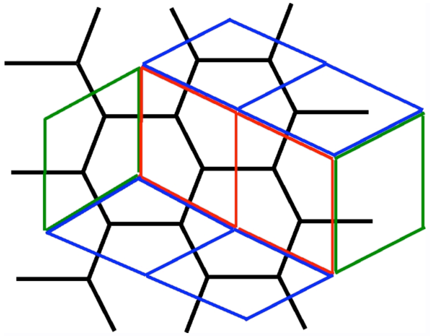
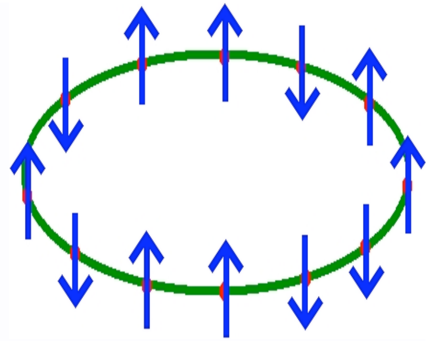


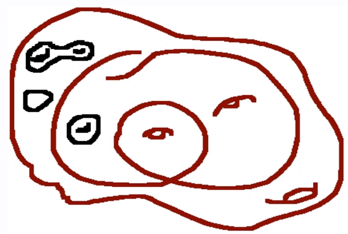


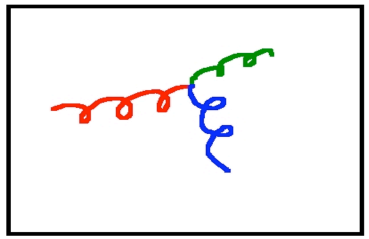
Supersymmetric Quantum Field and String Theories and Integrable Lattice Models

Nikita Nekrasov

Simons Workshop

July 2, 2008





Based on

NN, S.Shatashvili,

hep-th/0807...xyz

G.Moore, NN, S.Sh., arXiv:hep-th/9712241 ;

A.Gerasimov, S.Sh.

arXiv:0711.1472, arXiv:hep-th/0609024

The Characters of our play

2,3, and 4 dimensional susy gauge theories

with 4 supersymmetries

($\mathcal{N}=1$ $d=4$)

on the one hand

and

Quantum integrable systems

soluble by Bethe Ansatz

on the other hand

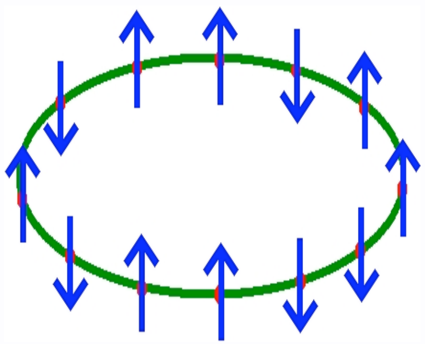
For example, we shall
relate
the XXX Heisenberg magnet
to
2d $\mathcal{N}=2$ SYM theory
with some matter

Dictionary

$U(N)$ gauge theory with
$N=4$ susy in two dimensions
with L fundamental hypermultiplets
softly broken to $N=2$ by giving
the generic twisted masses to the
adjoint, fundamental and antifundamental
chiral multiplets compatible with the
Superpotential inherited from the
$N=4$ theory

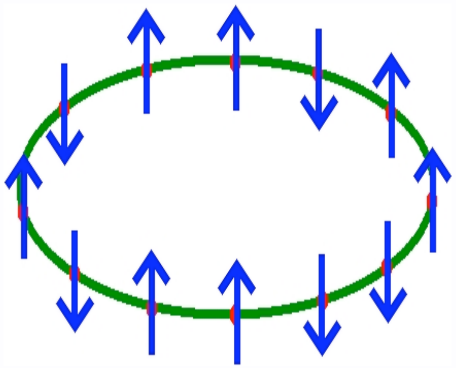
Dictionary

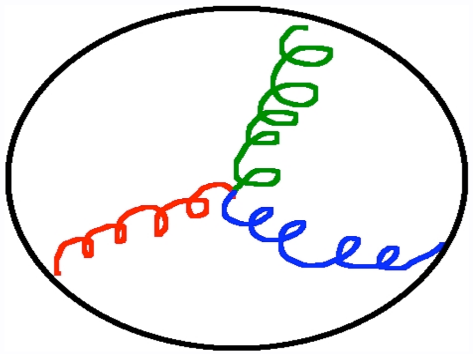
Becomes (in the vacuum sector)
the SU(2) XXX spin chain
on the circle with L spin sites
in the sector with
N spins flipped



