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BPS Dyons, Wall Crossing and Borcherds Algebra

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Based on work with Miranda Cheng

Outline

- N=4 Dyonic Black Holes and Walls of Marginal Stability
- Hyperbolic Kac Moody Algebra and its Weyl Group
- Discrete Attractor Flow and Arithmetic Decay
- Microscopic Description: Borcherds Algebra.

Heterotic Strings on T^n

Momentum and winding form an even Lorentzian lattice

$$P \in \Gamma_{n,n+16}, \qquad P^2 \in 2\mathbf{Z}$$

Given a point on the moduli space

$$SO(n)\backslash SO(n,n+16)/SO(n+16)$$

one can write

$$P = (P_L, P_R), \qquad P^2 = P_L^2 - P_R^2$$

Heterotic Strings on T^n

The leftmoving sector contains 16 supersymmetry charges.

8 of these annihilate the left-moving ground states.

Such states are called 1/2 BPS states. Their mass equals

$$M = \frac{1}{\sqrt{\tau_2}}|P_L|, \qquad \tau_2 = \text{string coupling}$$

Note that

$$\tau_2 M^2 \geq P^2, \qquad P^2 \equiv P_L^2 - P_R^2 \in 2\mathbf{Z}$$

Equality holds when $P_R = 0$, this is an **attractor point**.

Heterotic Strings on T^6

Momentum and winding correspond to electric charges

$$P = (P_L, P_R) \in \Gamma_{6,22}$$

Left ground states: 1/2 BPS states, counting of rightmovers

$$d(P) = \int_C d\rho \, \frac{e^{-i\pi\rho P^2}}{\eta^{24}(\rho)}$$

$$-\tfrac{1}{2} \leq \mathrm{Re}\,\rho \leq \tfrac{1}{2}$$
$$\mathrm{Im}\,\rho > 0$$

with

$$\eta(\rho) = q^{\frac{1}{24}} \prod_n (1 - q^n)$$

$$q = e^{2\pi i \rho}$$

BPS Dyons for Heterotic String on T^6

Electric and magnetic charge

$$(P, Q) \in \Gamma_{6,22} \oplus \Gamma_{6,22} ,$$

1/4 BPS states obey

$$M_{P,Q} = |Z_{P,Q}|$$

$Z_{P,Q}$= largest eigenvalue of the 4x4 anti-symm. matrix

$$\hat{Z} = \frac{1}{\sqrt{\tau_2}}(P_L - \tau Q_L)^m \Gamma_m \ ; \ m = 1, .., 6 ,$$

Dyonic Black Holes

Entropy

$$S = \pi |P \wedge Q| \equiv \pi \sqrt{Q^2 P^2 - (Q \cdot P)^2}\ ,$$

S-duality

$$\begin{pmatrix} P \\ Q \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}$$

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

S-duality + Parity: SL(2,Z) => PGL(2,Z)

$$\tau \longrightarrow \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \quad \text{for} \quad ad - bc = -1$$

Attractor Flow

At the horizon the moduli reach fixed point.

Flow of Narain moduli

$$\begin{pmatrix} P_L \\ Q_L \end{pmatrix} \to \begin{pmatrix} P \\ Q \end{pmatrix} \qquad \begin{pmatrix} P_R \\ Q_R \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Flow of axion dilaton

$$\frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} \to \frac{1}{|P \wedge Q|}\begin{pmatrix} P^2 & PQ \\ PQ & Q^2 \end{pmatrix}$$

Walls of Marginal Stability

In general one has

$$Z_{P,Q} = Z_{P,0} + Z_{0,Q} \ ,$$

Hence,

$$M_{P,Q} \leq M_{P,0} + M_{0,Q}$$

At the wall of marginal stability

$$|Z_{P,Q}| = |Z_{P,0}| + |Z_{0,Q}| \ .$$

=> real co-dimension 1 subspace of moduli space.

