Topological strings and non-supersymmetric black holes

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Based on: K.S.+ C.Vafa, hep-th/0703214

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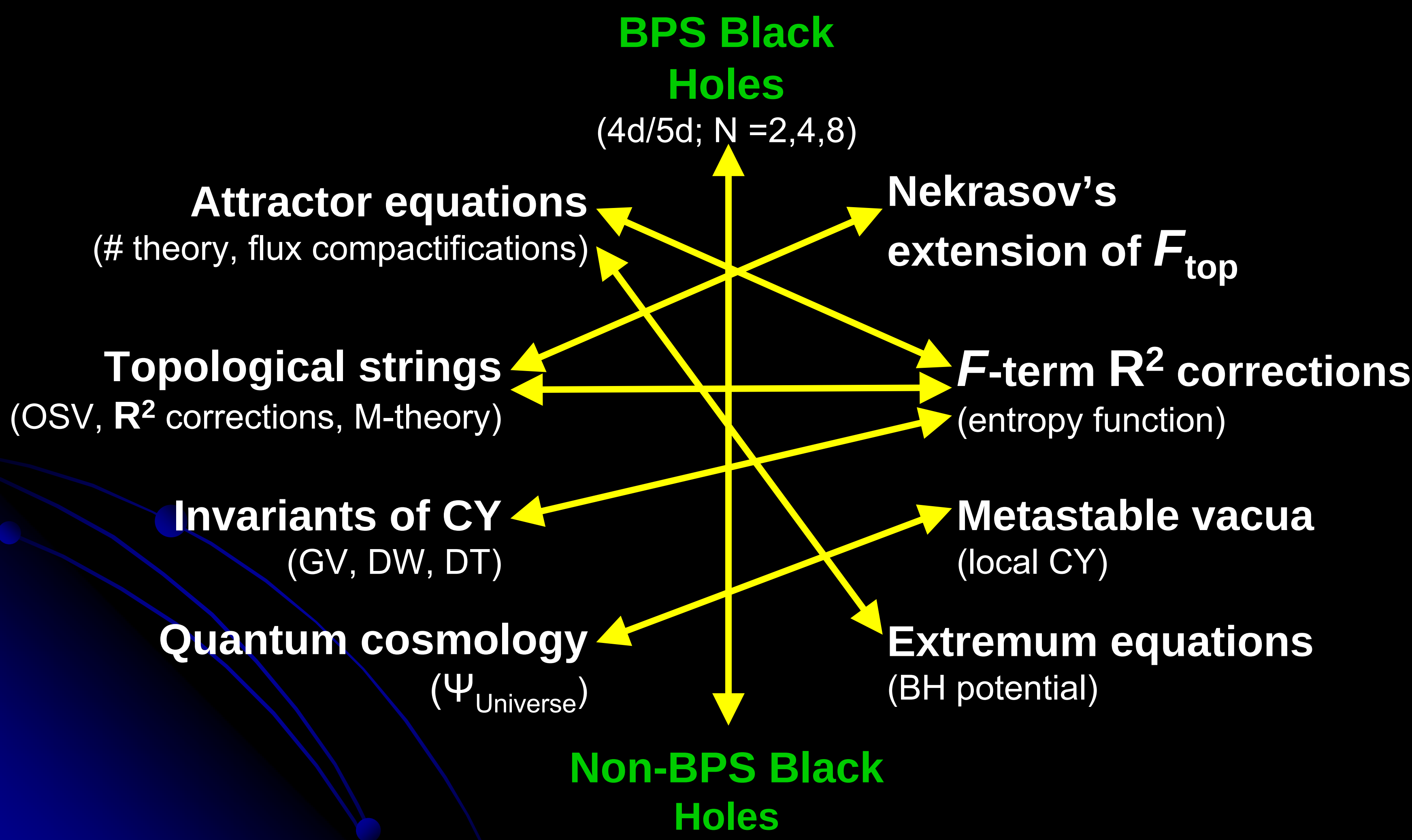
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String theory at work

Black hole thermodynamics: macroscopic entropy

$$S_{\mathrm{BH}} = \frac{\mathrm{Area}}{4G_{\mathrm{N}}}$$

[Bekenstein-Hawking'71-74]

D-brane microstates:

$$S_{\mathrm{BH}} = \ln(\mathrm{Degeneracy})$$

[Strominger-Vafa'96]

Good news: String theory is successful!

However: Only leading terms in Q>>1 limit, only BPS...

So, can we do better?

