

# Calabi-Yau Singularities

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## Introduction

Geometric engineering was the program of Vafa et al to embed (essentially) all quantum field theories into string theory.

The swampland program, also begun by Vafa, is the new idea that most quantum field theories actually belong to the swampland, not the string landscape. Here the focus is on the restrictions to what we can geometrically engineer.

These two points of view are reconciled by the fact that the local geometry which engineers a generic field theory will not have a global completion. Only those QFT's that are in the landscape actually come from a global (ie. compact) compactification.

These field theories are typically found in string theory compactifications at points in the moduli space where isolated singularities are formed by the shrinking of some cycles. We will look at the difference between the deformation theory and resolutions of such singularities in the local versus global case. We will find that typically (at least in the physically relevant case of CY 3-folds) the deformations / resolutions will be much more constrained when embedded in a global geometry.

## 6 and 5 spacetime dimensions

We will warm up with  $\mathbb{C}$ -dim 2. Globally only K3 surface is possible, while locally, there are all the ADE singularities.

The singularities of K3 can be deformed or resolved, which are similar operations in this case because of the hyperkahler structure. Let  $X \rightarrow \bar{X}$  be a resolution, containing local neighborhoods,  $U \rightarrow \bar{U}$  respectively. Here  $U$  is a tubular neighborhood of the exceptional divisor.

There is an injective map,  $H_2(U) \rightarrow H_2(X)$ , or equivalently a surjection on cohomology,  $H^2(X) \rightarrow H^2(U)$ . Hence every local resolution class appears in the global completion.

Similarly for complex structure deformations, it was shown by [1] that all local deformation extend to global ones; more precisely,  $Def \bar{X} \rightarrow Def \bar{U}$  is surjective.

The difference between global and local deformations of singularities begins to make an appearance as we go to higher dimensional compactifications. Consider next a 5 dimensional geometry of the form  $(X \times S^1)/(\mathbb{Z}/N)$ , where the group  $G = \mathbb{Z}/N$  acts by translation on  $S^1$  and preserves the holomorphic 2-form on  $X$  (there may be fixed points in the action on  $X$ ).

Now,  $H^2((\bar{X} \times S^1)/(\mathbb{Z}/N)) \rightarrow H^2(\bar{U} \times (-\epsilon, \epsilon))$ , and the image is only the  $G$  invariant subspace of the cohomology. Exactly the same is true for the deformations. Hence the local singularity can have more deformations or resolutions than in the global case.

### Calabi-Yau 3-folds

Recall the conifold transition in the usual local description:  $\bar{U}$  is the real cone over  $S^3 \times S^2$ , which has a family of deformations  $\bar{U}_t = D^3 \times S^3$ , and the can be blown up to  $U = S^2 \times D^4$ .

There are important global aspects of the resolutions that are missed in this local picture. Consider shrinking to points a collection of  $k$  rational curves,  $C_j \subseteq X$ , with generic normal bundles in the Calabi-Yau  $\mathcal{N}_{C_j/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ . The singular variety is  $\bar{X} \leftarrow X$ , containing  $k$  conifold points. Then we have the following result that the smooth deformations of the singular variety correspond to the relations among the homology cycles of the exception curves in the resolution.

**Theorem** (Friedman [2])

$\bar{X}$  has a deformation  $\phi$  into smooth  $\bar{X}_t \Leftrightarrow [C_1], \dots, [C_k]$  are not independent. More precisely, there is a map  $\mathbb{R}[C_1] \oplus \dots \oplus \mathbb{R}[C_k] \rightarrow H^2(X, \mathbb{R})$ , such that the dim of its kernel =  $\dim Def \bar{X}$  in smooth varieties.

For example, consider the singular quintic 3-fold,  $Q \subset \mathbb{P}^4$  defined by the equation  $x_0 f_4(x_0, \dots, x_4) + x_1 g_4(x_0, \dots, x_4) = 0$ . Near the 16 points  $f_4 = g_4 = 0$ , the geometry looks locally like a conifold point. There are 15 deformations into smooth varieties. There is also a blow up along  $x_0 = x_1 = 0$ , which simultaneously resolves all 16 points, hence in homology,  $[C_i] = [C_j]$ . These 15 relations give rise to the 15 deformations.

Key structure: The local Picard group (of  $\bar{X}$  or  $\bar{U}$ ) determines the resolutions of the singularities.

On a smooth  $\mathbb{C}$ -manifold, every  $\mathbb{C}$ -codim 1 subspace is defined by a single equation, locally. A *Weil divisor* is a codim 1 subspace, while a *Cartier divisor* is a subspace defined by 1 equation. On singular local varieties these can be different, hence the *local Picard group*,  $\text{Pic}(X) = \text{Weil}/\text{Cartier}(X)$  is nontrivial. (Of course, globally, Pic is non-vanishing even for smooth varieties).

For example, at the conifold point,  $xy = zt$ , the equations  $x = z = 0$  define a codim 1 subspace.