Walls of Marginal Stability

From the central charge matrix it follows that at

$$\frac{\tau_1}{\tau_2} + \frac{P_L \cdot Q_L}{|P_L \wedge Q_L|} = 0\ .$$

a 1/4 BPS state may decay in a pair of 1/2 BPS states

$$\begin{pmatrix} P \\ Q \end{pmatrix} \rightarrow \begin{pmatrix} P \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Q \end{pmatrix}$$

Supergravity analysis: bound state is stable when

$$P \cdot Q \left(\frac{\tau_1}{\tau_2} + \frac{P_L \cdot Q_L}{|P_L \wedge Q_L|} \right) < 0$$

Walls of Marginal Stability

Other walls are obtained by using that the matrix

$$X = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} + \frac{1}{|P_L \wedge Q_L|}\begin{pmatrix} P_L\cdot P_L & P_L\cdot Q_L \\ P_L\cdot Q_L & Q_L\cdot Q_L \end{pmatrix}$$

transforms under PGL(2,Z) as $X \to \gamma X \gamma^T$ $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

this leads to

$$ac\left(\frac{|\tau|^2}{\tau_2} + \frac{P_L\cdot P_L}{|P_L \wedge Q_L|}\right) - (ad+bc)\left(\frac{\tau_1}{\tau_2} + \frac{P_L\cdot Q_L}{|P_L \wedge Q_L|}\right) + bd\left(\frac{1}{\tau_2} + \frac{Q_L\cdot Q_L}{|P_L \wedge Q_L|}\right) = 0$$

for the decay

$$\begin{pmatrix} P \\ Q \end{pmatrix} = (-cP + dQ)\begin{pmatrix} b \\ a \end{pmatrix} + (aP - bQ)\begin{pmatrix} d \\ c \end{pmatrix}$$

The Walls

Let us introduce the space of symmetric matrices

$$M_2^{(s)}(\mathbb{R}) = \left\{ X \middle| \, X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \; X = X^T \,, x_{ab} \in \mathbb{R} \right\}$$

with norm $\|X\|^2 = (X, X) = -2\,\det X$

The locations of the walls can be written as

$$(X, \alpha) = 0 \qquad \alpha = \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma^T$$

$$(\alpha, \alpha) = 2 \qquad \alpha = \begin{pmatrix} 2bd & ad + bc \\ ad + bc & 2ac \end{pmatrix}$$

Walls and Weyl Chambers

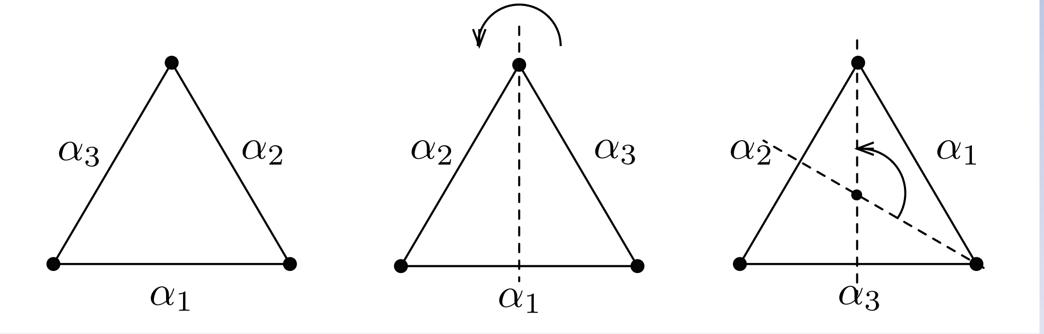
The basis $\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

obeys

$$(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

= Cartan matrix of a Generalized Kac Moody algebra

Dihedral group of outer automorphisms

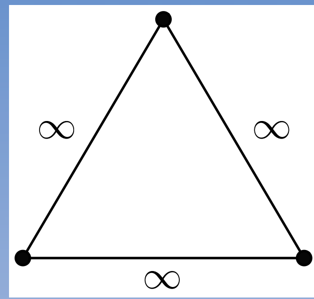


Weyl Group and Chambers

$$s_i : X \to X - 2\frac{(X,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i \quad , \quad i = 1,2,3$$