OSV conjecture

Type IIB/CY -> $\mathcal{N} = 2$ SUGRA with F-term R^2 corrections

macroscopic entropy can be computed via Wald's formula…

[Cardoso-de Wit-Mohaupt'98]

…and then can be reinterpreted using topological string

$$\Omega(p,q) = \int d\phi e^{\phi^I q_I} \, | Z_{\text{top}}(p,\phi) |^2$$

[Ooguri-Strominger-Vafa'04]

- Topological string partition function: $Z_{\text{top}}(p,\phi) = e^{F_{\text{top}}(p,\phi)}$
- Index of BPS states/ BH degeneracy: $\Omega(p,q)$
- CY moduli and g_{top} are fixed in terms of BH charges

OSV conjecture

$$\Omega(p,q) = \int d\phi e^{\phi^I q_I} \, |Z_{\text{top}}(p,\phi)|^2$$

asymptotic expansion,
all orders in 1/Q

- Tests:
  - Local CY: 2d qYM [Vafa'04, Aganagic-Saulina-Ooguri-Vafa'04,…]
  - Small black holes [Dabholkar-Denef-Moore-Pioline'05]
  - $\mathcal{N} = 4,8$ black holes [Shih-Yin'05]
- Derivation:

  [Gaiotto-Strominger-Yin'06]
  [Boer-Cheng-Dijkgraaf-Manschot-Verlinde'06]
  [Beasley-Gaiotto-Guica-Huang-Strominger-Yin'06]
  [Denef-Moore'07]
- Ambiguities: background dependence, integration contour and measure… [Verlinde'04]

OSV formula: problem with "non-susy" charges

$$\Omega(p,q) \stackrel{?}{=} \int d\phi\, e^{\phi^I q_I}\, |Z_{\mathrm{top}}(p,\phi)|^2$$

Something is missing even semiclassically:

- RHS is defined for any charge vector (p^I, q_I)
- However, not any (p^I, q_I) will support a BPS state

BPS degeneracy $\Omega_{\mathrm{susy}}(p^I, q_I)$
is not well-behaved on $H^3(M,\mathbb{R})$!

Discriminant hypersurface

Typically, in the space of charges $H^3(M,\mathbb{R})$

there is a special codimension one hypersurface

$$\mathcal{D}(p,q)=0$$

The BPS degeneracy behavior is

$$\Omega_{\rm susy}(p,q) \approx \exp \pi \sqrt{-\mathcal{D}(p,q)}$$

<u>Example:</u> Het/K3xT^2 or Het/T^6

$$\Omega_{\rm susy}(p,q) \approx \exp \pi \sqrt{P^2Q^2-(P\cdot Q)^2}$$ need $P^2Q^2 > (P\cdot Q)^2$

Discriminant hypersurface

When $\Omega_{\text{susy}}(p,q)$ goes bad,
non-BPS extremal black holes appear!

The non-BPS degeneracy behavior is

$$\Omega_{\text{n-susy}}(p,q) \approx \exp \pi \sqrt{+\mathrm{D}\,(p,q)}$$

Example: Het/K3xT^2

$$\Omega_{\text{n-susy}}(p,q) \approx \exp \pi \sqrt{-P^2 Q^2 + (P\cdot Q)^2}$$

Discriminant hypersurface

$$\begin{cases} \mathrm{D}(p,q) \le 0 & \text{Susy} \\ \mathrm{D}(p,q) > 0 & \text{Non-Susy} \end{cases}$$

Extremal black holes degeneracy

$$\Omega(p,q) = \Omega_{\text{susy}}(p,q) + \Omega_{\text{n-susy}}(p,q) \approx \exp \pi \sqrt{|\mathrm{D}(p,q)|}$$

This works semiclassically. What about OSV?

$$\Omega(p,q) = \int_{CY} d\phi \; e^{\phi^I q_I} \ldots ?$$

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Black Holes from IIB/CY

Consider Type IIB on $M \times \mathbb{R}^{3,1}$

This gives $\mathcal{N} = 2$ 4d SUGRA

graviphoton

- gravitational multiplet: $(e^a{}_\mu, A^0_\mu, \ldots)$
- $n_V = h^{2,1}(M)$ vector multiplets: $(X^I, A^I_\mu, \ldots)$
- $n_H = h^{1,1}(M) + 1$ hyper multiplets

Two types of geometries are involved:

- moduli space (special) geometry
- space-time geometry

Calabi-Yau geometry

Basic objects:

- Holomorphic 3-form $\Omega \in H^{(3,0)}(M)$
- Holomorphic prepotential $F = F(X)$ homogeneous, degree 2

Choose symplectic basis (A^I, B_I) of 3-cycles on M

$$X^I = \int_{A^I} \Omega, \quad I = 0, \ldots, n_V$$

homogeneous coordinates on Calabi-Yau moduli space M

$$F_I(X) = \int_{B_I} \Omega = \frac{\partial F(X)}{\partial X^I}$$

dual periods

inhomogeneous coordinates: $t^i = \dfrac{X^i}{X^0}, \quad i = 1, \ldots, n_V$

Black hole geometry

Example: Reissner-Nordstrom solution

$$ds^2 = -(r - r_+)(r - r_-)\frac{dt^2}{r^2} + \frac{r^2 dr^2}{(r - r_+)(r - r_-)} + r^2 d\Omega_2^2$$

