

The Geometry of AdS/CFT

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Introduction

These notes are based on a talk I gave at the Fourth Simons Workshop in Mathematics and Physics on July 28, 2006. The talk was based on joint work with Jerome Gauntlett, Dario Martelli, and Shing-Tung Yau [1, 2].

Earlier in the workshop, we heard a very nice review of the AdS/CFT correspondence by David Lowe. It is interesting to consider more general forms of this correspondence, where one studies type IIB string theory on the background $\text{AdS}_5 \times L$. Here (L, g_L) is a Sasaki-Einstein 5-manifold:

Definition: A compact Riemannian manifold (L, g_L) is said to be Sasakian iff its metric cone $(X_0 = \mathbb{R}_+ \times L, g_{X_0} = dr^2 + r^2 g_L)$ is Kähler. Moreover, (L, g_L) is said to be Sasaki-Einstein iff this metric cone is Kähler and Ricci-flat.

A simple calculation shows that if (L, g_L) is Sasaki-Einstein then $\text{Ric}_L = (2n - 2)g_L$, where $n = \dim_{\mathbb{C}} X_0$, so that (L, g_L) is a positively curved Einstein manifold. Of course, for the physical application we have in mind, $n = 3$. The subscript zero on X_0 denotes the open cone over L , with $r > 0$.

The AdS/CFT conjecture states that to every such manifold (L, g_L) there is a dual superconformal field theory in four dimensions. This may be thought of as living on the Penrose conformal boundary of anti-de Sitter space, in the usual way. The AdS/CFT correspondence then makes some remarkable connections between field theory and geometry, some of which I have uncovered over the last year or so. In this talk I'd like to focus on two topics:

Topic 1: Any conformal field theory in four dimensions has a central charge $a \in \mathbb{R}$. For superconformal field theories, a -maximisation [3] implies that this central charge is an algebraic number *i.e.* is the root of some polynomial with integer coefficients. The AdS/CFT correspondence then implies that $\text{vol}[L, g_L]/\text{vol}[S^{2n-1}]$ is also an algebraic number¹. From a geometrical point of view this is a rather mysterious statement, and the aim of [4, 1] was to understand this claim. Moreover, as we shall see, one can effectively compute these volumes without solving the Einstein equations, but instead solving a finite dimensional extremal problem, and assuming such an Einstein metric exists.

Topic 2: In the second part of the talk, I shall briefly describe two simple new obstructions to the existence of Sasaki–Einstein metrics. These also have a simple physical interpretation in conformal field theory.

I shall now discuss each topic in a little more detail.

Topic 1

We begin by briefly reviewing a -maximisation in $\mathcal{N} = 1$ superconformal field theories (SCFT), in order set up and motivate the geometrical problem. Any conformal field theory in four dimensions has a central charge $a \in \mathbb{R}$, defined via the one-point function of the trace of the stress–energy tensor in an arbitrary background 4-metric

$$\langle T_{\mu}^{\mu} \rangle = \frac{1}{120 (4\pi)^2} \left(c(\text{Weyl})^2 - \frac{a}{4}(\text{Euler}) \right). \quad (1)$$

Here Weyl and Euler schematically denote the Weyl tensor and Euler density of the background 4-metric, respectively. The coefficient a was conjectured by Cardy to count massless degrees of freedom. Moreover, there is an associated four dimensional a -conjecture, which states that the endpoints of all renormalisation group flows satisfy $a_{IR} < a_{UV}$. This expresses ones intuition about RG flows to the IR as a form of course-graining. For superconformal field theories, the a -central charge may be expressed in terms of the R-symmetry:

$$a = \frac{3}{32} (3\text{Tr}R^3 - \text{Tr}R) \quad (2)$$

where recall that any SCFT has a conserved global symmetry called the R-symmetry. The trace here is sum over fermions in the theory. The divergence of the R-current is related by supersymmetry to the trace of the stress–energy tensor. Thus the R-symmetry satisfies:

- It is conserved (non-anomalous).

¹This is the case only for $n = 3$ of course, although we shall develop the geometry in arbitrary dimension.

