Perils and Promises of Heterotic Standard Models

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1 Introduction

One of the main challenges in string phenomenology consists in constructing fully realistic low energy limits of string theory. Much progress has been done in recent years on this front: in particular, a globally consistent string compactification with low energy effective field theory reproducing precisely the massless spectrum of the Minimal Supersymmetric Standard Model (MSSM) was found in [1]. However, questions abound: Are there many such "realistic" vacua? Do they have common features? And so on and so forth. In this contribution, we propose a brief overview of recent progress in this particular sphere of string phenomenology, focusing on the corner of the string landscape in which the construction of [1] lies.

This contribution is divided in two parts. In the first part, we sketch the main lines of the type of compactifications under study. Since the pioneer work of Candelas, Horowitz, Strominger and Witten [2], four-dimensional compactifications of the $E_8 \times E_8$ heterotic string on smooth Calabi-Yau threefolds have provided a fertile ground for phenomenological studies. More precisely, we focus on compactifications of the $E_8 \times E_8$ heterotic string on smooth, non-simply connected Calabi-Yau threefolds, with $SU(4)$ (resp. $SU(5)$) gauge bundles. We describe the constraints necessary for supersymmetry to remain unbroken at low energies, and for global consistency of the string compactification (or in other words, "UV-completion" of the low energy effective field theory). Then, we study phenomenological properties of the low energy four-dimensional theory. The resulting theory is a $SO(10)$ (resp. $SU(5)$) Grand Unified Theory (GUT). We use a discrete Wilson line to break the GUT gauge group to the MSSM gauge group. We explain how to compute the massless spectrum of the four-dimensional theory, and the tri-linear couplings in the low energy Lagrangian.

As a result, we obtain a purely algebro-geometric translation of the physical problem of obtaining realistic physics out of the heterotic string.

In the second part, we propose two attempts at solving this mathematical problem. One of our motivations is to try to find as many "realistic" vacua as possible, in order to see if they have common features which could (perhaps) lead to model-independent predictions coming out of heterotic string theory.

The first construction was proposed by the author and Donagi in [1] a little more than two years ago, and further studied in [3]. It solves all of the consistency and phenomenological conditions (at the level of computation available to us in the heterotic string), hence providing so far the only known "realistic" limit of heterotic string theory. The low energy theory has the MSSM gauge group $SU(3) \times SU(2) \times U(1)$, and exactly the massless spectrum of the MSSM, with no exotic particles, up to a certain number of massless scalar moduli fields. The tri-linear couplings are semi-realistic, in the sense that tree level calculations do not produce results violating phenomenological bounds. Of course, much more phenomenology is needed to render this vacuum fully realistic; supersymmetry breaking must be understood, the massless moduli should be stabilized, *etc.*

The second construction was the subject of Bak's thesis [4] and further work is in progress by the author in collaboration with Bak and Donagi [5]. This new construction looks very promising, and may provide some new realistic vacua satisfying all of the consistency and phenomenological conditions. However, a few mathematical subtelties remain to be checked carefully. Nevertheless, we explain the main building blocks of the construction, pointing out some of the features that make this construction phenomenologically appealing.

2 Overview of heterotic compactifications

2.1 Heterotic vacua

We consider supersymmetry preserving compactifications of $E_8 \times E_8$ heterotic string theory on $M \times X$, where M is four-dimensional Minkowski space and X is an internal six-dimensional smooth compact manifold. A large class of such heterotic vacua are indexed by the data (X, ω, V), where X is a Calabi–Yau threefold, ω is a Kähler form on X, and $V \to X$ is a holomorphic vector bundle with structure group $H \subseteq E_8 \times E_8$ (see for instance [6], chapters 14–16, for more details). Consistency of the heterotic compactification imposes the two following requirements:

1. V must be polystable with respect to the Kähler class ω;
2. $c_2(TX) - c_2(V) = 0$.

Here, $c_2(V)$ is the second Chern class of V, and $c_2(TX)$ is the second Chern class of the tangent bundle of X.