Extremal BH: $r_+ = r_-$ (zero temperature)

Near horizon geometry: $AdS^2 \times S_2$

$$ds^2 = -r_+^2(-r^2 dt^2 + \frac{dr^2}{r^2}) + r_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$F_{rt} = \frac{q}{4\pi}, \quad F_{\theta\varphi} = \frac{p}{4\pi}\sin\theta, \quad r_+^2 = G_N \frac{p^2 + q^2}{4\pi}$$

Black holes from branes

Wrap 3-Brane on $\gamma = q_I A^I - p^I B_I \in H_3(M,\mathbb{Z})$

This leads to a black hole in 4d

Susy attractor equations:

$$\begin{cases} p^I = \mathrm{Re}\, C X^I \\ q_I = \mathrm{Re}\, C F_I \end{cases}$$

[Ferrara-Kallosh-Strominger'95]
[Strominger'96]
[Ferrara-Kallosh'96]

CY moduli are fixed near horizon
in terms of BH charges

Until further notice

For a while we will discuss only leading contribution to BH entropy, e.g.

$$S_{\mathrm{BH}}^{\mathrm{susy}} = -\frac{i\pi}{4}\int_M \Omega \wedge \bar{\Omega}$$

No higher-derivative terms in L

Classical CY geometry: tree-level prepotential

$$F = F^{(0)}$$

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Black hole potential

Attractor behavior is a consequence of extremality ⇓ expected both for BPS and non-BPS black holes

[Ferrara-Gibbons-Kallosh'97]

Here is how it works:

- Introduce $V_{\mathrm{BH}} = V_{\mathrm{BH}}(p, q, t, \overline{t}\,)$ CY moduli
- Look for the minima of V_{BH}: $\dfrac{\partial V_{\mathrm{BH}}}{\partial t^i} = 0$

This fixes CY moduli $t^i = t^i_*(p, q)$

and BH entropy $S_{\mathrm{BH}} = \pi V_{\mathrm{BH}}(p, q, t_*, \overline{t_*})$

Black hole potential

$$V_{\mathrm{BH}} = -\frac{1}{2}(q_I - \mathcal{N}_{IJ} p^J)\frac{1}{\mathrm{Im}\mathcal{N}_{IJ}}(q_J - \overline{\mathcal{N}}_{JL} p^L)$$

- $\mathcal{N}_{IJ}$ is coupling matrix of abelian gauge fields A^I_μ

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\frac{X^L \mathrm{Im}(F_{IL}) X^M \mathrm{Im}(F_{MJ})}{X^I \mathrm{Im}(F_{IJ}) X^J}, \quad F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J}$$

- $\mathrm{Im}\,\mathcal{N}_{IJ}$ is negative definite as opposed to $\mathrm{Im}F_{IJ}$

signature $(1, n_V)$

some identities:

$$F_I = \mathcal{N}_{IJ} X^J, \quad \mathcal{N}_{IJ} = \mathcal{N}_{JI}, \quad \mathcal{N}_{IJ} = \mathcal{N}_{IJ}(t, \bar{t})$$

Modified black hole potential

$$\tilde{V}_{\mathrm{BH}} = q_I \mathrm{Im} P^I + \frac{i}{4}(P^I - X^I)(P^J - X^J)\bar{F}_{IJ} + \\ + \frac{i}{2}(P^I - X^I)F_I + \frac{i}{2}F + c.c.$$

We introduced Lagrange multiplier $P^I = p^I + i\phi^I$

$$p^I = \mathrm{Re}(P^I), \quad \phi^I = \mathrm{Im}(P^I)$$

and homogeneous variables X^I instead of t^i

$\tilde{V}_{\mathrm{BH}}$ is equivalent to V_{BH} if we integrate out ϕ^I
and the scale λ: $X^I \to \lambda X^I$

What is it good for?

$$\tilde{V}_{\mathrm{BH}} = q_I \mathrm{Im} P^I + \frac{i}{4}(P^I - X^I)(P^J - X^J)\bar{F}_{IJ} + \\ + \frac{i}{2}(P^I - X^I)F_I + \frac{i}{2}F + c.c.$$

Alternative form of the attractor equations!

$$C_{IJK}(\bar{P}^J - \bar{X}^J)(\bar{P}^K - \bar{X}^K) = 4i\mathrm{Im}(F_{IJ})(P^J - X^J)$$

$\partial_I F_{JK}$

$$\begin{cases} p^I = \mathrm{Re}(P^I) \\ q_I = \mathrm{Re}(N_{IJ}P^J) \end{cases}$$

“New” attractor equations

The idea:

- First find $P^I = P^I(X)$ from

$$C_{IJK}(\bar{P}^J - \bar{X}^J)(\bar{P}^K - \bar{X}^K) = 4i\mathrm{Im}(F_{IJ})(P^J - X^J)$$

- Then find $X^I = X^I(p,q)$ from

$$\begin{cases} p^I = \mathrm{Re}(P^I) \\ q_I = \mathrm{Re}(\mathrm{N}_{IJ} P^J) \end{cases}$$

Susy black hole example

Note that $P^I(X) = X^I$ is a solution of

$$C_{IJK}(\bar{P}^J - \bar{X}^J)(\bar{P}^K - \bar{X}^K) = 4i\mathrm{Im}(F_{IJ})(P^J - X^J)$$

Since $\mathrm{N}_{IJ} X^J = F_I$ we get a well-known equations

$$\begin{cases} p^I = \mathrm{Re}(P^I) \\ q_I = \mathrm{Re}(F_I) \end{cases}$$

[Ferrara-Kallosh-Strominger'95]
[Strominger'96]
[Ferrara-Kallosh'96]

in the "gauge" $C = 1$

What else is it good for?

So far we just rewrote attractor equations and saw that BPS solutions appear naturally

There is one more
interesting thing
that we can do now…

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Reminder

Direct problem: for a given BH charge vector (p^I, q_I), find CY moduli X^I that solve the attractor equations

$$\partial \tilde{V}_{\mathrm{BH}} = 0$$

Hard because the equations are highly nonlinear: the prepotential $F(X)$ is known only indirectly, via Picard-Fuchs system solution

What is inverse problem?

Inverse problem: for a given pt X^I on a CY moduli space, find all charge vectors (p^I, q_I), such that attractor equations are satisfied

Much easier!

Why?

Solving inverse problem

BH potential and attractor equations are quadratic in charges!

$$\tilde{V}_{\mathrm{BH}} = q_I \operatorname{Im} P^I + \frac{i}{4}(P^I - X^I)(P^J - X^J)\bar{F}_{IJ} + \\ + \frac{i}{2}(P^I - X^I)F_I + \frac{i}{2}F + c.c.$$

$$C_{IJK}(\bar{P}^J - \bar{X}^J)(\bar{P}^K - \bar{X}^K) = 4i\operatorname{Im}(F_{IJ})(P^J - X^J)$$

$$\begin{cases} p^I = \operatorname{Re}(P^I) \\ q_I = \operatorname{Re}(\mathcal{N}_{IJ} P^J) \end{cases}$$

How many solutions?

$$C_{IJK}(\bar{P}^J - \bar{X}^J)(\bar{P}^K - \bar{X}^K) = 4i\mathrm{Im}(F_{IJ})(P^J - X^J)$$

n_V+1 quadratic equations for n_V+1 unknowns

- At most 2^{n_V+1} solutions

1	SUSY
$\leq 2^{n_V+1}$-1	Non-SUSY

One-modulus CY is solvable

Consider n_V=1 (e.g. quintic)

$$F=(X^0)^2 f(t), \quad t=\frac{X^1}{X^0}$$

We want to find $P^I(X)$ from the attractor equations

$$\begin{cases} C_{0JK}(\bar{P}^J-\bar{X}^J)(\bar{P}^K-\bar{X}^K)=4i\mathrm{Im}(F_{0J})(P^J-X^J) \\ C_{1JK}(\bar{P}^J-\bar{X}^J)(\bar{P}^K-\bar{X}^K)=4i\mathrm{Im}(F_{1J})(P^J-X^J) \end{cases}$$

One-modulus CY is solvable

$$\begin{cases} C_{0JK}(\bar{P}^J - \bar{X}^J)(\bar{P}^K - \bar{X}^K) = 4i\mathrm{Im}(F_{0J})(P^J - X^J) \\ C_{1JK}(\bar{P}^J - \bar{X}^J)(\bar{P}^K - \bar{X}^K) = 4i\mathrm{Im}(F_{1J})(P^J - X^J) \end{cases}$$

The homogeneity of the prepotential

$$F_0 = 2X^0 f - X^1 f' \qquad f' \equiv \partial_t f$$

implies $C_{0IJ} = -tC_{0IJ}$

This gives a linear relation between the equations solution is straightforward

One-modulus CY is solvable

We find:

$$(P^0 - X^0)^4 = \Theta(P^0 - X^0)$$

where

$$\Theta = -4i \frac{(X^0)^4 \det \left\| \operatorname{Im} F_{IJ} \right\|}{f'''(X^L X^M \operatorname{Im} F_{LM})^2} X^K \operatorname{Im} F_{K1}$$