$$PGL(2,\mathbb{Z}) \cong O^+(2,1;\mathbb{Z}) \cong W \rtimes D_3$$

Coxeter diagram

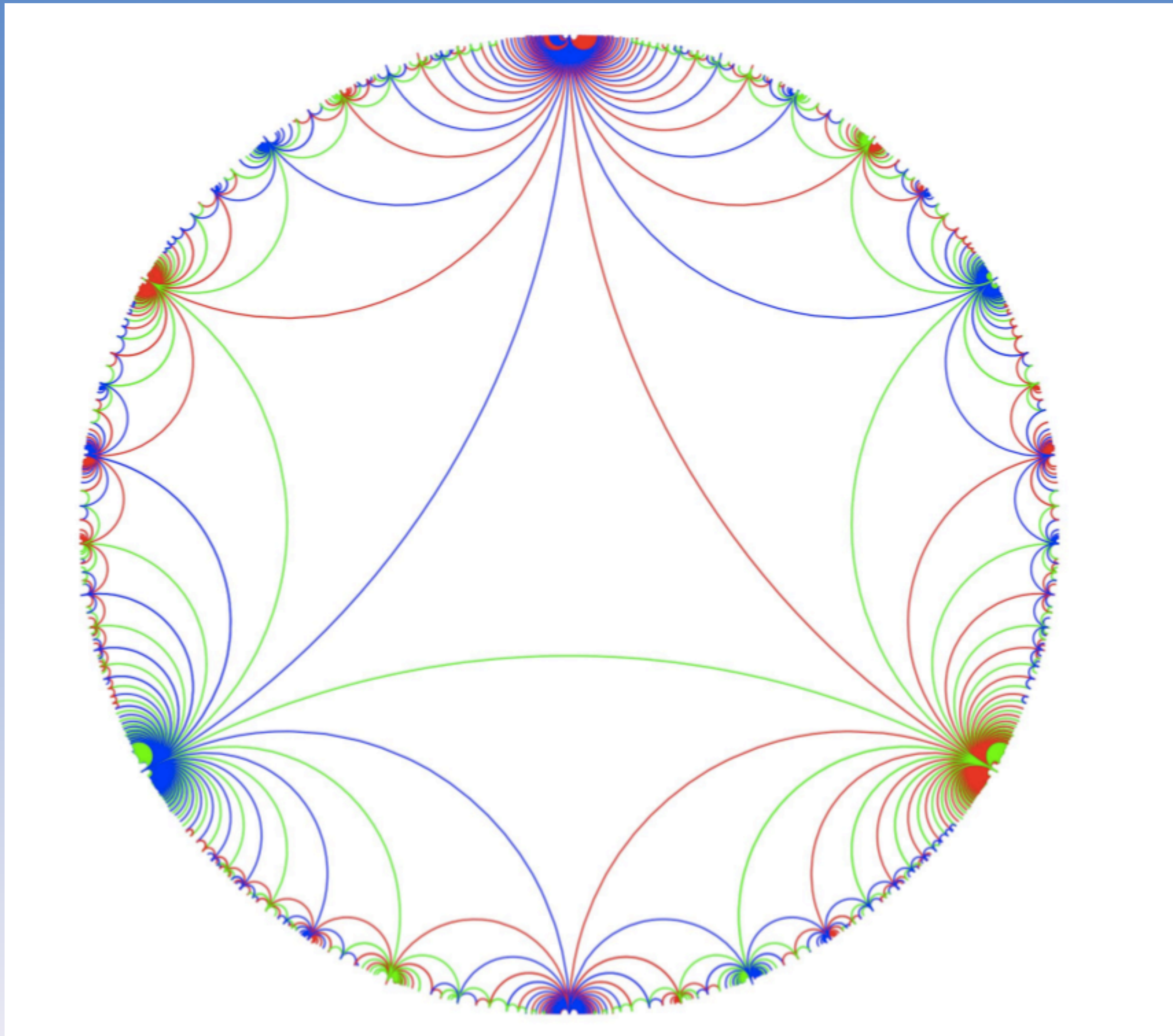


Weyl chambers can be visualized on Poincare disk

$$X = \frac{1}{\lambda_2}\begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix}$$