- The superpotential W , which specifies interactions in the theory, has R-charge 2.

The last assertion is true by definition of W . In general, these two conditions are insufficient to determine the R-symmetry of a theory at a conjectured IR fixed point: if there are other conserved flavour symmetries, under which W has charge zero, then one is free to add any linear combination of these to any symmetry satisfying the above constraints to obtain another symmetry satisfying the constraints.

The problem of identifying the exact R-symmetry was solved by Intriligator and Wecht [3]. The solution is simple: one considers a space of all symmetries R_{trial} satisfying the above two constraints, and then locally maximises the a -function

$$a(R_{\text{trial}}) = \frac{3}{32} (3\text{Tr}R_{\text{trial}}^3 - \text{Tr}R_{\text{trial}}) . \quad (3)$$

This function may be computed in the UV using 't Hooft anomaly matching, and the exact R-symmetry is the local maximum of this function, assuming there are no accidental symmetries in the IR. Since this a -function is a cubic function with rational coefficients², the R-charges are algebraic numbers. The AdS/CFT correspondence then states that

$$\frac{a}{a(\mathcal{N} = 4 \text{ SYM})} = \frac{\text{vol}[S^5]}{\text{vol}[L, g_L]} . \quad (4)$$

A beautiful example of how this formula holds is given by $Y^{p,q}$. This is an infinite family of explicit Sasaki-Einstein metrics on $L = S^2 \times S^3$, labelled by $p, q \in \mathbb{N}$, $q < p$, $\text{hcf}(p, q) = 1$ [5]. The volumes are given by

$$\frac{\text{vol}[Y^{p,q}]}{\text{vol}[S^5]} = \frac{q^2(2p + \sqrt{4p^2 - 3q^2})}{3p^2(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2})} . \quad (5)$$

The dual field theories, which are quiver gauge theories, were constructed in [6], aided by the toric description worked out in [7], and a -maximisation indeed reproduces the remarkable agreement (4). The aim of [4, 1] was to gain a general geometrical understanding of how one is effectively computing the volumes by a simple extremal problem, and why the resulting number is algebraic.

In order to proceed, we need to know a little more about Sasakian geometry. Recall that (L, g_L) is Sasakian iff its metric cone is Kähler. The Kähler form is simply

$$\omega = \frac{1}{2} i \partial \bar{\partial} r^2 . \quad (6)$$

²This is true, at least, for quiver gauge theories. This is a broad class of $\mathcal{N} = 1$ gauge theories where the matter content and superpotential may be specified combinatorially, by a type of directed graph called a quiver. When the Kähler cone X admits a *crepant resolution*, so that the nowhere zero holomorphic $(n, 0)$ -form Ω extends to the resolved space, then there are good reasons to expect that the dual SCFT is an IR fixed point of such a quiver gauge theory. Explaining this would take us too far afield. It is an important fact that not all Gorenstein isolated singularities (X, Ω) admit a crepant resolution, including examples that are known to admit Ricci-flat Kähler cone metrics.

Thus r^2 also serves as a global Kähler potential on the cone. We define the Reeb vector field as

$$\xi = J \left(r \frac{\partial}{\partial r} \right) \tag{7}$$

which is holomorphic, Killing, and has square norm $\|\xi\|^2 = r^2$. The orbits of ξ therefore foliate L , which we may regard as embedded in the cone at $r = 1$. The metric on L is then

$$g_L = \eta \otimes \eta + g_T \tag{8}$$

where g_T is a transverse Kähler metric, transverse to the orbits of ξ . Here η is the dual 1-form, which may be written

$$\eta = i(\bar{\partial} - \partial) \log r \tag{9}$$

while the transverse Kähler form is given by $2\omega_T = d\eta$. The orbits of ξ may then either all close, or else the generic orbit does not close. In the former case, there is an isometric $U(1)$ action on L . If this action is free, L is the total space of a circle bundle over a Kähler manifold (V, g_V) , where g_V is essentially g_T , now viewed as a metric on the quotient space. More generally, this $U(1)$ action will only be locally free, and (V, g_L) will be a Kähler orbifold. When the orbits don't generically close, there exists at least a \mathbb{T}^2 isometry acting on L : the closure of the orbit gives a compact abelian group, since the isometry group of (L, g_L) is necessarily compact. These three types of Sasakian structure are referred to as regular, quasi-regular and irregular, respectively. We note that when g_L is Sasaki-Einstein, g_T is also Einstein, of curvature $2n$.