Inverse map for one-modulus CY

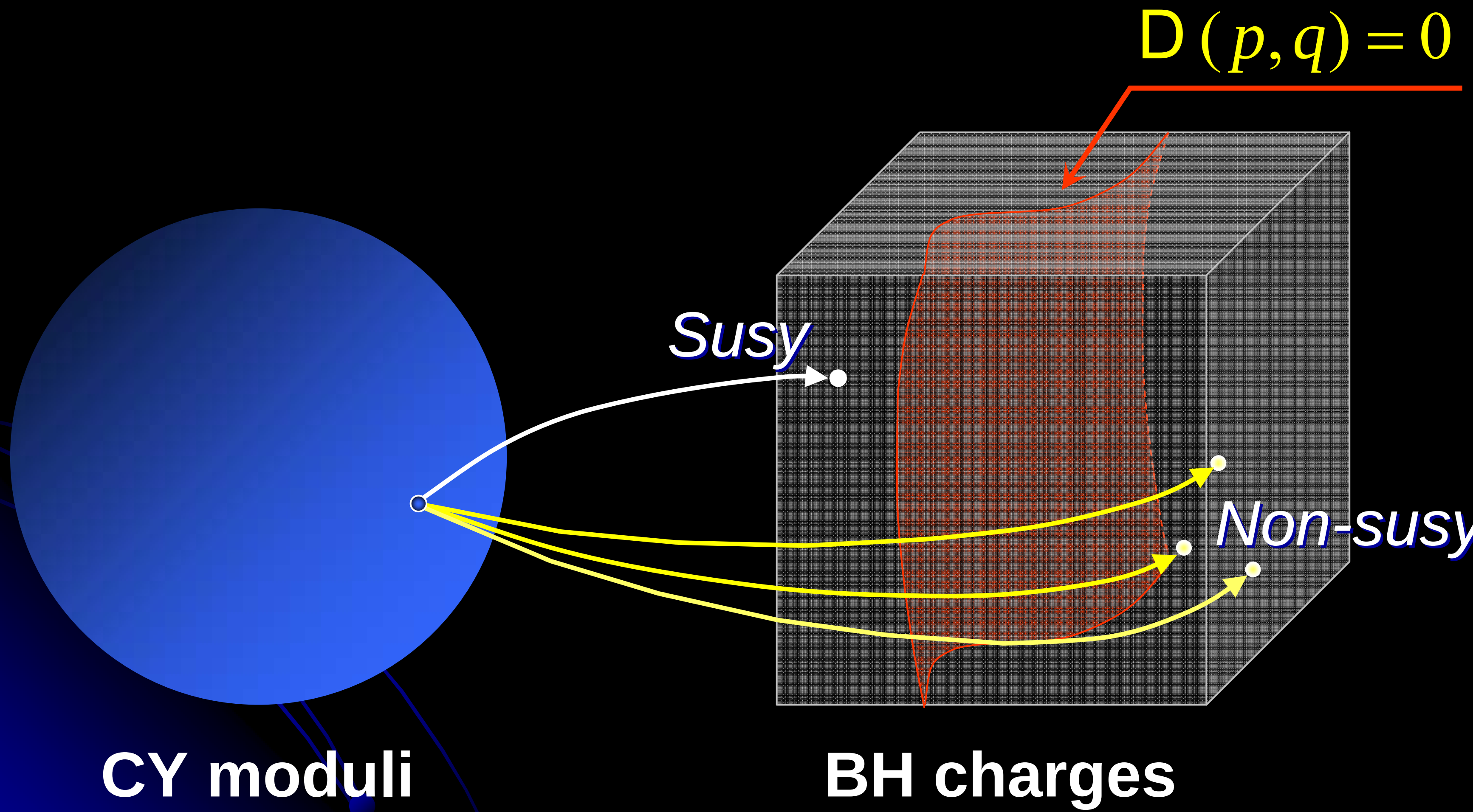
- 1 SUSY branch

$$\begin{cases} P^0 = X^0 \\ P^1 = X^1 \end{cases}$$

- 3 non-SUSY branches

$$\begin{cases} P^0 = X^0 + (\Theta\bar{\Theta}^2)^{1/3} e^{2\pi i k/3}, \ \ k = 1,2,3 \\ P^1 = X^1 - \dfrac{X^I \operatorname{Im} F_{I0}}{X^J \operatorname{Im} F_{J1}} (\Theta\bar{\Theta}^2)^{1/3} e^{2\pi i k/3} \end{cases}$$

Inverse map



Recap

Rewrite BH potential as

$$\pi\tilde{V}_{\mathrm{BH}} = q_I \,\mathrm{Im} P^I - \pi \,\mathrm{Im}\, \mathsf{G}$$

$$\mathsf{G} = \frac{1}{2}(P^I - X^I)(P^J - X^J)\bar{F}_{IJ} + (P^I - X^I)F_I + F$$

Then semiclassical entropy in OSV ensemble is

$$\Omega(p,q) = \int d\phi e^{\phi^I q_I} \sum_{\text{solutions}} \left| \exp\frac{i\pi}{2}\mathsf{G}_* \right|^2$$

where G_* is obtained by substituting attractor problem solution $X_*^I = X^I(P)$ into G