$$z = i\left(\frac{\lambda + e^{-\frac{i\pi}{3}}}{\lambda + e^{\frac{i\pi}{3}}}\right)$$

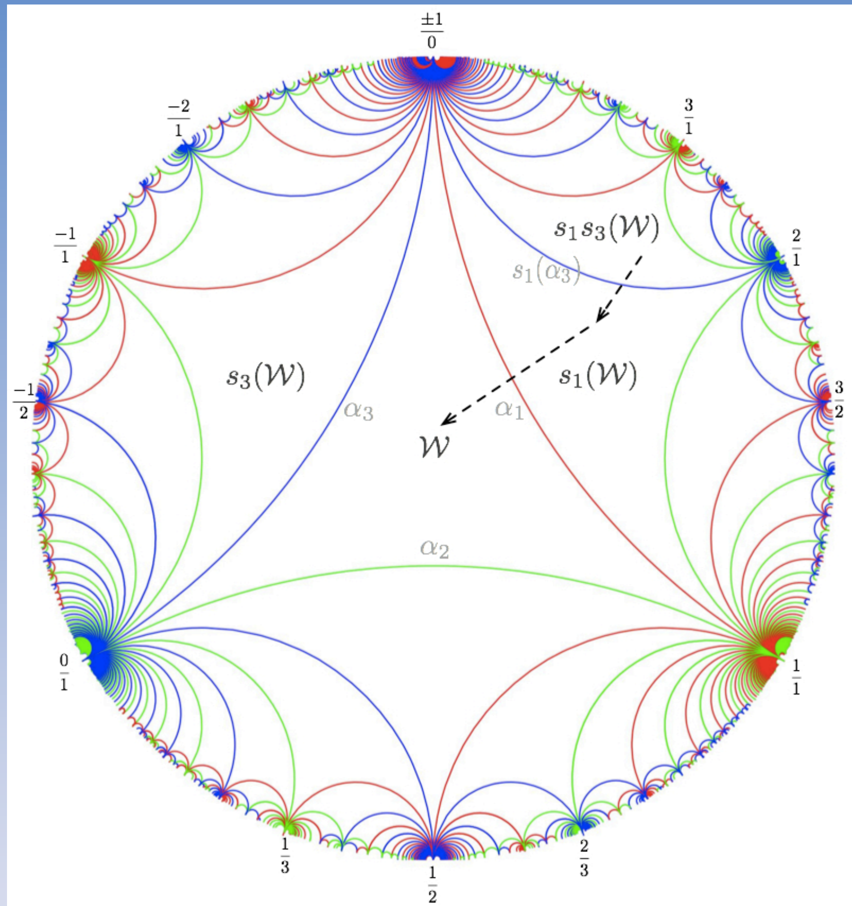
Walls and Weyl Chambers



Discrete Attractor flow

Matrix of T-duality invariants => integral weight vector

$$\Lambda_{P,Q} = \begin{pmatrix} P\cdot P & P\cdot Q \\ P\cdot Q & Q\cdot Q \end{pmatrix}$$