The first condition comes from the Donaldson-Uhlenbeck-Yau theorem. The heterotic compactification has a gauge field (which is represented by a holomorphic connection on V) which must satisfy the Hermitian-Yang-Mills equations for supersymmetry to be unbroken. Donaldson-Uhlenbeck-Yau theorem states that there exists such a solution if V is polystable with respect to a Kähler class ω. But what is polystability? Recall that the slope $\mu_\omega(V)$ of a vector bundle V with respect to a Kähler class ω is given by

$$\mu_\omega(V) = \frac{c_1(V) \cdot \omega^2}{\mathrm{rank}(V)}, \tag{1}$$

where $c_1(V)$ is the first Chern class of V. A vector bundle V is *stable* with respect to ω if and only if $\mu_\omega(V') < \mu_\omega(V)$ for all sub-bundles V' of V. It is *polystable* if it is the direct sum of stable vector bundles of equal slope.

The second condition comes from the Bianchi identity for the gauge-invariant field strength of the B-field. More precisely, the Bianchi identity must be satisfied at the form level, but here we only require that it is satisfied at the cohomology level, from which we extract the above condition on the second Chern classes of the tangent bundle of X and of V.

Note that the second condition can be relaxed slightly. Following Hořava and Witten [7], it is believed that the strong coupling limit of heterotic string theory is 11-dimensional M-theory compactified on the interval $S^1/\mathbb{Z}_2$. However, heterotic M-theory vacua can also include $M5$-branes wrapping holomorphic curves in X, and located at points on the interval. Those contribute to the Bianchi identity, which becomes

$$c_2(TX) - c_2(V) = [M] \tag{2}$$

in the presence of an $M5$-brane, where $[M]$ is the effective cohomology class Poincaré dual to the holomorphic curve wrapped by the $M5$-brane. Therefore, if one is willing to consider the strong coupling limit of heterotic string theory, the second condition can be relaxed to the requirement that $c_2(TX) - c_2(V)$ be an effective cohomology class.

2.2 Phenomenology

Once we are given a consistent heterotic vacuum (X, ω, V), the next step is to study phenomenology of the low energy four-dimensional theory.

2.2.1 Gauge group

The structure group H of V is a subgroup of $E_8 \times E_8$. In fact, the bundle V splits as a direct sum $V = V_v \oplus V_h$, where V_v is the *visible* bundle and V_h is the *hidden* bundle, each of which has structure group contained in one of the two E_8 factors. For simplicity, here we only consider trivial hidden bundles V_h. Then, $V = V_v$, and $H \subseteq E_8$.

The ten-dimensional visible E_8 gauge group of the heterotic string is then broken down to the subgroup that commutes with the gauge fields present in the vacuum; that is, the gauge group G of the low energy effective theory in the visible sector is the commutant of H in E_8.[1] In particular, we will focus on the two following cases:

- $H = SU(5)$, which implies that $G = SU(5)$;
- $H = SU(4)$, which implies that $G = SO(10)$.

Those two choices provide interesting candidates for four-dimensional Grand Unified Theories (GUT). Notice that to get a bundle V with structure group $G = SU(5)$ (resp. $G = SU(4)$), one must require that $\mathrm{rank}(V) = 5$ (resp. $\mathrm{rank}(V) = 4$) and $c_1(V) = 0$.

2.2.2 GUT breaking

We now have four-dimensional GUT theories, which we must break down to the Minimal Supersymmetric Standard Model (MSSM). However, in the context of heterotic compactifications we cannot use standard field theory mechanisms to break the gauge symmetry, since the massless spectrum of the heterotic string does not contain the required GUT Higgs field. We need a "stringy" mechanism.

One approach that has proven fruitful is to break the GUT gauge group using a discrete Wilson line F on X (see [6]). This however requires that F be a subgroup of the fundamental group $\pi_1(X)$ of X. That is, we must require that X be non-simply connected. This is the approach that we will follow in this paper.

For example, one could consider a Wilson line $F = \mathbb{Z}_2$ to break the $SU(5)$ GUT group to the MSSM gauge group $SU(3) \times SU(2) \times U(1)$. In the case of $SO(10)$, one could consider $F = \mathbb{Z}_6$ or $F = (\mathbb{Z}_3)^2$ to break $SO(10)$ to the MSSM gauge group with an extra $U(1)$. Such breaking patterns can be obtained for many other finite groups F.

[1] Since we assume V_h to be trivial, the vacuum also includes a hidden E_8 super-Yang-Mills theory.