For example, in susy case $\mathsf{G}_* = F$

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Higher derivative corrections

The macroscopic black hole entropy in the presence of higher derivative terms is computed by the Wald's formula.

$$S_{\mathrm{BH}} = -2\pi \int_{\mathrm{H}} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\lambda\rho}} \varepsilon_{\mu\nu} \varepsilon_{\lambda\rho}$$

For spherically symmetric extremal black holes with $AdS^2 \times S_2$ near horizon geometry it simplifies drastically. [Sen'05]

The entropy function

Consider the most general $SO(2,1) \times SO(3)$ ansatz for a field configuration consistent with the $AdS^2 \times S_2$ near horizon geometry of a black hole.

The entropy function is defined as

$$\mathcal{E} = 2\pi(-e^I q_I - \int_{\mathrm{H}} d\theta d\phi \sqrt{-\det g}\,)\mathcal{L}$$

[Sen'05]

Macroscopic computation

The entropy is computed in two steps:

- Extremize E with respect to free parameters

$$\partial \mathsf{E} = 0$$

- The entropy is equal to the value of E at the extremum

$$S_{\mathrm{BH}} = \mathsf{E}\,\big|_{\partial \mathsf{E}=0}$$

This should be compared to the black hole potential story:

$$\frac{\partial V_{\mathrm{BH}}}{\partial t^i} = 0, \quad S_{\mathrm{BH}} = \pi V_{\mathrm{BH}}(p,q,t_*,\overline{t_*})$$

Computing the entropy function

$SO(2,1)\times SO(3)$ ansatz with $AdS^2 \times S_2$ near horizon geometry:

[Sen'05]
[Sahoo-Sen'06]

$$ds^2 = v_1^2\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$x^I = X^I, \quad F_{rt}^I = e^I, \quad F_{\theta\varphi}^I = p^I \sin\theta, \quad T_{rt}^- = v_1 w$$

Generalized prepotential

$$F = F(X^I, \hat{A}) = \sum_{g=0}^{\infty} F_g(X^I)\hat{A}^g$$

g-loop topological string amplitude

Computing the entropy function

$$\frac{1}{\pi}\mathrm{E} = q_I e^I - v_1 v_2 [-i(v_1^{-1} - v_2^{-1})\bar{x}^I F_I +$$

$$+\frac{i}{4}(e^I v_1^{-1} + ip^I v_2^{-1} - \tfrac{1}{2} x^I \bar{w})(e^J v_1^{-1} + ip^J v_2^{-1} - \tfrac{1}{2} x^J \bar{w})\bar{F}_{IJ} +$$

$$+\frac{i\bar{w}}{4}(e^I v_1^{-1} + ip^I v_2^{-1} - \tfrac{1}{2} x^I \bar{w})F_I + \frac{i|w|^2}{4} F +$$

$$+ i\Big(|w|^4 - 8|w|^2 (v_1^{-1} + v_2^{-1}) + 64(v_1^{-1} - v_2^{-1})^2\Big) F_{\hat{A}} - c.c.]$$

$$F = F(X^I, \hat{A})$$

Analyzing the entropy function

$$\frac{1}{\pi}\mathrm{E} = q_I e^I - v_1 v_2\Big[-i(v_1^{-1} - v_2^{-1})\bar{x}^I F_I +$$

$$+\frac{i}{4}(e^I v_1^{-1} + ip^I v_2^{-1} - \tfrac{1}{2}x^I\bar{w})(e^J v_1^{-1} + ip^J v_2^{-1} - \tfrac{1}{2}x^J\bar{w})\bar{F}_{IJ} +$$

$$+\frac{i\bar{w}}{4}(e^I v_1^{-1} + ip^I v_2^{-1} - \tfrac{1}{2}x^I\bar{w})F_I + \frac{i|w|^2}{4}F +$$

$$+i\Big(|w|^4 - 8|w|^2(v_1^{-1} + v_2^{-1}) + 64(v_1^{-1} - v_2^{-1})^2\Big)F_{\hat{A}} - c.c.\Big]$$

Compare to

$$\tilde{V}_{\mathrm{BH}} = q_I \operatorname{Im} P^I + \frac{i}{4}(P^I - X^I)(P^J - X^J)\bar{F}_{IJ}^{(0)} + \\ + \frac{i}{2}(P^I - X^I)F_I^{(0)} + \frac{i}{2}F^{(0)} + c.c.$$