Attractor flow

$$X \to \frac{\Lambda_{P,Q}}{|\Lambda_{P,Q}|}$$

BPS counting jumps at walls

$$(X,\alpha) = 0 \quad , \quad \alpha \in \Delta_+^{re}$$

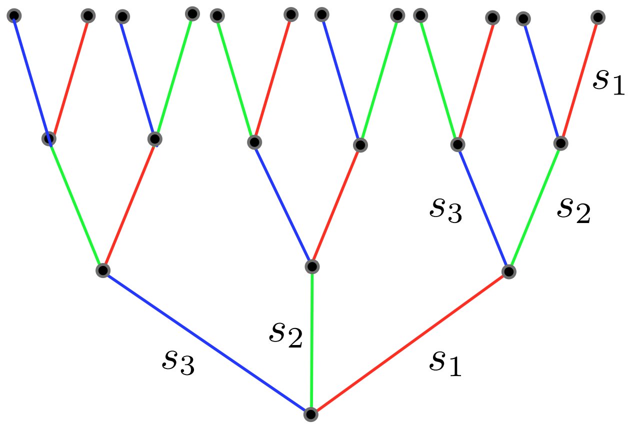
Arithmetic Decay

Decay labeled by pair of fractions

$$\left\{ \frac{b}{a}, \frac{d}{c} \right\} \quad , \quad ad - bc = 1 \quad , \quad a, c \geq 0$$

(Farey series)

$$\left\{ \frac{b}{a}, \frac{d-b}{c-a} \right\} \text{ if } c \geq a, bd \geq b^2$$



$\frac{-1}{0}$	$\frac{-2}{1}$	$\frac{-1}{1}$	$\frac{-1}{2}$	$\frac{0}{1}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{1}{0}$
$\frac{-1}{0}$		$\frac{-1}{1}$		$\frac{0}{1}$		$\frac{1}{2}$		$\frac{1}{1}$		$\frac{2}{1}$		$\frac{1}{0}$
$\frac{-1}{0}$				$\frac{0}{1}$				$\frac{1}{1}$				$\frac{1}{0}$

Arithmetic Decay

Decay labeled by

$$\alpha = \begin{pmatrix} 2bd & ad+bc \\ ad+bc & 2ac \end{pmatrix}$$

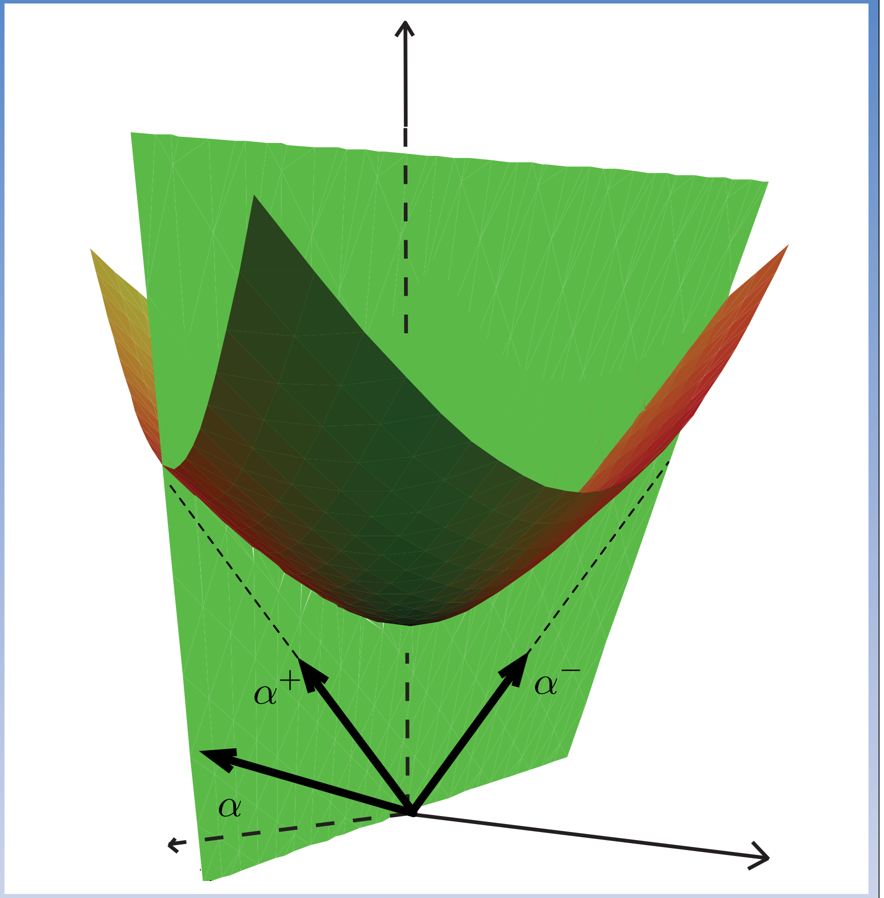
$$\{\alpha^+, \alpha^-\} = \left\{ \begin{pmatrix} b^2 & ab \\ ab & a^2 \end{pmatrix}, \begin{pmatrix} d^2 & cd \\ cd & c^2 \end{pmatrix} \right\}$$