2.2.3 Massless particle spectrum

The zero modes of the ten-dimensional Dirac operator on X in the background with V give rise to four-dimensional massless particles. Since X is a Calabi-Yau threefold, the zero modes of the Dirac operators become zero modes of the Dolbeault operator on X coupled to V. That is, four-dimensional massless particles can be represented by cohomology classes of certain bundles on X.

More precisely, the zero modes of the Dirac operator are counted by the cohomology groups $H^q(X, \mathrm{ad}\mathcal{V})$, $q = 0, 1, 2, 3$, where $\mathrm{ad}\mathcal{V}$ is the rank 248 bundle associated to the E_8 bundle via the adjoint representation of E_8. By Serre duality, we can restrict ourselves to $q = 0, 1$ (the dual spaces index anti-particles). We obtain that the massless particle spectrum is given by

$$Z = H^0(X, \mathrm{ad}\mathcal{V}) \oplus H^1(X, \mathrm{ad}\mathcal{V}). \tag{3}$$

We must however now embed a non-trivial H-bundle V in the E_8 bundle. That is, we must decompose the bundle $\mathrm{ad}\mathcal{V}$ via $H \times G \subset E_8$. Under this subgroup, the adjoint of E_8 decomposes into a sum of irreducible representations:

$$\mathrm{ad}E_8 = \sum_a R_a(H) \otimes R'_a(G). \tag{4}$$

We can similarly decompose the bundle $\mathrm{ad}\mathcal{V}$,

$$\mathrm{ad}\mathcal{V} = \sum_a V_{R_a(H)} \otimes R'_a(G), \tag{5}$$

where the $V_{R_a(H)}$ are the bundles associated to the H-bundle V via the representation $R_a(H)$ of H. The representations $R'_a(G)$ give the massless particles in the GUT theory with gauge group G. Combining with (3), we see that the multiplicity of these particles is given by the dimension of the cohomology groups of the bundles $V_{R_a(H)}$.

Let us work out an example explicitly for concreteness. Consider $H = SU(5)$ and $G = SU(5)$. Under $SU(5) \times SU(5) \subset E_8$, the adjoint **248** of E_8 decomposes as

$$\mathbf{248} = (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{10}, \mathbf{5}) \oplus (\overline{\mathbf{10}}, \overline{\mathbf{5}}) \oplus (\overline{\mathbf{5}}, \mathbf{10}) \oplus (\mathbf{5}, \overline{\mathbf{10}}). \tag{6}$$

As a result, we obtain that the massless spectrum is given by

$$\begin{aligned} Z =& (H^0(X, \mathcal{O}_X) \otimes \mathbf{24}) \oplus (H^1(X, \mathrm{ad}V \otimes \mathbf{1}) \oplus (H^1(X, \wedge^2 V) \otimes \mathbf{5}) \oplus (H^1(X, \wedge^2 V^*) \otimes \overline{\mathbf{5}}) \\ & \oplus (H^1(X, V^*) \otimes \mathbf{10}) \oplus (H^1(X, V) \otimes \overline{\mathbf{10}}), \end{aligned} \tag{7}$$

where we kept only the non-vanishing cohomology groups.[2] In particular, the net number of generations is given by

$$|h^1(X, V) - h^1(X, V^*)| = |h^1(X, V) - h^2(X, V)| = \frac{|c_3(V)|}{2}, \tag{8}$$

[2] Indeed, for any stable vector bundle V on a Calabi-Yau threefold X, we have that $H^0(X, V_{R_a(H)}) = 0$ for any representation $R_a(H)$, except for the trivial representation, in which case $H^1(X, \mathcal{O}_X) = 0$.

where $h^i(X,V)$ denotes the dimension of $H^i(X,V)$, and we used Serre duality and footnote 2. This turns out to be a general result: the net number of generations is always given by $\frac{|c_3(V)|}{2}$. Hence, to obtain three generations, we must require that $c_3(V) = \pm 6$.

More generally, requiring that the low-energy theory has precisely the MSSM massless particle spectrum, with no exotic particles, imposes a set of numerical constraints on the cohomology groups arising in the decomposition of the kernel of the Dirac operator, as in (7). Note however that it is generally not possible to require that $H^1(X, \mathrm{ad}V) = 0$; hence the massless spectrum will generally contain a certain number of massless scalar fields, which are called *vector bundle moduli.*[3] Those should acquire mass somehow to render the compactification phenomenologically viable; this is what is usually called *moduli stabilization* in string theory. At present it is unclear how to do so in the heterotic context.

