Heterotic Fluxes and Non-Kähler Geometries II

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Introduction

We start by giving the topological data needed for formulating Strominger's equations describing heterotic string compactifications on non–Kähler manifolds [1]. Let M^3 be a complex, compact manifold such that

- (1) $\exists \Omega^{3,0}$ closed holomorphic (3,0) form,
- (2) $\exists \widetilde{J} > 0$, a (1,1) form such that $d(\widetilde{J}^2) = 0$,
- (3) and with a stable vector bundle V, such that $c_1(V) = 0, c_2(V) \equiv c_2(M) \mod \partial \overline{\partial}(\ldots)$.

Recall that a vector bundle $V \to M$ is stable with respect to \widetilde{J} if for all proper subsheaves $\mathcal{F} \subset V$ the following inequality holds:

$$\frac{\int c_1(\mathcal{F}) \wedge \widetilde{J}^2}{\operatorname{rank}(\mathcal{F})} < \frac{\int c_1(V) \wedge \widetilde{J}^2}{\operatorname{rank}(V)}.$$

Notice that we only need a closed \widetilde{J}^2 as opposed to \widetilde{J} in algebraic geometry.

The $\partial \bar{\partial}$ cohomology $\frac{\ker \partial \bar{\partial}}{\operatorname{im} \partial + \operatorname{im} \bar{\partial}}$ is always finite-dimensional for complex manifolds and is in many cases more suitable for non-Kähler manifolds. It is isomorphic to the standard cohomology for

Kähler manifolds. In Kähler geometry, we would require that $c_2(V)$ is equal to $c_2(M)$ as a cohomology class. The $\partial \bar{\partial}$ cohomology is more refined hence we impose condition (3) in non-Kähler geometry.

In this setup Strominger's equations read

- (1') There is a curvature F on V that satisfies the Hermitian Yang–Mills equations and $F^{2,0} = F^{0,2} = 0$.
- (2') $d(\|\Omega\|_J J^2) = 0.$
- (3') $\sqrt{-i}\partial\bar{\partial} J = \frac{\alpha'}{4} (\operatorname{tr} R \wedge R \operatorname{tr} F \wedge F).$

From the point of view of differential equations, if M is a Calabi–Yau manifold then (2') is the analogue of the Kähler condition and (3') is the analogue of the Ricci–flatness condition. We want to know when these equations are satisfied. Conditions (1), (2), (3) are necessary for (1'), (2'), (3').

Conjecture: Conditions (1), (2), (3) are also sufficient for (1'), (2'), (3').

Example: Consider a locally trivial fiber bundle with fiber T^2 over a complex manifold with vanishing first Chern class. The same construction gives a Calabi–Yau manifold fibered over a del Pezzo surface with elliptic fibers. In this case, however, the fibers are not fixed, i.e. the fibration is not locally trivial, but the fibers are twisted. There are many examples known of such Calabi– Yau manifolds. Probably, one can do a similar construction with del Pezzo surfaces to obtain new non–Kähler manifolds, but with a different twisting. This is a very special example because one can never fiber a Calabi–Yau manifold non–trivially over a K3 surface otherwise the Chern class increases.

One more important class of non-Kähler manifolds is the following. Start with a Calabi-Yau manifold, say the quintic, and consider a \mathbb{P}^1 inside it with normal bundle $N_{\mathbb{P}^1/M} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, i.e. the small resolution of a conifold singularity. We can blow down these curves in many ways to obtain manifolds with conifold singularities, smooth them out and obtain non-Kähler complex manifolds. These manifolds will not have second cohomology and therefore be diffeomorphic to connected sums $S^3 \times S^3 \# \dots \# S^3 \times S^3$. This can be viewed as a topology-changing transition from one Calabi-Yau manifold to another "Calabi-Yau" manifold according to a conjecture of Miles Reid. This is beautiful since it is just like building a Riemann surface as a connected sum. Note that these connected sums are not Kähler since there is no second cohomology. There is an upper bound on the number of summands but it is not easy to get down to a low number of summands. For $S^3 \times S^3$ the complex structure would be very different from the standard one. The holomorphy in condition (1) is not automatic.