decomposition along root

$$\Lambda_{P,Q} \quad = \quad P_\alpha^2\, \alpha^+ + Q_\alpha^2\, \alpha^- - |(P\cdot Q)_\alpha|\, \alpha$$

Degeneracy should jump by

$$(-1)^{(P\cdot Q)_\alpha + 1}\, |(P\cdot Q)_\alpha|\, d(P_\alpha)\, d(Q_\alpha)$$



Weyl-Kac-Borcherds Formula

In addition to real roots there are null and imaginary roots.

The product

$$\Phi(\Omega) = e^{(\rho,\Omega)} \prod_{\alpha\in\Delta^+} \left(1 - e^{(\alpha,\Omega)}\right)^{\frac{1}{2}c(|\alpha|^2)}$$

$$\Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix}$$

with c determined by the elliptic genus of K3 is a

Sp(2,Z)-modular form of weight k =10.

$$\Phi(\Omega) \to \left(\det(C\Omega + D)\right)^k \Phi(\Omega) .$$

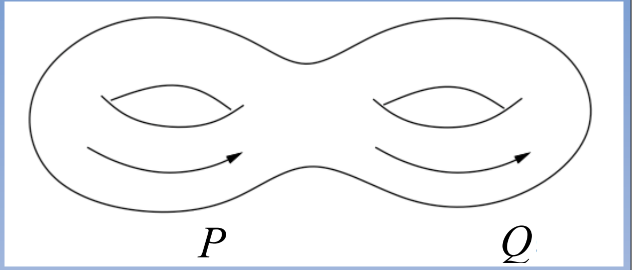
$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}$$

The Counting Formula

The # of dyonic BPS states with charge (P,Q) equals

$$D(P,Q) = \int_C d\Omega \, \frac{e^{(\Lambda_{P,Q},\Omega)}}{\Phi^2(\Omega)} (-1)^{PQ}$$

(DVV, 1996)



Reproduces entropy

$$\log D(P,Q) \sim \pi |P \wedge Q|$$

$$\Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix}$$

and has double poles

$$\frac{1}{\Phi(\rho,\sigma,\nu)} = \frac{1}{4\pi^2} \frac{1}{\nu^2} \frac{1}{\eta^{24}(\rho)} \frac{1}{\eta^{24}(\sigma)} \left(1 + \mathcal{O}(\nu^2)\right) .$$