Analyzing the entropy function

$$\frac{1}{\pi}\mathrm{E} = q_I e^I - v_1 v_2 \Big[-i(v_1^{-1} - v_2^{-1})\bar{x}^I F_I +$$

$$+\frac{i}{4}(e^I v_1^{-1} + ip^I v_2^{-1} - \tfrac{1}{2}x^I \bar{w})(e^J v_1^{-1} + ip^J v_2^{-1} - \tfrac{1}{2}x^J \bar{w})\bar{F}_{IJ} +$$

$$+\frac{i\bar{w}}{4}(e^I v_1^{-1} + ip^I v_2^{-1} - \tfrac{1}{2}x^I \bar{w})F_I + \frac{i|w|^2}{4}F +$$

$$+ i\Big(|w|^4 - 8|w|^2(v_1^{-1} + v_2^{-1}) + 64(v_1^{-1} - v_2^{-1})^2\Big)F_{\hat{A}} - c.c.\Big]$$

- 2 extra parameters v_1 , v_2 (w can be gauged away)
- Appear only at one loop
- Geometry is not anti-selfdual: $\mathrm{T}^{+}_{\mu\nu} \neq 0$

Lessons from the SUGRA computation

Supergravity variables: x^I, v_1, v_2

Topological string variables: X^I

We need to incorporate two extra parameters into the topological string!

How?

Nekrasov's extension of the topological string

Никита Некрасов

There is a refinement of topological string with two equivariant parameters!

$$F = F(X^I, \varepsilon_1, \varepsilon_2)$$

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Review of Nekrasov's extension

Reduction to the ordinary topological string

$$F(X^I, \varepsilon_1, -\varepsilon_1) = \sum_g g_{\text{top}}^{2g-2} F^{(g)}(X^I), \quad g_{\text{top}} = \varepsilon_1$$

Homogeneity

$$\left[X^I \frac{\partial}{\partial X^I} + \varepsilon_1 \frac{\partial}{\partial \varepsilon_1} + \varepsilon_2 \frac{\partial}{\partial \varepsilon_2} \right] F(X^I, \varepsilon_1, \varepsilon_2) = 0$$

Original formulation involved not anti-selfdual supergravity background

$$\mathrm{T} = \varepsilon_1 dx^1 \wedge dx^2 + \varepsilon_2 dx^3 \wedge dx^4$$

Review of Nekrasov's extension

Expansion

$$F(X^I,\varepsilon_1,\varepsilon_2)=-\frac{1}{\varepsilon_1\varepsilon_2}F^{(0)}+\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}H_1+\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}G_1+$$

$$+F^{(1)}+\mathrm{O}(\varepsilon_1,\varepsilon_2)$$

[Nekrasov-Nakajima-Yoshioka'03]

$$F(X^I,\varepsilon_1,-\varepsilon_1)=\frac{1}{\varepsilon_1^2}F^{(0)}+F^{(1)}+\ldots$$

Refinement of the BH potential

Start with $$\tilde{V}_{\mathrm{BH}} = q_I \mathrm{Im} P^I + \frac{i}{4}(P^I - X^I)(P^J - X^J)\bar{F}_{IJ} + \\ + \frac{i}{2}(P^I - X^I)F_I + \frac{i}{2}F + c.c.$$

Deform $$F(X) \to F(X, \varepsilon_1, \varepsilon_2)$$

When $$\varepsilon_1 = -\varepsilon_2 = g_{\mathrm{top}}, \quad F(X) \to F(X, g_{\mathrm{top}})$$

Deform $$P^I = p^I + i\phi^I \to P^I_\varepsilon = -\varepsilon_2 p^I + i\varepsilon_1 \phi^I$$

When $\varepsilon_1 \neq -\varepsilon_2$, this changes complex structure on $H^3(M, \mathbb{Z})$

Refinement of the BH potential

Require that after the deformation

$$\tilde{V}_{\mathrm{BH}} \rightarrow \tilde{V}_{\mathrm{BH}}^{\varepsilon}$$

we can still reproduce the supersymmetric attractor solutions and the entropy value

This fixes the form of $\tilde{V}_{\mathrm{BH}}^{\varepsilon}$ up to the terms of order $(\varepsilon_1+\varepsilon_2)^2$

Deformed BH potential

$$\tilde{V}_{\mathrm{BH}}^{\varepsilon} = q_I \phi^I + \frac{i}{2}[\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)\bar{X}^I F_I +$$

$$+\tfrac{1}{2}(P_\varepsilon^I - X^I)(P_\varepsilon^J - X^J)\bar{F}_{IJ} +$$

$$+(P_\varepsilon^I - X^I)F_I + F -$$

$$-\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 \frac{\partial}{\partial \varepsilon_1} - \varepsilon_2 \frac{\partial}{\partial \varepsilon_2})F - c.c.]$$

where $F = F(X, \varepsilon_1, \varepsilon_2)$

Deformed BH potential

It is easy to check that

$$\frac{\partial}{\partial X^I}\tilde{V}^{\varepsilon}_{\mathrm{BH}} = \frac{\partial}{\partial \varepsilon_1}\tilde{V}^{\varepsilon}_{\mathrm{BH}} = \frac{\partial}{\partial \varepsilon_2}\tilde{V}^{\varepsilon}_{\mathrm{BH}} = 0$$

can be solved by

$$P^I_{\varepsilon} = X^I, \quad \varepsilon_1 = -\varepsilon_2 = 1$$

and moreover,

$$\pi \tilde{V}^{\varepsilon}_{\mathrm{BH}}\Big|_{\partial \tilde{V}^{\varepsilon}_{\mathrm{BH}}=0} = -\operatorname{Im} F_{\mathrm{top}} = S^{\mathrm{susy}}_{\mathrm{BH}}$$