The AdS/CFT correspondence relates the R-symmetry to the symmetry generated by ξ . Thus a -maximisation suggests that we look for a geometric extremal problem in which we vary this vector field. This means we must look at a space of Sasakian metrics, or equivalently Kähler cone metrics, on some fixed CR manifold L , or equivalently complex manifold X . The latter point of view is more straightforward, and we may set up the problem algebraically as follows.

Set-up: Fix an affine algebraic variety³ X , with an isolated singular point. We require that X , minus this singular point, is diffeomorphic as a real manifold to $X_0 = \mathbb{R}_+ \times L$, where L is compact. We also assume that X has a holomorphic $(\mathbb{C}^*)^r$ action⁴. Clearly all of this is necessary just to get started. We shall give some concrete constructions of such X momentarily. We also assume that there exists a space of Sasakian metrics \mathcal{S} on X , such that every Reeb vector field ξ lies in the Lie algebra \mathfrak{t}_r of $\mathbb{T}^r \subset (\mathbb{C}^*)^r$, with the real torus acting isometrically on each metric. Here $\mathbb{T} = U(1)$ is the multiplicative group of unit norm complex numbers.

We now give two classes of examples:

Ex 1: Isolated quasi-homogeneous hypersurface singularities

³Roughly, the zero locus of a set of polynomials in \mathbb{C}^m for some m .

⁴Recall that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is the multiplicative group of complex numbers.

We consider a polynomial $F(z_1, \dots, z_{n+1})$ on \mathbb{C}^{n+1} such that

$$F(q^{w_1} z_1, \dots, q^{w_{n+1}} z_{n+1}) = q^d F(z_1, \dots, z_{n+1}) \quad (10)$$

with $q \in \mathbb{C}^*$ and $\vec{w} \in \mathbb{N}^{n+1}$. Then X is constructed as the zero set of F :

$$X = \{F = 0\} \subset \mathbb{C}^{n+1}. \quad (11)$$

Note that the weighted \mathbb{C}^* action on \mathbb{C}^{n+1} then descends to X , so that X has at least a \mathbb{C}^* action. A simple set of examples satisfying our requirements is given by the so-called Brieskorn–Pham singularities:

$$F = \sum_{i=1}^{n+1} z_i^{a_i} \quad (12)$$

where $w_i = d/a_i$, $\vec{a} \in \mathbb{N}^{n+1}$. One easily checks that $z_1 = \dots = z_{n+1} = 0$ is the only singular point of X , and it is known that the link L is $(n-2)$ -connected, meaning that the homotopy groups $\pi_a(L)$ are trivial for all $a = 1, \dots, n-2$. In fact, the homology groups of L are all known. A familiar example is to take $n = 3$, and $a_1 = a_2 = a_3 = a_4 = 2$. Then X is the conifold singularity for physicists, or the ordinary double point singularity for mathematicians.

Ex 2: Affine toric varieties

When $r = n$ is maximal, $\text{sp } (\mathbb{C}^*)^n$ acts on X with an open dense orbit, then X is called an affine toric variety. Any such X may be constructed as a gauged linear sigma model, or Kähler quotient of \mathbb{C}^d :

$$\begin{aligned} X &= \mathbb{C}^d // \mathbb{T}^{d-n} \times \Gamma \\ &= \left\{ \sum_{i=1}^d Q_I^i |z_i|^2 = 0 \right\} / \mathbb{T}^{d-n} \times \Gamma. \end{aligned} \quad (13)$$

Here $Q_I^i \in \mathbb{Z}$ specifies an embedding of tori $\mathbb{T}^{d-n} \subset \mathbb{T}^d$, $I = 1, \dots, d-n$, and Γ is a finite abelian group⁵. The conifold is again an example, with $d = 4$ and $Q = (1, 1, -1, -1)$.