Walls in the Space of Contours

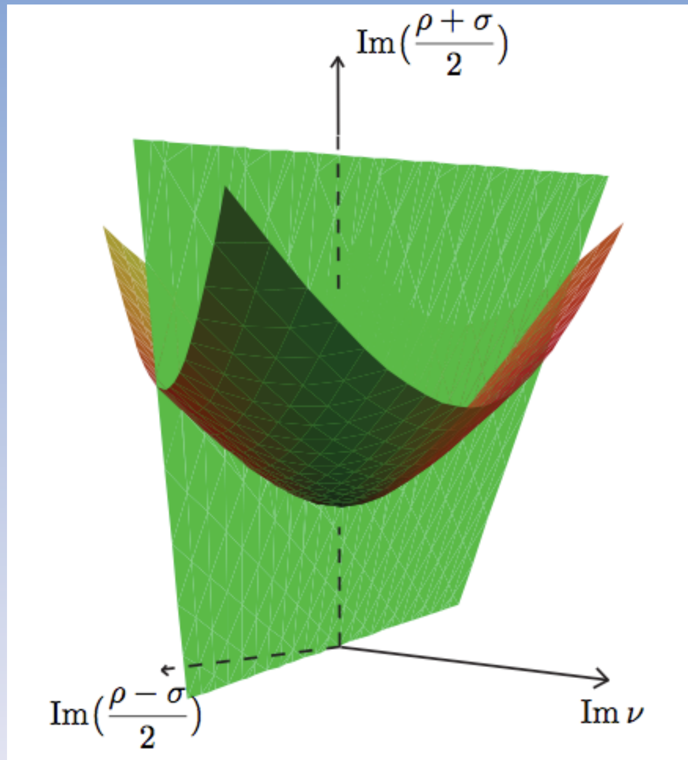
Hence, the counting formula is contour dependent!

The contours are determined by the moduli

$$\mathrm{Im}\Omega = \epsilon^{-1} X$$

$$(\mathrm{Im}\rho, \mathrm{Im}\sigma, \mathrm{Im}\nu) \in \mathbb{R}^{1,2}$$

Poles coincide with walls of Weyl Chambers



The Wall Crossing Formula

The contribution at the pole at *v =0* is given by

$$\frac{(-1)^{P\cdot Q}}{4\pi^2} d(P) d(Q) \oint_{\mathcal{C}_\nu} d\nu \, \frac{e^{-2\pi i (P\cdot Q)\nu}}{\nu^2} \, ,$$

$$d(P) = \int_C d\rho \, \frac{e^{-i\pi\rho P^2}}{\eta^{24}(\rho)}$$

which can be further evaluated to give

$$(-1)^{(P\cdot Q)+1} \, |P \cdot Q| \, d(P) \, d(Q) \qquad \text{when} \quad (P\cdot Q) \, \mathrm{Im}\nu > 0$$

This describes exactly the contribution due to the decay of 1/2 BPS bound states!

$$\begin{pmatrix} P \\ Q \end{pmatrix} \rightarrow \begin{pmatrix} P \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Q \end{pmatrix}$$

Towards Microscopic Description

Defining relations of Generalized Kac Moody algebra

$$
\begin{aligned}
&[e_i, f_j] = \delta_{ij} h_i \\
&[h, h'] = 0 \\
&[h, e_i] = (h, h_i)\, e_i \quad , \quad [h, f_i] = -(h, h_i)\, f_i \\
&(\mathrm{ad} e_i)^{1-A_{ij}} e_j = (\mathrm{ad} f_i)^{1-A_{ij}} f_j = 0 \quad \text{if } A_{ii} = 2, i \neq j \\
&[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } A_{ij} = 0 \,.
\end{aligned}
$$

No representations are known.

Towards Microscopic Description

Weyl-Kac-Borcherds formula counts the degeneracy of highest weight modules

$$\prod_{\alpha\in\Delta^+} (f_\alpha)^{n_\alpha} |\Lambda\rangle$$

The Weyl group acts on the highest weight

$$\Lambda_w = \varrho + w^{-1}(\Lambda - \varrho)$$

One can show that for a dominant weight

$$M(\Lambda) \subset M(\Lambda_w)$$

Conclusions

- The occurence of a Hyperbolic KM Algebra understood from supergravity perspective (cf. Cosmic Billiards).
- Wall crossings are related to Weyl reflections.
- Microscopic counting in terms of Borcherds Formula incorporates wall crossing through moduli dept. contour.
- Can one obtain representation of Borcherds Algebra?
- Counting is more elaborate for non-primitive charges.
- Can one generalize this to all N=4 theories? Or N=2?