Finally, it is relatively straightforward to extend the discussion of the massless spectrum after introducing a non-trivial Wilson line F. We refer the reader to [6, 8] for more details.

2.2.4 Tri-linear couplings

Once we know the massless spectrum, we can ask what the tri-linear couplings in the low-energy Lagrangian look like. Since massless particles correspond to cohomology classes, it follows that tri-linear couplings can be computed through tri-linear products of cohomology groups. Specifically, consider the cohomology groups arising in the decomposition of the kernel of the Dirac operator, as in (7), and form all possible triple cohomology products; those give the tri-linear couplings of the associated particles in the low energy Lagrangian.

Coming back to our previous example with $G = SU(5)$ and $H = SU(5)$, we obtain the massless particle spectrum given by (7). Consider for instance the triple product

$$H^1(X, V^*) \times H^1(X, V^*) \times H^1(X, \wedge^2 V) \to H^3(X, \mathcal{O}_X) \simeq \mathbb{C}. \tag{9}$$

According to (7), this should compute the tri-linear coupling **10**–**10**–**5** in the $SU(5)$ GUT theory. The same can be done for all other cohomology groups in (7).

Two remarks are now in order. First, these triple products are valid only at tree level; they should, generically, receive worldsheet instanton corrections. But at the moment, it is unclear how to compute these corrections explicitly. Second, the triple products actually give functions of the massless moduli fields. Hence, to get numbers, we need to stabilize moduli. But since moduli stabilization is out of reach in heterotic compactifications for the moment, all that we can say is whether these triple products identically vanish or not. In particular, since some of these couplings

[3] Other massless moduli fields also come from the complex structure and Kähler parameters of X.

must vanish for the low energy theory to be phenomenologically viable (for instance for proton stability), we must impose a set of vanishing constraints on these triple products.

2.3 Summary

To end this section, let us summarize in list form the data that needs to be constructed in order to obtain a phenomenologically viable supersymmetric compactification of heterotic string theory. This list provides an algebro-geometric translation of the problem, which we will try to solve in the next section. We need:

- a Calabi-Yau threefold X with a Kähler class ω;
- a stable holomorphic vector bundle V satisfying the condition that $c_2(TX) - c_2(V)$ be an effective class;
- $\mathrm{rank}(V) = 4$ or 5;
- $c_1(V) = 0$, $c_3(V) = \pm 6$;
- $\pi_1(X) = F$ where F is a finite group that can be used to break $G = \mathrm{struct}(V)$ to the MSSM gauge group (with perhaps an extra $U(1)$);
- various numerical conditions on the dimensions of the cohomology groups $H^1(X, V_{R_a(H)}) = 0$ arising in the decomposition of the kernel of the Dirac operator to be satisfied;
- some triple products of these cohomology groups to vanish.

The reader may have noticed that we have not discussed supersymmetry breaking at all. There are various supersymmetry breaking scenarios that could in principle be embedded in this type of compactification, involving the hidden E_8 sector. However in this paper we will confine ourselves to the supersymmetric realm. Let us also restate that satisfying the above conditions is not the end of the story, since more phenomenology is needed to make the vacua fully realistic. But this set of conditions is already very hard to solve, and we will no try to go beyond those in the present contribution.

3 Two constructions with promising phenomenology

In this section we try to solve the mathematical problem formulated in subsection 2.3 in two different ways. The first construction is the only known model satisfying all conditions of subsection

2.3. The second construction is still work in progress, but there is hope that it may also provide compactifications satisfying the above conditions.

3.1 Heterotic Standard Models on Schoen's threefolds

3.1.1 The Calabi-Yau threefold

The first step consists in constructing a Calabi-Yau threefold X with non-trivial fundamental group $\pi_1(X) = F$. Generally speaking, X has a universal cover $\tilde{X}$ which is also Calabi-Yau, on which the finite group F acts freely; X is then recovered as the quotient $X = \tilde{X}/F$. Thus, the problem of constructing a non-simply connected Calabi-Yau threefold may be recast as the problem of constructing a Calabi-Yau threefold $\tilde{X}$ with a free finite group action.