Notice that in both cases X inherits a natural Kähler cone metric from the flat metric on \mathbb{C}^m . In neither case is this metric ever Ricci-flat⁶. When X is toric, one can describe the space of all Sasakian metrics \mathcal{S} on the link L rather explicitly [4].

Before proceeding to our variational problem, we must impose one further restriction: since we wish to find a Ricci-flat Kähler metric on X_0 , we require X to be Gorenstein. This means simply

⁵This is trivial when X_0 is simply-connected.

⁶Unless X is toric and $d = n$, obviously.

that there is a nowhere zero holomorphic $(n, 0)$ -form Ω on X_0 . A Ricci-flat Kähler metric then satisfies

$$\Omega \wedge \bar{\Omega} \sim \omega^n \tag{14}$$

up to an irrelevant dimension-dependent factor. Thus we see that, for a Sasaki-Einstein metric, Ω is homogeneous degree n , which may be written

$$\mathcal{L}_\xi \Omega = in\Omega . \tag{15}$$

Morally, this is equivalent to fixing the R-charge of the superpotential to be 2.

Ex 1: Define

$$\Omega = \frac{dz_1 \wedge \dots \wedge dz_n}{\partial F / \partial z_{n+1}} \tag{16}$$

on the open set where the denominator is non-zero. One easily checks that similar expressions on other such open sets glue together to give a globally defined nowhere zero holomorphic $(n, 0)$ -form on X_0 . The charge of Ω under the generator ζ of $U(1) \subset \mathbb{C}^*$ is

$$\mathcal{L}_\zeta \Omega = i(|\vec{w}| - d)\Omega . \tag{17}$$

Thus we require also that

$$|\vec{w}| - d > 0 \tag{18}$$

where $|\vec{w}| = \sum_{i=1}^{n+1} w_i$ denotes the sum of the weights. Indeed, given any Sasakian metric with $\mathcal{L}_\xi \Omega = i\gamma\Omega$, $\gamma > 0$, one can always rescale⁷ the transverse metric and Reeb vector field in such a way as to set $\gamma = n$. Another way of understanding (18) is that this is precisely the condition that the orbit space V of the \mathbb{C}^* action is a Fano orbifold.

Ex 2: Here we must impose an additional constraint:

$$\sum_{i=1}^d Q_I^i = 0, \quad \forall I = 1, \dots, d - n . \tag{19}$$

Note this says that the holomorphic $(d, 0)$ -form $dz_1 \wedge \dots \wedge dz_d$ on \mathbb{C}^d has charge zero under the torus \mathbb{T}^{d-n} .

A central fact, with which every physicist is familiar, is that Einstein metrics are critical points of the Einstein-Hilbert action. This is a functional

$$I : \text{Met}(L) \rightarrow \mathbb{R} \tag{20}$$

⁷Boyer and Galicki call this a \mathcal{D} -homothety [8].

on metrics on L , given by

$$I[g_L] = \int_L [s(g_L) + 2(n-1)(3-2n)] d\mu . \quad (21)$$

Here $s(g_L)$ is the scalar curvature of g_L , and $d\mu$ is the usual Riemannian measure. Critical points of I are Einstein metrics of Ricci curvature $2n-2$.

The key idea is to restrict I to our space of Sasakian metrics \mathcal{S} on L . We thus now define \mathcal{S}_0 to be the space of Sasakian metrics in \mathcal{S} satisfying (15). Then a central result is that [1]

$$I[g_L] = 4(n-1)\text{vol}[L, g_L] \quad (22)$$

for $g_L \in \mathcal{S}_0$. Thus the Einstein–Hilbert action is essentially just the volume. This is a key step: Einstein metrics are critical points of I , and we have succeeded in relating this to the volume of (L, g_L) , which is the object of interest. The next key fact is that:

vol $[L, g_L]$ **depends only on the Reeb vector field ξ .**

Thus it is independent of g_T , the transverse Kähler metric. The reason for this is essentially that the volume of a Kähler manifold, given by $\int_V \omega_V^m/m!$, depends only on the Kähler class $[\omega_V]$. Thus we may regard $\text{vol}[L, g_L]$ as $\text{vol}[L, \xi]$, which is a function on (a conical subset of) the Lie algebra \mathfrak{t}_r . This is clearly just what we wanted: the Reeb vector field ξ for a Sasaki–Einstein metric is a critical point of this function, provided we satisfy the constraint (15). Moreover, it’s not difficult to prove that this function is strictly convex, which allows one to prove statements about uniqueness of critical points.