Now, we go back to the equations and the example of the torus fibration M over a K3 surface.

We have a holomorphic 1-form θ from the torus and the Kähler form J_{K3} from the K3 surface. $J = e^u J_{K3} + \frac{\sqrt{-1}}{2} \theta \wedge \overline{\theta}$ is a hermitian form on the total space M that satisfies (2) for an arbitrary function $u : K3 \to \mathbb{R}$. Therefore we want to solve (3') using this freedom in choosing u:

$$\sqrt{-1}\,\partial\bar{\partial}\,e^{u}\wedge J_{\mathrm{K3}} - \frac{\alpha'}{2}\,\partial\bar{\partial}\,(e^{u}\rho) - \alpha'\,\partial\bar{\partial}\,u \wedge \partial\bar{\partial}\,u + \mu J_{\mathrm{K3}}^{2} = 0$$

We will set $\alpha' = 1$. ρ is a (1,1) form given in terms of the data of the background, i.e. the curvature of two line bundles L_1, L_2 on K3 (cf. Tseng's talk). μ is a function that can be computed from the background data, too. It turns out that there is a constant ambiguity $A = \int_{K3} e^{-4u} J_{K3}^2$ for u that allows a scaling. We require A to be small. This amounts to scaling the base large. In fact, we have to assume A to be small enough in order to be able to solve the equation.

We can linearize this equation by putting artificially a constant t

$$\sqrt{-1}\,\partial\bar{\partial}\,e^{u}\wedge J_{\mathrm{K}3} - \frac{t}{2}\,\partial\bar{\partial}\,(e^{u}\rho) - \partial\bar{\partial}\,u\wedge\partial\bar{\partial}\,u + t\mu J_{\mathrm{K}3}^{2} = 0$$

For t = 0 the equation is trivially solved by u = const. which corresponds to the scale of the K3, i.e to the choice of the ambiguity A. Now, we vary t from 0 to 1 and want to prove that a solution always exists. In other words, we want to prove that if there is a solution to $L_{t_0}u = \text{at } t = t_0$ then there exist a solution to $L_{t_0+\epsilon}u = 0$ at $t = t_0 + \epsilon$ by perturbation theory. In order to to this the first order equation $\delta L|_{t=t_0}(\delta u) = 0$ has to be solved. We can only solve this equation [2] if we impose that the differential operator δL be elliptic for all t. We do not know why we would want this to be true from the point of view of physics but it is natural from the point of view of geometry. Furthermore we have to prove that the solution does not blow up when varying t, i.e. if $L_{t_i}u_i = 0$ and $t_i \to \bar{t}$ then $L_{\bar{t}}\bar{u} = 0$ and u_i converges smoothly to \bar{u} . This is a hard problem, and it is in the required estimates that the large base comes in. The linearized operator δL gives rise to the metric $a_{i\bar{j}}$ on the solution space (cf. Tseng's talk). The estimate on its determinant is the most difficult part. This and other estimates will not be given here (see [2]).

In the general case the fibration will have singular fibers where the torus degenerates. For this situation we will need a different argument. The moduli space of these non-Kähler manifolds (in particular their special geometry) should be easy to understand once the equation is solved in the most general case. But one can map this background via heterotic-F-theory duality to a Calabi-Yau fourfold whose moduli space is better understood and presumably learn from this something about the structure of the moduli space on the heterotic side [3].

References

[1] A. Strominger, "Superstrings with Torsion," Nucl. Phys. B 274 (1986) 253.

- [2] J. X. Fu and S. T. Yau, "The theory of superstring with flux on non-Kaehler manifolds and the complex Monge-Ampere equation," arXiv:hep-th/0604063.
- [3] K. Becker, M. Becker, J. X. Fu, L. S. Tseng and S. T. Yau, "Anomaly cancellation and smooth non-Kaehler solutions in heterotic string theory," Nucl. Phys. B 751 (2006) 108 [arXiv:hepth/0604137].