

Heterotic Fluxes and Non-Kähler Geometries

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This talk is based on work with K. Becker, M. Becker, J.X. Fu, and S.T. Yau [1].

Introduction

In connecting ten (or eleven) dimensional string theory to our low-energy four dimensional observable world, a standard geometrical approach is to wrap the extra dimensions around an internal compact manifold X_6 . The four dimensional physical observables are then determined by the characteristics of X_6 . For many years, X_6 was commonly taken to be a Calabi-Yau (CY) manifold, mainly because a CY preserves minimal four dimensional supersymmetry without turning on any additional background fields. However, a CY manifold typically has many scalar moduli which result in four dimensions unwanted massless scalars. Thus, in the past few years, there have been much work in incorporating fluxes, i.e. non-zero p -form background fields, into the compactification. Adding fluxes while preserving supersymmetry typically requires the geometry to be non-Kähler and hence non-CY.

Below, I shall describe flux compactification in heterotic theory and present a smooth geometrical model that preserves N=1 supersymmetry in four dimensions. Before proceeding, let me mention some of the advantages of studying flux compactification in heterotic theory as compared to type II theories:

1. The only flux present is the three-form H . Hence, the supersymmetry conditions incorporating the fluxes are "simpler" and the possible types of geometry are still rather constrained and more tractable.
2. The internal geometry X_6 can be smooth since no sources or branes need to be present. In type II theories, supergravity no-go theorems for compactification to four dimensional

Minkowski spacetime stipulate that fluxes can only be non-zero if the compact geometry has singularities (which can arise from the presence of branes). The heterotic theory bypass the no-go theorem with the anomaly cancellation condition which modifies the Bianchi identity for the three-form H .

3. Gauge fields are naturally present. This allows for the possibility of constructing models with interesting phenomenology and possibly reproducing the standard model.

Review of $N = 1$ SUSY Constraints

The background fields of the heterotic theory are $\{g_{mn}, H_{mnp}, \phi, F_{mn}\}$. In the string frame, preserving supersymmetry requires the ten dimensional geometry to be the product space $M = M^{3,1} \times X_6$.

Preserving $N = 1$ supersymmetry is the requirement that there exists a no-where vanishing spinor η that satisfies the following equations

$$\begin{aligned} \delta\psi_m &= \nabla_m \eta + \frac{1}{8} H_{mnp} \gamma^{np} \eta = 0 , \\ \delta\lambda &= \gamma^m \partial_m \phi \eta + \frac{1}{12} H_{mnp} \gamma^{mnp} \eta = 0 , \\ \delta\chi &= \gamma^{mn} F_{mn} \eta = 0 . \end{aligned}$$

With a non-vanishing spinor η , we can write down a complex structure $J_m{}^n = -i\eta^\dagger \gamma_m{}^n \eta$ such that $J^2 = -1$ and the Nijenhuis tensor, $N_{mn}{}^p = 0$. Furthermore, the hermitian $(1,1)$ -form and the holomorphic $(3,0)$ -form which together define the geometry of X_6 can also be simply expressed as fermion bilinears as follows.

$$\begin{aligned} J_{mn}^{(1,1)} &= -i\eta^\dagger \gamma_{mn} \eta , \\ \Omega_{mnp}^{(3,0)} &= e^{-2\phi} \bar{\eta}^\dagger \gamma_{mnp} \eta . \end{aligned}$$

We now list the supersymmetry constraints on the background fields.

A. Geometry

The constraints on the six dimensional geometry can be expressed as differential equations on J and Ω . We compare the case of CY versus non-Kähler case in the following.

<u>CY</u>	<u>Non-Kähler</u>
$dJ = 0$ (Kähler condition)	$d(\ \Omega\ * J) = 0$ (conformally balanced)
$d\Omega^{(3,0)} = \bar{\partial}\Omega^{(3,0)} = 0$	$d\Omega^{(3,0)} = 0$
$H = 0$	$H = i(\bar{\partial} - \partial)J \neq 0$
$\phi = \phi_0 = \text{constant}$	$e^{-2(\phi - \phi_0)} = \ \Omega\ $

Note that since $H^{(1,2)} = i\bar{\partial}J$, if $H \neq 0$, then the metric $J_{a\bar{b}} = ig_{a\bar{b}}$ must be non-Kähler. However, not any non-Kähler geometry is allowed, the metric with non-zero H-flux turned on must be conformally balanced. The balanced condition is defined to be

$$d(*J) = \frac{1}{2}d(J \wedge J) = 0 .$$

Notice that Kähler metrics are also balanced since $dJ = 0$ implies $d(J \wedge J) = 0$. Indeed, the balanced condition can be thought as a relaxation of the Kähler condition. Consider the number of constraint equations. For Kähler, $dJ = 0$ gives 9 complex constraint equations. For balanced, $*d(*J) = 0$ gives only 3 complex constraint equations. Heuristically, the additional "degrees of freedom" in a conformally balanced geometry can be thought of as being parametrized by (H, ϕ) . Or conversely, one can just forget about the H -flux and ϕ and simply consider the compactification geometry as being defined by J and Ω which are required to satisfy the above constraints.

B. Gauge Field

The background gauge field satisfies the Hermitian-Yang-Mills condition:

$$F_{(2,0)} = F_{(0,2)} = 0 , \quad F_{mn}J^{mn} = 0' .$$

The first part is the condition that the gauge bundle is holomorphic. The second is a primitivity condition. Together, the Hermitian-Yang-Mills condition is known from the work J. Li and Yau [2] to correspond to the gauge bundle being "stable."