Here we run into a problem: The holomorphic coordinate transformations of an analytic  $\mathbb{C}$ -manifold need not be birational. Hence not all  $\mathbb{C}$ -analytic neighborhoods of a point have the same local Picard group! For example, any double point in dimension 3 can be put into the form  $xy = zv$ , by a holomorphic change of coordinates, but this will not be true at the level of algebraic varieties.

Between the notions of global and  $\mathbb{C}$ -local lies the idea of Zariski-local. An algebraic variety can be covered by  $X = \bigcup Z_i$ , where  $Z_i$  is an algebraic subvariety (which will always have some codim). The transition functions are birational  $\phi_{ij} : \mathbb{C}^{m_i} \rightarrow \mathbb{C}^{m_j}$ , where  $Z_i \hookrightarrow \mathbb{C}^{m_i}$ , and the image is defined by polynomial equations.

Fact: The local Picard group is the same in all Zariski-local neighborhoods of a point. But Zariski neighborhoods are very large, so this is not the ideal framework for us.

### The goal

The goal is to isolate the properties of CY singularities of relevance to physics, encoded in another structure which does not require the full baggage of Zariski-local.

Consider D5 branes wrapping the  $S^2$  of resolved conifold in IIB theory. These become singular D3 branes as the  $S^2$  shrinks to become the conifold singularity. There are two points of view:

AdS / CFT exploits the conical structure to use the geometry  $S^2 \times S^3 \times AdS_5$ .

We will take another approach due to Douglas and Moore. Near an orbifold point, we can use the orbifold structure to write down the open string boundary conditions.

### McKay correspondence

The orbifold singularities in 2  $\mathbb{C}$  dimensions are classified locally by  $\bar{U} = \mathbb{C}^2/G$ , where  $G \subseteq SU(2)$  is a finite subgroup. We will look at the resolutions,  $\pi : U \rightarrow \bar{U}$  of these rational double points. Recall the McKay correspondence of irreducible representations of  $G$  with vertices of the affine Dynkin diagram of ADE Lie groups.

Following the geometrical interpretation found by Gonzalez-Sprinberg and Verdier. Given a representation  $\rho : G \rightarrow \mathbb{C}^n$ , then they constructed  $M_\rho : \mathbb{C}^n \times_\rho (\mathbb{C}^2/G) \rightarrow \mathbb{C}^2/G$ .

$\pi^*(M_\rho)/(tors.) =$  vector bundle on  $U$  with  $C_1$  the corresponding component. Note that the trivial representation gets associated with  $C_1 = 0$ , the extended node of the affine Dynkin diagram.

$$\mathbb{C}[s, t] = \bigoplus_{\rho} \mathbb{C}[s, t]_{\rho}$$

This can be computed explicitly using generators and relations.  $\mathbb{C}[s, t]_{\rho}$  are modules on  $\mathbb{C}[s, t]^G$ .

Example:  $\mathbb{Z}/N$  acting on  $\mathbb{C}^2$  via  $(s, t) \mapsto (e^{2\pi i/N} s, e^{-2\pi i/N} t)$ . It is easy to work out the description of this orbifold as an affine variety: In terms of the variables on  $\mathbb{C}^3$  given by  $x = s^N$ ,  $y = t^N$ ,  $z = st$ , the variety is defined by  $xy = z^N$ .

The irreps are given by the action  $\rho_k(e^{2\pi i/N})\alpha = e^{2\pi i k/N} \alpha$ . Then  $a = s^k$ ,  $b = t^{N-k}$  generate  $M_{\rho_k}$ , with relations  $ya = z^k b$ ,  $xb = z^{N-k} a$ . These can be encoded in the matrix

$$\Psi = \begin{pmatrix} y & -z^k \\ -z^{N-k} & x \end{pmatrix},$$

whose cokernel is exactly the relations of the module. Together with a companion matrix,

$$\Phi = \begin{pmatrix} x & z^k \\ z^{N-k} & y \end{pmatrix},$$

this gives a factorization of the superpotential,

$$\Phi\Psi = (xy - z^N)I.$$

If the rank of  $M_{\rho_k} = \ell$ , then the matrix is  $s\ell \times 2\ell$ . Matrix factorization can be used to specify open string boundary conditions. From the worldsheet boundary theory this results from the fact that the Warner term can be cancelled by

$$\int_{\partial\Sigma} dt d\theta \Gamma \Phi(x),$$

where  $\Gamma$  is the boundary fermion, satisfying  $\bar{D}\Gamma = \Psi(x)\Gamma$ .