A fertile family of Calabi-Yau threefolds was constructed by Schoen [9]. Consider smooth Calabi-Yau threefolds $\tilde{X} = B \times_{\mathbb{P}^1} B'$ obtained as fiber products of two rational elliptic surfaces B and B'. All such $\tilde{X}$ are elliptically fibered, and have Euler characteristic $\chi(\tilde{X}) = 0$. For non-generic rational elliptic surfaces, it turns out that such fiber products admit many free actions. In fact, we classified in [10] all free finite group actions on Schoen's threefolds. By taking the quotients, we obtained threefolds X with fundamental group:

$$\pi_1(X) \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, (\mathbb{Z}_2)^2, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_4, (\mathbb{Z}_3)^2\}. \tag{10}$$

In all these cases, F preserves the elliptic fibration of $\tilde{X}$ (although not the section).

On these non-simply connected Calabi-Yau threefolds we can use Wilson lines to break the GUT group to the MSSM gauge group. We will now focus on the Calabi-Yau threefold X with $\pi_1(X) = \mathbb{Z}_2$, which is the one on which a viable $SU(5)$ bundle was constructed in [1].

3.1.2 The bundle

The next step consists in constructing a bundle V on X satisfying all of the conditions listed in subsection 2.3, where X is the threefold with $\pi_1(X) = \mathbb{Z}_2$ constructed above. In this case, it turns out to be easier to first construct a bundle $\tilde{V}$ on $\tilde{X}$, then show that it is invariant under the action of F, so that it descends to a vector bundle V on X.

Since $\tilde{X}$ is elliptically fibered, one can use the standard technique known as the spectral cover construction to construct the bundle $\tilde{V}$ [11, 12]. Let us now give a heuristic description of the spectral cover construction, without getting into the mathematical details, which can be found in the previous references.

Let E be an elliptic curve, and $E^\vee \simeq \mathrm{Pic}^0(E)$ be the dual elliptic curve, which parameterizes isomorphism classes of flat line bundles on E. It is well known that E is isomorphic to $E^\vee$. Now consider a rank n flat bundle $U \to E$, which splits as a direct sum of n flat line bundles. Each flat line bundle corresponds to a point in the dual elliptic curve $E^\vee$. Thus, the rank n flat bundle $U \to E$ has a dual description as a set of n points in $E^\vee$.

Now fiber the construction over a two-dimensional base B. That is, consider a threefold $\tilde{X}$ which is an elliptic fibration over B, and a rank n bundle $\tilde{V} \to \tilde{X}$ which is flat on the elliptic fiber. The bundle then has an equivalent description in terms of the dual data (C, L) on $\tilde{X}$, where the *spectral surface* C is a n-cover of the base B (it intersects the elliptic fibers at n points), and L is a line bundle on C. The spectral cover construction then consists in constructing a bundle $\tilde{V} \to \tilde{X}$ by specifying the dual data (C, L) on $\tilde{X}$.

The clear advantage of this construction is due to a theorem of Friedman-Morgan-Witten [11]. They show that if C is an irreducible surface, then V is necessarily stable in a certain region of the Kähler cone. Since proving stability of a bundle is in general rather difficult, this provides a straightforward way to construct stable bundles. However, the irreducilibity condition is only sufficient for stability, not necessary, and requiring irreducibility may be too strong a condition for phenomenological purposes. Indeed, in our case it turns out that we cannot construct bundles $\tilde{V}$ on $\tilde{X}$ with irreducible spectral surfaces which satisfy all the required phenomenological constraints. Hence we are led to consider reducible spectral surfaces.

Instead of doing so, it was suggested in [13, 14] to combine the spectral cover construction with another approach to constructing bundles, which is through extensions of two other bundles V_1 and V_2 on $\tilde{X}$:[4]

$$0 \to V_1 \to \tilde{V} \to V_2 \to 0. \tag{11}$$

A sufficient condition for stability of $\tilde{V}$ was also derived in [14], following the original line of argument in [11], in the case where V_1 and V_2 are manufactured through the spectral cover construction (up to twisting by line bundles).

This is precisely the construction that we used in [1] to obtain a viable $SU(5)$ bundle on $\tilde{X}$. We used an extension of the form (11), and showed that $\tilde{V}$ satisfies the sufficient condition for stability. We also showed, following the explicit calculations in [13, 14], that $\tilde{V}$ is invariant under the free $\mathbb{Z}_2$ action on $\tilde{X}$, so that it descends to a bundle V on X. We refer the reader to [1] for the explicit data describing V_1 and V_2 and the proofs of these claims.