Aside: As an aside comment, let us consider the derivative $d\text{vol}[L, \xi](Y)$ along a holomorphic Killing vector field $Y \in \mathfrak{t}_r$. From our comments thus far, $d\text{vol}[L, \xi]$ is a linear map on a space of holomorphic vector fields that depends only on ξ and vanishes when (L, g_L, ξ) is a Sasaki–Einstein manifold. Those readers familiar with Kähler–Einstein geometry will immediately recognise these as properties of the *Futaki invariant*. In fact, one can show that, when ξ is quasi-regular and $\mathcal{L}_Y\Omega = 0$, we have [1]

$$d\text{vol}[L, \xi](Y) = -n \int_L \eta(Y) = -\frac{\ell}{2} F [J_V(Y_V)] . \quad (23)$$

Here ℓ is the length of the generic ξ orbit, J_V is the complex structure on the Kähler orbifold (V, g_V) , Y_V is the push-down of Y to V , and F is the Futaki invariant. A nice example of how this works is when $V = \mathbb{F}_1$, the first del Pezzo surface. This is the blow-up of $\mathbb{C}P^2$ at a point. The Futaki invariant of \mathbb{F}_1 is non-zero, so that this Kähler manifold cannot admit a Kähler–Einstein metric. However, one can construct the complex cone X from the canonical line bundle over V . This *does* have a Ricci-flat Kähler cone metric, which is irregular. In fact, this metric is the explicit

metric $Y^{2,1}$ [7]. The key point here is that the holomorphic vector field that rotates the fibre of the canonical line over $V = \mathbb{F}_1$ is simply not a critical point of I – the correct critical point is a linear combination of this and a holomorphic vector field on \mathbb{F}_1 . For further details, the reader is referred to [1, 7].

The remaining question is now: how do we compute this function $\text{vol}[L, \xi]$? There are various approaches. The approach I took in the talk was via the Duistermaat–Heckman formula. There’s an initial trick, which is to rewrite the volume as

$$\text{vol}[L, g_L] = \frac{1}{2^{n-1}(n-1)!} \int_X e^{-r^2/2} \frac{\omega^n}{n!} . \quad (24)$$

This is simple to prove, by writing the volume form on X in polar coordinates. The strange prefactor arises from doing the integral over r . The function $H = r^2/2$ is the *Hamiltonian function* for ξ . This means that

$$dH = -\xi \lrcorner \omega , \quad (25)$$

as is easily checked. The integral in (24) now looks like a classical canonical partition function, where H is the Hamiltonian and (X, ω) is the phase space. Indeed, it *is* the canonical partition function – of a BPS D3–brane wrapping a three–sphere in AdS₅ [9]. The Duistermaat–Heckman formula, from symplectic geometry, allows one to evaluate such an integral by computing things at the *fixed point set* of ξ . Now, recall that $\|\xi\|^2 = r^2$, so that ξ vanishes only at the tip of the cone, which is singular. Thus, in order to get a useful expression from the theorem, we must (partially) *resolve* the singularity. This is immediately quite interesting, since the dual CFT may also be thought of as a form of resolution of the singularity. Mathematically, the set–up is as follows.