For example, consider the singularity associated to the central node of the  $E_8$  Dynkin diagram. Then the superpotential,  $x^2 + y^3 + z^3$  has a  $12 \times 12$  matrix factorization,  $(XI_{12} + \Xi_{12})(XI_{12} - \Xi_{12}) = (X^2 + Y^3 + Z^5)I_{12}$ , where

$$\Xi_{12} = \begin{pmatrix} 0 & \phi_6 \\ -\psi_6 & 0 \end{pmatrix},$$

is expressed in terms of

$$\varphi_6 = \begin{pmatrix} 0 & 0 & 0 & -Y^2 & -YZ^2 & -Z^4 \\ 0 & 0 & 0 & -Z^3 & Y^2 & YZ^2 \\ 0 & 0 & 0 & -YZ & -Z^3 & Y^2 \\ -Y & -Z^2 & 0 & 0 & 0 & -Z^3 \\ 0 & Y & -Z^2 & Z^2 & 0 & 0 \\ -Z & 0 & Y & 0 & Z^2 & 0 \end{pmatrix},$$

and

$$\psi_6 = \begin{pmatrix} 0 & 0 & Z^3 & Y^2 & YZ^2 & Z^4 \\ -Z^2 & 0 & 0 & Z^3 & -Y^2 & -YZ^2 \\ 0 & Z^2 & 0 & YZ & Z^3 & -Y^2 \\ Y & Z^2 & 0 & 0 & 0 & 0 \\ 0 & -Y & Z^2 & 0 & 0 & 0 \\ Z & 0 & -Y & 0 & 0 & 0 \end{pmatrix}.$$

### Matrix factorizations and deformations of singularities

We want to study the deformation theory of these D-brane boundary condition data. To see how things work, consider the  $A_1$  conifold singularity,  $xy = z^2 - t^2 = (z+t)(z-t)$ . Then we can deform the matrix  $\Psi$  (for this case of  $N = 2, k = 1$ ) to the matrix,

$$\Psi_t = \begin{pmatrix} y & -(z+t) \\ -(z-t) & x \end{pmatrix},$$

which factorizes the deformed equation,  $\Phi_t \Psi_t = (xy - (z+t)(z-t)) I_2$ , where

$$\Phi_t = \begin{pmatrix} x & z+t \\ z-t & y \end{pmatrix}.$$

**Principle:** Deformations of matrix factorizations capture the local Picard group in the D-brane boundary condition data on the deformed singularity.

We will see this for further examples with the coefficient of the longest root,  $\ell = 1$ , which corresponds to  $2 \times 2$  factorization. Consider the singular geometry,  $xy = z^{2n} - t^2 = (2z^n + \tilde{t})\tilde{t}$ . This has a matrix factorization,

$$\begin{pmatrix} x & f_k(z, t) \\ g_{N-k}(z, t) & y \end{pmatrix} \begin{pmatrix} y & -f_k \\ -g_{N-k} & x \end{pmatrix} = (xy - f_k g_{N-k}) I_2,$$

which depends on a degree  $k$  and a degree  $n - k$  polynomial,  $f_k$  and  $g_{n-k}$ . To have a blow up, it is necessary that  $k, n - k > 1$ , which leads to  $\mathcal{N}_{C/X} = \mathcal{O} \oplus \mathcal{O}(-2)$ .

The only other local geometry possible for a contractible  $\mathbb{P}^1$  in a Calabi-Yau 3-fold is the  $\mathcal{O}(1) \oplus \mathcal{O}(-3) \rightarrow \mathbb{P}^1$  studied by Katz and Morrison.

$\bar{X} \rightarrow \Delta$ , where the fibers are ADE. We blow up,  $X \rightarrow \bar{X}$ , a single node of the Dynkin diagram. Depended on the value of  $\ell =$  coefficient of the longest root, there will be a different "universal" singularity.

For length  $\ell = 1$ , the general form is  $xy = z^2 - t(\dots)^2$ , with deformation theory as described above.

For  $\ell = 2$ , we have a quadratic in 4 variables whose discriminant is a perfect square,

$$Q = \begin{pmatrix} x & y & z & t \end{pmatrix} \begin{pmatrix} 1 & & & \\ & u & v & \\ & v & w & \\ & & & uw - v^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

The total space (ie. including the deformation parameters geometrically) of the deformations of this singularity is the quadratic hypersurface  $Q = 0$  in  $\mathbb{C}^7$ , which includes the coordinates of the affine embedding of the Calabi-Yau in  $\mathbb{C}^4$ , as well as the three parameter moduli space of deformations.

The deformations are all captured by the matrix factorizations, of the form  $(xI - \Xi)(xI + \Xi) = QI_4$ , where

$$\Xi = \begin{pmatrix} -vt & y & z & t \\ -uy - 2vz & vt & -ut & z \\ -wz & wt & -vt & -y \\ -uwt & -wz & uy + 2vz & vt \end{pmatrix}.$$

A global deformation can be constructed similarly if you can write the CY globally in this form. This method allows for the explicit computation of the superpotential of wrapped D5 branes, for example in  $\mathcal{O}(1) \oplus \mathcal{O}(-3) \rightarrow \mathbb{P}^1$ .

**Extensions:**

- 1) For singularities of length  $\ell = 3, 4, 5, 6$  - we need the universal deformation.
- 2) More general Calabi-Yau singularities, such as del Pezzo?

**References**

[1] M. Burns Jr. and J. M. Wahl "Local contributions to global deformations of surfaces," *Inventiones Mathematicae* **26**, 1, March, 1974.

- [2] R. Friedman "Simultaneous resolution of threefold double points," Math. Ann. **274**, 1986.