Comparison

$$\tilde{V}^{\varepsilon}_{\rm BH} = q_I \phi^I + \frac{i}{2}[\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)\bar{X}^I F_I +$$
$$+\tfrac{1}{2}(P^I_\varepsilon - X^I)(P^J_\varepsilon - X^J)\bar{F}_{IJ} +$$
$$+(P^I_\varepsilon - X^I)F_I + F -$$
$$-\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 \frac{\partial}{\partial \varepsilon_1} - \varepsilon_2 \frac{\partial}{\partial \varepsilon_2})F - c.c.]$$

$$\frac{1}{\pi}\mathrm{E} = q_I e^I - v_1 v_2 \Big[-i(v_1^{-1} - v_2^{-1})\bar{x}^I F_I +$$
$$+\frac{i}{4}(e^I v_1^{-1} + i p^I v_2^{-1} - \tfrac{1}{2}x^I \bar{w})(e^J v_1^{-1} + i p^J v_2^{-1} - \tfrac{1}{2}x^J \bar{w})\bar{F}_{IJ} +$$
$$+\frac{i\bar{w}}{4}(e^I v_1^{-1} + i p^I v_2^{-1} - \tfrac{1}{2}x^I \bar{w})F_I + \frac{i|w|^2}{4}F +$$
$$+i\Big(|w|^4 - 8|w|^2(v_1^{-1} + v_2^{-1}) + 64(v_1^{-1} - v_2^{-1})^2\Big)F_{\hat{A}} - c.c.\Big]$$

Deformed G-function

Define

$$\mathsf{G} = \tfrac{1}{2}(P_\varepsilon^I - X^I)(P_\varepsilon^J - X^J)\bar{F}_{IJ} + (P_\varepsilon^I - X^I)F_I + F +$$

$$+\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)\bar{X}^I F_I - \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 \frac{\partial}{\partial \varepsilon_1} - \varepsilon_2 \frac{\partial}{\partial \varepsilon_2})F$$

so that

$$\pi \tilde{V}_{\mathrm{BH}}^{\varepsilon} = \phi^I q_I - \pi \operatorname{Im} \mathsf{G}$$

Extended OSV formula

$$\Omega(p,q) = \int d\phi e^{\phi^I q_I} \sum_{\text{solutions}} \left| \exp\frac{i\pi}{2} \mathsf{G}_* \right|^2$$

where $\mathsf{G} = \frac{1}{2}(P_\varepsilon^I - X^I)(P_\varepsilon^J - X^J)\bar{F}_{IJ} + (P_\varepsilon^I - X^I)F_I + F +$
$+ \frac{1}{2}(\varepsilon_1 + \varepsilon_2)\bar{X}^I F_I - \frac{1}{2}(\varepsilon_1 + \varepsilon_2)(\varepsilon_1\partial_{\varepsilon_1} - \varepsilon_2\partial_{\varepsilon_2})F$

and $\mathsf{G}_* = \mathsf{G}\big(X_*(P)\big)$ is obtained by solving X^I-part of the extremum equations $\partial_I \operatorname{Im} \mathsf{G} = 0$

Note that we treat CY moduli semiclassically and chemical potentials quantum mechanically

Contents

- Motivation: BPS vs Non-BPS
- Black hole potential and attractors
  - Black holes from IIB/CY
  - Attractor equations revisited
  - The inverse problem
- Turning on higher derivative corrections
  - Entropy function results
  - Extending OSV formula
- Conclusion and future directions

Summary

- We study non-BPS, extremal 4d black holes which arise in IIB/CY compactification
- Inverse problem of fixing charges in terms of the attractor value of CY moduli can be explicitly solved
- We propose a generalization of the OSV conjecture for higher derivative corrections of the non-BPS black hole entropy, in terms of Nekrasov's refinement of topological string

Open questions and conclusion

- Can we fix $(\varepsilon_1 + \varepsilon_2)^2$ ambiguity in $\tilde{V}^{\varepsilon}_{\mathrm{BH}}$?
- Can we do more tests? How about a derivation?
- What exactly a higher genus terms in Nekrasov's refinement of topological string are computing?

It is clear that we are just touching the top of an iceberg
⇓ more fun lies ahead!