Let

$$\pi : W \rightarrow X \quad (26)$$

be a partial resolution of X , which is equivariant with respect to the holomorphic $(\mathbb{C}^*)^r$ action. Thus the inverse image of the singular point is some exceptional set $E \subset W$, and $W \setminus E \cong X_0$ is a biholomorphism. We allow W to have orbifold singularities in general, which is what we mean by a partial resolution. Any such resolution always exists, since we may simply pick any quasi–regular ξ and take W to be the total space of the complex line bundle over the corresponding orbifold V . Clearly this will be equivariant, and $E = V$ is an exceptional divisor. However, any other resolution

will do. Our main result⁸ is then the following formula:

$$\frac{\text{vol}[L, \xi]}{\text{vol}[S^{2n-1}]} = \sum_{\{F\}} \frac{1}{d_F} \int_F \prod_{i=1}^R \frac{1}{(\xi, u_i)^{n_i}} \left[\sum_{a \geq 0} \frac{c_a(\mathcal{E}_i)}{(\xi, u_i)^a} \right]^{-1}. \quad (27)$$

We must now explain all the symbols:

- $\{F\}$ is the set of connected components of the fixed point set of ξ , which is a generic vector in the Lie algebra of \mathbb{T}^r acting on W . Note this fixed point set is supported in E , since ξ is nowhere vanishing on X_0 .
- For fixed connected component F , its normal bundle in W is \mathcal{E} , and this splits as a direct Whitney sum

$$\mathcal{E} = \bigoplus_{i=1}^R \mathcal{E}_i \quad (28)$$

where $\text{rank } \mathcal{E}_i = n_i$ and $\sum_{i=1}^R n_i = k$ is the rank of \mathcal{E} .

- The splitting arises from the induced action of \mathbb{T}^r on \mathcal{E} : since F is fixed, \mathbb{T}^r acts linearly on its normal bundle. Any such action is specified by a set of *weights*, $u_1, \dots, u_R \in \mathbb{Z}^r \subset \mathfrak{t}_r^*$. A vector field $\zeta \in \mathfrak{t}_r$, then acts on the fibres of \mathcal{E} as the matrix

$$\begin{bmatrix} e^{i(\zeta, u_1)} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & e^{i(\zeta, u_R)} \end{bmatrix} \quad (29)$$

where each block is $n_i \times n_i$, in the decomposition (28), with $\exp i(\zeta, u_i) \in \mathbb{T}$ acting by multiplication.

- $c_a(\mathcal{E}_i)$ are the Chern classes of \mathcal{E}_i . These may be defined in the usual way using curvature forms. The notation in (27) is standard in index theory: one should formally expand the sum over Chern classes, which is $1 + \dots$, and take the formal inverse using Taylor's theorem. The integral over F then picks out the term of degree $\dim_{\mathbb{R}} F$. The reader should convince himself/herself that (27) is indeed homogeneous degree $-n$ in ξ .

⁸Strictly speaking, in order to apply the Duistermaat–Heckman theorem, we assumed in [1] that W admits a 1-parameter family of \mathbb{T}^r -invariant Kähler metrics, that approach the cone metric as the parameter $t \rightarrow 0$. One then applies the theorem to this family and takes the limit. In the limit the dependence on the choice of family of resolving metrics drops out, leaving (27). However, one can also get to that formula via a different route, which replaces the volume by a holomorphic invariant at an earlier stage and uses the (equivariant) Riemann–Roch theorem [1]. Then one does not need to make this assumption. Hence the following theorem holds as stated.

- Finally, when W has orbifold singularities, the normal fibre to a *generic* point on F is not a complex vector space, but rather an orbifold \mathbb{C}^k/Γ . Thus \mathcal{E} is more generally an orbibundle – the Chern classes should then be defined in terms of the curvature of an Hermitian connection on \mathcal{E} . Then $d_F = |\Gamma|$ is the order of Γ .

The upshot of all this is that the final formula (27) for $\text{vol}[L, \xi]/\text{vol}[S^{2n-1}]$ is manifestly a rational function of ξ with *rational* coefficients, namely certain weights and Chern classes. This allows one to prove that the critical point of I is indeed an algebraic number. We have thus achieved our goal of understanding the geometry underlying a -maximisation.