We then moved to phenomenology, and showed in [1, 3] that the resuling V on X satisfies *all* the conditions of subsection 2.3. That is, we obtain a low energy effective theory with not only the

[4]Here $0 \to V_1 \to \tilde{V} \to V_2 \to 0$ is a short exact sequence of bundles, which means that V_1 is a sub-bundle of $\tilde{V}$, and V_2 is the quotient bundle $\tilde{V}/V_1$.

MSSM gauge group, but precisely the massless spectrum of the MSSM, with no exotic particles, up to moduli fields. We also showed that dangerous tri-linear couplings identically vanish at tree level (for instance, R-parity is preserved at tree level), thus making sure that the model survives phenomenological constraints. We further showed that in some regions of the moduli space, non-zero Higgs μ-terms can be generated, as well as neutrino mass terms.

3.1.3 Some comments

So far we only focused on the threefold X with $\pi_1(X) = \mathbb{Z}_2$; but our classification in [10] provides many other non-simply connected Calabi-Yau threefolds that could presumably be used for phenomenological purposes.

On this note, let us mention that we also tried in [15] to construct phenomenologically viable $SU(4)$ and $SU(5)$ bundles on the Calabi-Yau threefold X with $\pi_1(X) = \mathbb{Z}_6$. However, it turns out that all our attempts, using the techniques described above, have been unsuccessful. The moral of the story is that the two consistency conditions of the heterotic compactification, namely the stability condition and the Bianchi identity, are very difficult to satisfy simultaneously. Relaxing one or the other gives infinite families of viable models; but those are *not* globally consistent string compactifications. So it seems that requiring global consistency of the string compactification (that is, "UV-completion" of the low energy effective theory) is a very strong constraint. This gives a concrete realization of the common statement that the landscape of effective field theories that can be obtained as low energy limits of string theory is much smaller than the infinite landscape of effective field theories.

3.2 A construction in progress

3.2.1 The Calabi-Yau threefold

Another threefold with a large fundamental group recently appeared in the mathematics literature. Gross and Popescu constructed a threefold X with fundamental group $\pi_1(X) = (\mathbb{Z}_8)^2$, which is the Calabi-Yau threefold with the largest fundamental group known at the moment [16, 17]. From a phenomenological perspective, such a big fundamental group offers many different symmetry breaking patterns that can be used to provide interesting low energy phenomenology. Hence, it seems natural to try to construct viable bundles on this threefold. Attempts at constructing phenomenologically interesting bundles on Gross' threefold form the core of Bak's thesis [4]; further work is in progress by the author in collaboration with Bak and Donagi [5].

First, let us briefly explain the geometry. Start with a complete intersection X' of four particular quadrics in $\mathbb{P}^7$ (see [16, 17] for the explicit equations of the quadrics). It turns out that for this choice of quadrics, X' has 64 double-point singularities, but one can show that there exists a small resolution $\tilde{X} \to X'$, where $\tilde{X}$ is a smooth Calabi-Yau threefold. Moreover, $\tilde{X}$ is a fibration over $\mathbb{P}^1$, where the fiber is an Abelian surface with $(1, 8)$ polarization.

$\tilde{X}$ admits a free $F = (\mathbb{Z}_8)^2$ action; the quotient $X = \tilde{X}/F$ is smooth and has fundamental group $\pi_1(X) = (\mathbb{Z}_8)^2$. Moreover, the quotient map reduces on the fiber to the isogeny $A \to A^\vee$, where $A^\vee$ is the Abelian surface dual to A.[5] Hence, X is also an Abelian surface fibration, with fiber dual to the Abelian fiber of $\tilde{X}$.

It may be worth noting here that it was shown by Borisov and Hua that two other smooth threefolds with order 64 fundamental group (although non-Abelian) can be obtained in a similar fashion, as quotients of $\tilde{X}$ [18]. Hua also classified all Calabi-Yau threefolds with non-trivial fundamental group that can be obtained as free quotients of small resolutions of singular complete intersections of four quadrics in $\mathbb{P}^7$ [19]. Those threefolds provide other new manifolds which could be used to construct $SU(5)$ and $SO(10)$ models; but for now we will restrict ourselves to Gross' original threefold.