C. Anomaly Cancellation

In the heterotic theory, this is the modified Bianchi identity

$$dH = 2i\partial\bar{\partial}J = \frac{\alpha'}{4}(\text{tr}R \wedge R - \text{tr}F \wedge F) .$$

This condition further relates the gauge bundle with the geometry.

A Smooth Compact Model

Now we present a compactification model that satisfy all of the above conditions. The smooth geometry has been called the FSY geometry (after Fu, Strominger, and Yau) [3]. The manifold is a T^2 bundle over a CY two-fold. The metric ansatz is

$$ds^2 = e^{2\phi} ds_{CY_2}^2 + |dz + \alpha|^2 .$$

We shall let ϕ and α depend only on the base CY_2 coordinates. Further, $\theta = dz + \alpha$ will be taken to be a $(1,0)$ -form and we define $\omega = d\theta = \omega_S^{(2,0)} + \omega_A^{(1,1)}$.

Now in order for the metric to be globally defined, ω is "quantized," that is it must be an element of the integer class $H^2(CY_2, \mathbf{Z})$. If one treats α as the gauge field of a Kaluza-Klein T^2 reduction, then ω is the field strength and the quantization of ω is just the Dirac quantization.

The conformally balanced condition can be easily satisfied as long as we require

$$\omega \wedge J_{CY_2} = 0 ,$$

i.e. ω is primitive with respect to the CY_2 base. Interestingly, the conformally balanced condition then holds for *any* function ϕ .

The $(3,0)$ -form can be simply taken to be $\Omega^{(3,0)} = \Omega_{CY_2}^{(2,0)} \wedge \theta$. It is holomorphic since

$$d\Omega^{(3,0)} = \Omega_{CY_2}^{(2,0)} \wedge d\theta = \Omega_{CY_2}^{(2,0)} \wedge (\omega_S^{(2,0)} + \omega_A^{(1,1)}) = 0 ,$$

recalling that $\omega^{S,A}$ are two-forms on the CY_2 base.

Now we must impose the anomaly cancellation. But before doing so, let us take note of the parameters currently present in the model.

1. The CY_2 base can be either T^4 or $K3$.
2. There are an infinite number of solutions since ω is only required to be in the integer class $H^2(CY_2, \mathbf{Z})$ but any non-zero integer is allowed.
3. We have an infinite number of scalar moduli due to ϕ . This is because ϕ can be any smooth function, and therefore, we can expand ϕ in a basis of functions i.e. $\phi = \sum_{n=0}^{\infty} a_n \phi_n$. But even though the space is compact, the basis would be infinite dimensional.

The anomaly cancellation equation

$$2i\partial\bar{\partial}J = \frac{\alpha'}{4}(\text{tr}R \wedge R - \text{tr}F \wedge F) \tag{1}$$

will constrain each of the above freedom.

- 1'. Integrating $J \wedge [eq.(1)]$ over X_6 , we obtain the condition when $\omega \neq 0$

$$\int_{X_6} \text{tr}R_{CY_2} \wedge R_{CY_2} \wedge J > 0 .$$

This implies that $\text{tr}R_{CY_2} \wedge R_{CY_2} \neq 0$, or that $CY_2 = K3$.

- 2'. Since all fields depends on the base coordinates, we can integrate the four-form anomaly equation over $K3$ and obtain a condition (dropping factors of 2π and α')

$$-\frac{p_1(F)}{2} + \int_{K3} (\|\omega_S\|^2 + \|\omega_A\|^2) = 24 .$$

This is a topological condition that relates the twisting of the T^2 bundle to that of the gauge bundle (as given by $p_1(F)$). Furthermore, since both terms on the left hand side is positive semi-definite, the number of possible solutions becomes finite.

- 3'. The anomaly equation is also a differential equation. Here specifically, it becomes a second-order highly non-linear differential equation for ϕ

$$D_2(\phi) = \psi ,$$

where ψ can be considered as a source term. Thus an important question is whether this difficult differential equation allows for a solution for ϕ at all. As was proved by Fu and Yau [4], there exists a smooth solution for ϕ if the topological condition mentioned in (2') is satisfied. The existence proof assumes an elliptic condition which effectively guarantees that the moduli of space of solution for ϕ is finite. (It is not known whether a solution can exist that doesn't satisfy the elliptic condition.)

We see that the anomaly condition severely constrains the parameters of the model.

Finally, we describe the Hermitian-Yang-Mills gauge bundle, or equivalently, the stable gauge bundle on X_6 . The primitivity condition $F_{mn}J^{mn} = 0$ is equivalent to $F \wedge J \wedge J = 0$ in six dimensions and $F \wedge J = 0$ in four dimensions. It is not hard to show that any stable bundle on $K3$ can be lifted to a stable bundle on X_6 . The stable bundle on $K3$ has been studied by Mukai [5] and for $SU(r)$ group, they satisfy $c_2(F) = -p_1(F)/2 \geq r - \frac{1}{r}$. Therefore, one can easily turned on stable $SU(r)$ gauge bundle for $r = 4, 5$ that breaks one of the E_8 gauge group down to the phenomenologically interesting gauge groups $SO(10)$ and $SU(5)$, respectively.

References

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