Example: Since the general formula (27) is a little unwieldy, it is useful to give a simpler set of examples. We here describe toric varieties. Any affine toric variety may be specified by a strictly convex rational polyhedral cone $\mathcal{C}^* \subset \mathbb{R}^n \cong \mathfrak{t}_n^*$. This is the image of the moment map $\mu : X \rightarrow \mathfrak{t}_n^*$ arising via the \mathbb{T}^n action on (X, ω) . For further details, the reader may consult [4]. A (partial) resolution of X may be obtained by resolving this polytope, which amounts to cutting it with some number of rational hyperplanes, in such a way that every vertex of the resulting non-compact polytope has n edges coming out of it. Since everything is rational, these may be taken to be primitive vectors in \mathbb{Z}^n , and we require them to span \mathbb{Q}^n over \mathbb{Q} . Readers familiar with the topological string should be familiar with this, if not the way I’ve described it: any crepant resolution⁹ of X , when $n = 3$, may be drawn as a “(p,q)-web”. This is a doubly T-dual picture in which the geometry is replaced by a web of 5-branes in type IIB string theory. These 5-branes arise via fixed point sets of the torus action. The vertices of this web arise in the topological vertex in the topological string. Let P be the resolving polytope described above, corresponding to the image $\mu(W)$ of the moment map of \mathbb{T}^n on W , and let A be a label for its vertices. Then (27) reduces to

$$\frac{\text{vol}[L, \xi]}{\text{vol}[S^{2n-1}]} = \sum_{A \in P} \prod_{i=1}^n \frac{1}{(\xi, u_i^A)}. \quad (30)$$

Here the weights u_i^A are precisely the n outward-pointing edge vectors at the A th vertex of the polytope. When W is smooth, these are all primitive. When vertex A is an orbifold point, the weights are generally not primitive vectors, but are rather in \mathbb{Q}^n . For some detailed examples, including an example which is only partially resolved, the reader is directed to [1].

I should finally mention that another method for arriving at (27) is via the (equivariant) Riemann–Roch theorem. This is related to what we called the holomorphic partition function in [2]. Interestingly, this is precisely the *quantum* canonical partition function for the BPS D3-branes we mentioned earlier [9]. Thus the proof we summarised above may be viewed as a “classical” proof of (27), whereas the route for the character in [1], alluded to in the footnote above, may be viewed as a “quantum” proof, and taking a classical limit. For the details, together with some explicit examples, the reader is again referred to [1].

⁹A resolution over which Ω extends smoothly, so that W has zero first Chern class.

Topic 2

In the final part of the talk, I shall briefly describe some new obstructions to the existence of Sasaki–Einstein metrics on links of Gorenstein singularities (X, Ω) . This is based on the recent paper [2].

Yau’s famous theorem states that, for a *compact* (X, Ω) with a Kähler metric ω_0 , there exists a unique Ricci–flat Kähler metric ω on X in the same Kähler class. Thus $\omega = \omega_0 + i\partial\bar{\partial}\phi$, where ϕ is a smooth real function on X .

When (X, Ω) is non–compact, this theorem fails. Of course, once we allow X to be non–compact, we must specify some boundary condition. In the present set–up, this is replaced by requiring (X, ω) to be a cone. There are then two very simple holomorphic obstructions to the existence of a Ricci–flat Kähler cone metric on (X, Ω) , based on classical theorems in Riemannian geometry.

Let (L, g_L) be an Einstein manifold, of real dimension¹⁰ $2n - 1$, with Ricci curvature $2n - 2$. Let Δ_L denote the scalar Laplacian on L . Then

Bishop’s Theorem: $\text{vol}[L, g_L] \leq \text{vol}[S^{2n-1}]$

Lichnerowicz’s Theorem: The smallest positive eigenvalue E_1 of Δ_L is bounded from below by $E_1 \geq 2n - 1$, with equality iff (L, g_L) is the round sphere.

These theorems are both very straightforward to prove, and may be found in almost any textbook on Riemannian geometry. As we’ve already seen, $\text{vol}[L, g_L] = \text{vol}[L, \xi]$ is a holomorphic invariant, and may be computed without knowing the explicit metric on L . Thus if we compute the volume of a putative Sasaki–Einstein metric and find that it is bigger than that of the round sphere, we get a contradiction. We call this the **Bishop obstruction**.