3.2.2 Constructing bundles

The threefold X is an Abelian surface fibration over $\mathbb{P}^1$. As such, we cannot use the standard spectral cover construction, which demands that X be an elliptic fibration over a two-dimensional base. We will however use a modified version of the spectral cover construction, which was developed in Bak's thesis [4]. We give here an intuitive description of the construction, leaving aside the mathematical details.

Consider first a rank n flat bundle U on an Abelian surface A. Suppose that U splits as a sum of flat line bundles. It is then represented by a set of n points on the dual Abelian surface $A^\vee$, which is *not* isomorphic to A, but only isogenous, as explained in footnote 5.

Now fiber the construction over $\mathbb{P}^1$. That is, consider a rank n bundle V over Gross' threefold X, which is an Abelian surface fibration over $\mathbb{P}^1$ with fiber A. Suppose that V restricted to the Abelian fiber splits as a sum of flat line bundles. Then V can be described in terms of the dual data (C, L) on the *cover* threefold $\tilde{X}$ which is an Abelian surface fibration with fiber $A^\vee$, where C is an n-cover of the $\mathbb{P}^1$ base (intersecting the Abelian fiber $A^\vee$ in n points), and L is a line bundle on C.

[5] The dual Abelian surface $A^\vee \simeq \mathrm{Pic}^0(A)$ parameterizes isomorphism classes of flat line bundles on A. In contrast with the elliptic curve case, $A^\vee$ is not isomorphic to A; rather, the map $A \to A^\vee$ is a surjective morphism with finite kernel (*i.e.* an isogeny). For an Abelian surface with $(1, 8)$ polarization, the kernel is $(\mathbb{Z}_8)^2$.

An important point here is that there is no invariance to prove for the bundle V. Indeed, as we have seen, we can construct a bundle V on the quotient X by specifying the dual spectral data (C, L) on the *cover* threefold $\tilde{X}$! This is a consequence of the fact that the Abelian fibers of X and $\tilde{X}$ are dual. Hence, we can work on the simply connected cover $\tilde{X}$ to specify the spectral data, and we obtain directly a bundle V on the quotient threefold X. This reduces considerably the complexity of the construction.

Now we must also address the issue of stability of V. It turns out that one can prove a sufficient criterion for stability which is very similar to the Friedman-Morgan-Witten theorem for the elliptic fibration case, involving irreducibility of the spectral curve C; we refer the reader to [4, 5] for more details.

However, as in the elliptic fibration case, it turns out that the spectral cover construction for irreducible spectral curves is too restrictive for phenomenology: we can only construct bundles V with $c_3(V) = 0$ in this way, which produce low energy theories with the same number of generations and anti-generations. To overcome this difficulty, we combine the spectral cover construction with an elementary transform along an Abelian surface A.

More specifically, we construct a bundle E on X through an elementary transform

$$0 \to E \to V \to i_* Q \to 0, \tag{12}$$

where V is obtained through the spectral cover construction, and Q is a vector bundle supported on a surface A (i is the inclusion in X). For the elementary transform to be well defined, one must show that the map $V \to i_* Q$ is surjective.

At first sight, using this technique we obtain many candidates for stable $SU(4)$ and $SU(5)$ bundles E's with $c_3(E) = \pm 6$, $c_1(E) = 0$ and $c_2(E)$ satisfying the Bianchi identity, which seems very promising.[6] However, at the moment we cannot prove whether the map $V \to i_* Q$ is surjective for these interesting Q's and V's, which is required for the elementary transform to be well defined. Therefore, we cannot state yet whether these promising bundles E really exist or not. We hope to report conclusive results very soon. If surjectivity can be proved for some of these bundles, the next step consists in computing the massless spectrum and the tri-linear couplings. The large fundamental group $(\mathbb{Z}_8)^2$ provides a lot of freedom for implementing symmetry breaking mechanisms, which gives us hope that some of these bundles may yield precisely the MSSM spectrum and quasi-realistic tri-linear couplings.

[6] Although we should note that satisfying these conditions is already rather non-trivial. We must consider Q's which are rank 2 bundles over a non-generic Abelian surface A with extra polarization classes. We construct these Q's using the Serre construction, which defines Q as an extension of $N \otimes I_Z$ by L, where L and N are line bundles on the Abelian surface and I_Z is the ideal sheaf of a set of points Z on the surface.

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