ , \qquad (1)$$

where covariant derivatives D_0 are replaced by $D_0 + i\mathcal{A}_0$. The Euclidean action includes the terms

$$S_E = \frac{1}{g^2}\int d^4x\, \mathrm{Tr}\Big[\tfrac{1}{2}(D_\mu\phi_n)^2 + i\mu(\phi_1 D_0\phi_2 - \phi_2 D_0\phi_1) + \tfrac{1}{2}(R^{-2}-\mu^2)\phi_n^2 + \cdots\Big] \ . \qquad (2)$$

Notice that this is not real and further that the chemical potential acts as a negative squared mass. Hence if $\mu > R^{-1}$ the theory has a tachyonic mode and there is no ground state (at least in perturbation theory).

The zero modes of the theory consist of the constant mode of the gauge field around the thermal circle, the Polyakov line. Using the gauge symmetry we can diagonal this constant mode,

$$A_0 = \frac{1}{\beta}(\theta_1, \dots, \theta_N) \ , \tag{3}$$

with $\theta_i \sim \theta_i + 2\pi$ by large gauge transformations. However, when $\mu \sim R^{-1}$ the fields ϕ_1 and ϕ_2 have a light mode which must be included and so in this regime we also allow for a VEV for (say) ϕ_1:

$$\phi_1 = (\varphi_1, \dots, \varphi_N) \ . \tag{4}$$

The aim is to calculate the effective action $I(\theta_i, \varphi_i)$ at the one-loop level by integrating out all the massive modes. The phase structure is then determined by extremizing $I(\theta_i, \varphi_i)$. The fact that distinct phases can exist when the theory is in finite volume is ensured by taking the large N limit.

In the strong 't Hooft coupling limit, the theory is described by supergravity on $AdS_5 \times S^5$. The effective five-dimensional gauged supergravity description is often sufficient and the phase structure in the grand canonical ensemble is governed by five dimensional Euclidean geometries with an $S^3 \times S^1$ boundary and an electric potential for one of the $U(1)$ symmetries equal to μ.

On the QFT side, the VEVs φ_i vanish unless $R^{-1} - \mu \sim \mathcal{O}(g^2 N)$, since it is only then that loop corrections can be of the same order as the tree-level potential. For low T, the angles θ_i repel and the ground state consists of a uniform distribution around the circle. As T is increased, for fixed μ, a critical temperature $T_c(\mu)$ is reached and a phase transition occurs to a phase in which the θ_i are all equal. This is analogous to a confining-deconfining phases transition. $T_c(\mu)$ decreases with increasing μ. For $\mu < 1$ this is qualitatively matched on the gravity side by the Hawking-Page transition between two geometries: "thermal AdS" (consisting of Euclidean AdS_5 with a periodic identification) and an R-charge black-hole. The low temperature confined phase is the thermal AdS phase, while the high temperature deconfined phase is the black hole.

On the gravity side, the black hole phase persists into the $\mu > R^{-1}$ region up to a boundary which has $\mu \propto T$ (for large T) beyond which the solution develops pathologies. Yaffe and Yamada recently showed that an analogous region exists in the QFT as a metastable phase with a very long lifetime—at least at high temperature.

An interesting question concerns the situation at low temperature just below the $\mu \sim R^{-1}$ line. When $R^{-1} - \mu \sim \mathcal{O}(g^2 N)$ the tree-level potential for the light modes φ_i are of the same order as the one-loop contribution and there is the possibility for the VEV to develop. It is easy to argue that this effect is subdued once T is increased so the effect can be analyzed at $T = 0$. In this case quantum correction to the tree-level potential is precisely the Casimir energy density of the system. The effective potential, including the tree-level and one-loop terms has the form

$$V(\varphi_i, \mu) = \sum_{ij}^{N} f(\varphi_i - \varphi_j, \mu) \ , \qquad f(\varphi, \mu) = \frac{R^{-2} - \mu^2}{g^2 N}\varphi^2 + \mathcal{E}(\varphi, \mu) \ . \tag{5}$$

Although it is difficult to calculate the 1-loop Casimir energy for arbitrary φ and μ we can use the fact that $\mu \sim R^{-1}$ to order g^2N and replace μ in the 1-loop term by R^{-1}. In this case the calculation is tractable. The resulting function $\mathcal{E}(\varphi, R^{-1})$ decreases from $\mathcal{E}(0, R^{-1}) = 3/16R$ as φ increases. Hence, it is clear that there is some critical value of μ above which a VEV for the scalar field develops. This is a Bose-Einstein condensate.

The question is whether this new phase persists at strong coupling. If we imagine that it does then since part of the R-symmetry is spontaneously broken it suggests that the new phase corresponds to some Euclidean black hole solution which is localized on the S^5.

I would like to thank the organizers of the Simons Workshop for providing a stimulating environment for developing some of these ideas (in collaboration with Prem Kumar and Asad Naqvi).

References

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