The **Lichnerowicz obstruction** works similarly. Let f be a holomorphic function on X , with charge $\lambda > 0$ under a holomorphic vector field ξ , which is a putative Reeb vector field for a Sasaki–Einstein metric on L . Thus

$$\mathcal{L}_\xi f = i\lambda f . \tag{31}$$

This immediately implies that

$$f = r^\lambda \tilde{f} \tag{32}$$

where \tilde{f} is a function on the link $L = \{r = 1\} \subset X$. Since f is holomorphic, an easy calculation shows that

$$\Delta_L \tilde{f} = E \tilde{f} \tag{33}$$

¹⁰The following theorems hold in any dimension, odd or even.

where

$$E = \lambda(\lambda + (2n - 2)) . \tag{34}$$

Thus if we can find a holomorphic function on X with charge $\lambda < 1$ we contradict Lichnerowicz's Theorem, and we conclude that no Ricci-flat Kähler cone metric can exist on X with ξ as its Reeb vector field.

As examples of this, let us quote the results for isolated quasi-homogeneous hypersurface singularities. Recall these have a natural \mathbb{C}^* action – we take our Reeb vector field to lie in the Lie algebra of the $U(1)$ part, appropriately normalised so that Ω has charge n (see (15)). Then two necessary conditions for existence of a Ricci-flat Kähler cone metric on X with this Reeb vector field are:

$$d(|\vec{w}| - d)^n \leq wn^n \tag{35}$$

and

$$|\vec{w}| - d \leq w_{\min}n . \tag{36}$$

Here $|\vec{w}| = \sum_{i=1}^{n+1} w_i$ denotes the sum of the weights, $w = \prod_{i=1}^{n+1} w_i$ denotes their product, and w_{\min} denotes the smallest weight.

There are clearly lots of examples that fail to satisfy these bounds, including examples studied by string theorists, who mistakenly assumed that the metrics exist. For example, the singularities

$$F = z_1^2 + z_2^2 + z_3^2 + z_4^k = 0 \tag{37}$$

do not admit Ricci-flat Kähler cone metrics for any $k > 20$ by Bishop, and for any $k > 3$ by Lichnerowicz. The case $k = 3$ is left open, and at present cannot be ruled out by any known methods. We shall get on to the physics of these obstructions in a moment. Before that, we note that the Lichnerowicz bound is strikingly similar to a known *sufficient* condition for existence of a Sasaki-Einstein metric, as reviewed in [8]. This is

$$|\vec{w}| - d \leq \frac{w_{\min}n}{n - 1} . \tag{38}$$

Obviously this sufficient condition is implied the necessary one.

Finally, let us turn to the physics. The Lichnerowicz obstruction is easier. As we reviewed, any holomorphic function f on X gives rise to an eigenfunction of the Laplacian on L of eigenvalue $\lambda(\lambda + 4)$. In Kaluza-Klein reduction, such eigenfunctions give rise to massive scalar fields on AdS_5 . The AdS/CFT correspondence relates their mass to the conformal dimension of dual operators by $m^2 = \Delta(\Delta - 4)$. In particular, holomorphic eigenfunctions give rise to chiral primary operators in

the dual SCFT, and a careful analysis [2] reveals that $\lambda = \Delta$ relates the charge of the holomorphic function to the conformal dimension of the dual operator. It is then well-known that in any conformal field theory, the dimensions of scalar operators are bounded from below by the unitarity bound

$$\Delta \geq 1 . \tag{39}$$

Thus the classical Lichnerowicz theorem is just the unitarity bound in field theory. Note that chiral primary operators saturate the BPS bound $\Delta \geq 3R/2$ by definition, analogously to holomorphic functions on X .

The Bishop obstruction may be understood roughly as follows¹¹. By Higgsing the SCFT, and integrating out any massive fields, one expects to be able to flow to the $\mathcal{N} = 4$ super Yang–Mills theory. In string theory this should correspond to moving the D3–branes to a smooth point of X . In this process, one expects the central charge to decrease, in line with our earlier discussion of the a –conjecture. Thus one expects that $a(\mathcal{N} = 4 \text{ SYM}) < a$, which is precisely Bishop’s bound.

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