Geometric methods in (0,2) theories and a generalization of quantum cohomology

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This talk is based on work with Eric Sharpe [2]. Utilizing mirror symmetry, Adams, Basu & Sethi[1] computed certain correlators in heterotic (0,2) theories, which exhibit a topological nature. The ring generated by the underlying operators constitutes a generalization of quantum cohomology. We would like to give a mathematical description of these topological correlators, but unfortunately an analog of the mathematical device which enables computations in (2,2) theories, the virtual fundamental class, is not known for (0,2). We content ourselves with examples where the virtual fundamental class is not needed.

We wish to consider a $(0,2)$ sigma model whose target is a compact Kähler manifold X and whose left-movers couple to a rank r holomorphic vector bundle $E \to X$ (the heterotic string is an important example). Cancellation of $U(1)$-charge anomalies demands that $c_1(E) = c_1(TX)$ and that $c_2(E) = c_2(TX)$. We also restrict our attention to those bundles that satisfy $\Lambda^r E \cong K_X^\vee \cong \Lambda^{\dim X} TX$, which allows the final step in the computation of the correlation functions. The computation can and will be carried out more simply in an associated half-twisted theory. There is a BRST operator corresponding to $\bar\partial$ acting on bundle-valued differential forms, but the underlying CFT is not topological. Nevertheless, certain observables have a topological nature.

The relevant topological observables correspond to cohomology classes $\omega_i \in H^{q_i}(X, \Lambda^{p_i} E^\vee)$, which can be classically multiplied via the usual cup product. There will also be quantum corrections. Correlation functions of these operators define a deformation of quantum cohomology, and as such they fill out a ring structure on the space of observables. The CFT considerations of [3] have confirmed the ring structure for heterotic bundles of rank $r < 8$.

One may ask how to obtain bundles satisfying the above constraints: deformations of tangent bundles provide easy examples. Consider, for example, a deformation of $T(\mathbb{P}^1 \times \mathbb{P}^1)$. We may present the tangent bundle as the cokernel of a complex;

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{\begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ 0 & y_1 \\ 0 & y_2 \end{pmatrix}} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow T(\mathbb{P}^1 \times \mathbb{P}^1) \longrightarrow 0. \tag{1}$$

Here, $\mathcal{O}(1,0)$ is the bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ whose sections are the linear polynomials on the first $\mathbb{P}^1$ and $\mathcal{O}(0,1)$ is the corresponding bundle for the second $\mathbb{P}^1$, while the x's and y's are homogeneous coordinates on the first and second $\mathbb{P}^1$s, respectively.

We deform the tangent bundle by changing the map: the heterotic bundle E is the cokernel of

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{\begin{pmatrix} x_1 & \epsilon x_1 \\ x_2 & 0 \\ 0 & y_1 \\ 0 & y_2 \end{pmatrix}} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow E \longrightarrow 0. \tag{2}$$

Due to their properties with respect to exact sequences, the Chern classes of the bundle E satisfy $c_1(E) = c_1(TX)$ and $c_2(E) = c_2(TX)$.

To actually compute the correlation function computations, we use the (0,2) linear sigma model. One of the phases of the linear sigma model is the geometric phase, and this is where we do the computations. In the deformed $T(\mathbb{P}^1 \times \mathbb{P}^1)$ example, the gauge group is $U(1) \times U(1)$.

We take the fields of the linear sigma model to be valued in bundles $\mathcal{O}(d)$ on the worldsheet, i.e. degree d polynomials in the worldsheet coordinates, corresponding to the instanton background we want to compute in. Now, instead of working in some complicated instanton moduli space, we are simply dealing with a projective space, or more generally a toric variety. However, unlike the Gromov-Witten theory of (2,2) where we use de Rham cohomology, we must work with cohomology of a bundle on the moduli space. Specifically, the zero modes of the bundle E fill out a bundle F in the instanton background. The presentation of F derives from that of the original bundle E. In particular, the fields defining F inherit the $U(1) \times U(1)$ charges of the fields that define E.

For example, the degree (1,0) instanton sector for strings in the deformed bundle (2) leads to a sequence defining the bundle F on the moduli space $\mathcal{M} = \mathbb{CP}^{2(d+1)-1} \times \mathbb{CP}^1 = \mathbb{CP}^3 \times \mathbb{CP}^1$ for $d = 2$:

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{\begin{pmatrix} \alpha_1 & \epsilon\alpha_1 \\ \alpha_2 & \epsilon\alpha_2 \\ \alpha'_1 & 0 \\ \alpha'_2 & 0 \\ 0 & \beta_1 \\ 0 & \beta_2 \end{pmatrix}} \mathcal{O}(1,0)^4 \oplus \mathcal{O}(0,1)^2 \longrightarrow F \longrightarrow 0. \tag{3}$$

Here, the α's are coordinates on the $\mathbb{P}^3$ and the β's are coordinates on the $\mathbb{P}^1$.

The topological operators whose correlation functions we would like to compute correspond classically to elements of $H^1(X, E^{\vee})$. In the degree (1,0) instanton background the operators instead live in $H^1(\mathcal{M}, F^\vee)$, which is in fact isomorphic to $H^1(X, E^\vee)$. To compute correlators, we first compute the cup product of the corresponding cohomology classes. For example, a classical 2-point function corresponds to an element of $H^2(X, \Lambda^2 E^\vee)$, and the degree $(1,0)$ instanton contribution to a 4-point function corresponds to an element of $H^4(M, \Lambda^4 F^\vee)$.

This is where our final condition on the bundle, $\Lambda^r E \cong K_X^\vee \cong \Lambda^{\dim X} TX$, comes into play. In the example above, $\Lambda^2 E^\vee \cong \mathcal{O}(-2,-2) \cong K_X^\vee$, so that

$$H^2(X, \Lambda^2 E^\vee) \cong H^2(X, K_X^\vee) \cong \mathbb{C},$$

which allows the classical correlators to be computed as a complex number. Similarly, $H^4(M, \Lambda^4 F^\vee) \cong H^4(M, K_{\mathcal{M}}^\vee) \cong \mathbb{C}$ for the (1,0) instanton sector, allowing the computation of the $(1,0)$ contribution as a complex number. A Riemann-Roch calculation establishes the identity $\Lambda^{\text{top}} F^\vee \cong K_M$ for any instanton background.

The details of the cup product computation are most conveniently computed using Čech cohomology. The computational methods can be generalized to more interesting situations [5].

References

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[3] A. Adams, J. Distler and M. Ernebjerg, "Topological heterotic rings," arXiv:hep-th/0506263.

[4] D. R. Morrison and M. Ronen Plesser, Nucl. Phys. B **440**, 279 (1995) [arXiv:hep-th/9412236].

[5] J. Guffin and S. Katz, work in progress.