Dictionary

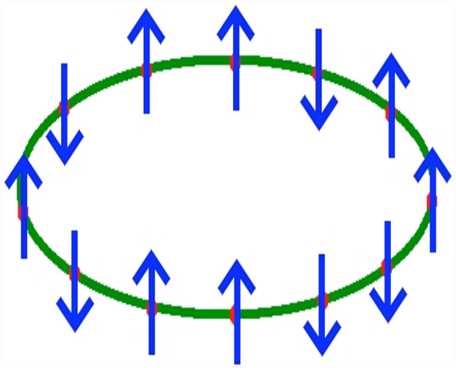
Eigenstates of the spin chain Hamiltonian(s) are in one-to-one correspondence with the supersymmetric vacua of the gauge theory

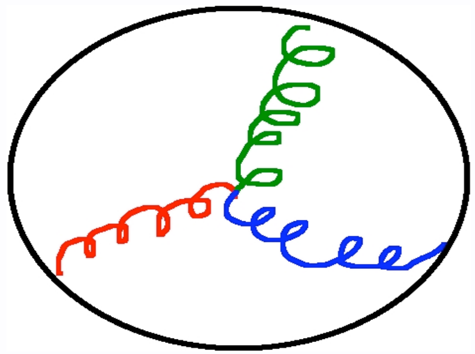




Dictionary

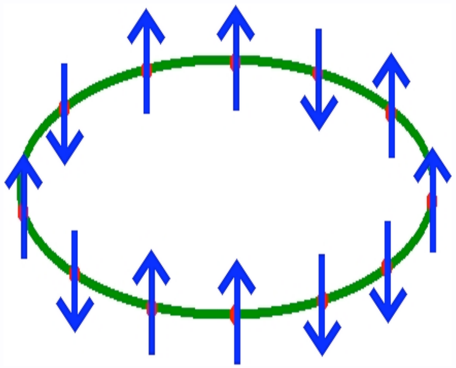
Eigenvalues of the spin chain Hamiltonians
coincide with the
vacuum expectation values
of the chiral ring operators
of the susy gauge theory

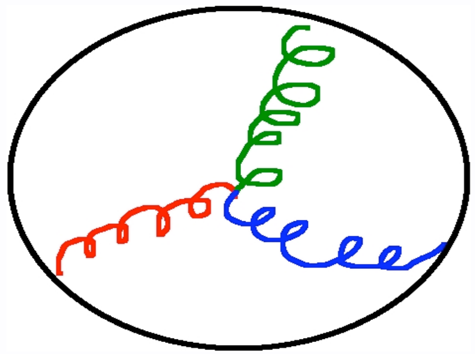




Dictionary

$$H_k \, \Psi_\lambda = E_k(\lambda) \, \Psi_\lambda \quad \Longleftrightarrow \quad E_k(\lambda) = \langle \, \lambda \, | \, \mathcal{O}_k \, | \, \lambda \, \rangle$$





Reminder on N=2 supersymmetry in two dimensions: Basic multiplets

Vector multiplet contains a gauge field and adjoint complex scalar;

Chiral multiplet contains a (charged) complex scalar (plus auxiliary bosonic field);

Twisted chiral multiplet contains a complex scalar and a gauge field strength;

Reminder on N=2 supersymmetry in two dimensions: Basic Multiplets

Vector multiplet contains a gauge field and a complex scalar

$$V = \theta^- \overline{\theta}^- (v_0 - v_1) + \theta^+ \overline{\theta}^+ (v_0 + v_1) - \sqrt{2}\sigma\theta^- \overline{\theta}^+ - \sqrt{2}\overline{\sigma}\theta^+ \overline{\theta}^-$$
$$+ 2i\theta^- \theta^+ \left(\overline{\theta}^- \overline{\lambda}_- + \overline{\theta}^+ \overline{\lambda}_+ \right) + 2i\overline{\theta}^+ \overline{\theta}^- (\theta^+ \lambda_+ + \theta^- \lambda_-) + 2\theta^- \theta^+ \overline{\theta}^+ \overline{\theta}^- D.$$

Reminder on N=2 supersymmetry in two dimensions: Basic Multiplets

Chiral multiplet contains a complex scalar (plus auxiliary bosonic field);

$$\Phi = \phi + \sqrt{2}\theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha F.$$

Reminder on N=2 supersymmetry in two dimensions: Basic Multiplets

Chiral multiplet contains a complex scalar (plus auxiliary bosonic field);

$$\overline{\mathcal{D}}_\alpha \Phi = 0.$$

Reminder on
N=2 supersymmetry in two dimensions: Basic Multiplets

Chiral multiplets will be denoted by

$$\mathbf{Q},\ \widetilde{\mathbf{Q}},\ \text{and}\ \mathbf{\Phi}$$

Reminder on N=2 supersymmetry in two dimensions: Basic Multiplets

Twisted chiral multiplet contains a gauge field strength and a complex scalar

$$\Sigma = \frac{1}{2\sqrt{2}}\{\overline{\mathcal{D}}_+, \mathcal{D}_-\} = \sigma + i\sqrt{2}\theta^+\overline{\lambda}_+ - i\sqrt{2}\overline{\theta}^-\lambda_- + \sqrt{2}\theta^+\overline{\theta}^- D$$

$$\ldots\ldots \qquad - i\sqrt{2}\theta^+\overline{\theta}^- F_{01}$$

$$\overline{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0.$$

Reminder on N=2 supersymmetry in two dimensions: Basic Multiplets

Twisted chiral multiplet contains a gauge field strength and a complex scalar:

Full expansion

$$\Sigma = \frac{1}{2\sqrt{2}}\{\overline{\mathcal{D}}_+, \mathcal{D}_-\} = \sigma + i\sqrt{2}\theta^+\overline{\lambda}_+ - i\sqrt{2}\overline{\theta}^-\lambda_- + \sqrt{2}\theta^+\overline{\theta}^- D$$
$$- i\overline{\theta}^-\theta^-(D_0 - D_1)\sigma - i\theta^+\overline{\theta}^+(D_0 + D_1)\sigma$$
$$+ \sqrt{2}\overline{\theta}^-\theta^-\theta^+(D_0 - D_1)\overline{\lambda}_+ - \sqrt{2}\theta^+\overline{\theta}^+\overline{\theta}^-(D_0 + D_1)\lambda_- - i\sqrt{2}\theta^+\overline{\theta}^- F_{01}$$
$$- 2i\theta^-\overline{\theta}^-\theta^+[\sigma, \overline{\lambda}_-] - 2i\overline{\theta}^-\theta^+\overline{\theta}^+[\sigma, \lambda_+]$$
$$- \overline{\theta}^-\theta^-\theta^+\overline{\theta}^+\left((D_0{}^2 - D_1{}^2)\sigma - [\sigma, [\sigma, \overline{\sigma}]]\right) + i\theta^-\overline{\theta}^-\theta^+\overline{\theta}^+[\sigma, \partial_m v^m].$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Gauge kinetic terms

$$
\begin{aligned}
L_g &= -\frac{1}{4e^2}\int d^2x\, d^4\theta\, \mathrm{Tr}\,\overline{\Sigma}\Sigma \\
&= \frac{1}{e^2}\int d^2x\ \mathrm{Tr}\left(\frac{1}{2}F_{01}^2 + |D_0\sigma|^2 - |D_1\sigma|^2 + i\overline{\lambda}_-(D_0+D_1)\lambda_- + i\overline{\lambda}_+(D_0-D_1)\lambda_+ \right. \\
&\qquad \left. +\frac{1}{2}D^2 - \frac{1}{2}[\sigma,\overline{\sigma}]^2 - \sqrt{2}\lambda_+[\sigma,\overline{\lambda}_-] + \sqrt{2}[\overline{\sigma},\lambda_-]\overline{\lambda}_+\right).
\end{aligned}
$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Fayet-Iliopoulos and theta terms

$$\mathcal{L}_{\mathrm{FI},\theta} = \frac{i\,\tau}{4}\int \mathrm{d}^2\widetilde{\theta}\ \mathrm{Tr}\Sigma\ +\ h.c.$$

$$= \int d^2x \left(-r\,\mathrm{Tr}\,D + \frac{\theta}{2\pi}\,\mathrm{Tr}\,F_{01}\right)$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Fayet-Iliopoulos and theta terms

$$\mathcal{L}_{\mathrm{FI},\theta} = \frac{i\,\tau}{4}\int \mathrm{d}^2\widetilde{\theta}\ \mathrm{Tr}\Sigma\ +\ h.c.$$

Give an example of the twisted superpotential

$$\tau = ir + \theta/2\pi .$$

Reminder on
N=2 supersymmetry in two dimensions: Lagrangians

Matter kinetic terms

$$L_{ch} = \frac{1}{4}\int d^2x\, d^4\theta\ \overline{\Phi}\Phi$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Matter kinetic terms

$$
= \int d^2x \; \Big(|D_0\phi|^2 - |D_1\phi|^2 + |F|^2 + i\overline{\psi}_+(D_0 - D_1)\psi_+
$$

$$
+ i\overline{\psi}_-(D_0 + D_1)\psi_- + \overline{\phi}D\phi - \overline{\phi}\{\sigma, \overline{\sigma}\}\phi
$$

$$
-\sqrt{2}\overline{\psi}_+\overline{\sigma}\psi_- - \sqrt{2}\overline{\psi}_-\sigma\psi_+ + i\sqrt{2}\overline{\psi}_+\overline{\lambda}_-\phi - i\sqrt{2}\overline{\psi}_-\overline{\lambda}_+\phi
$$

$$
+ i\sqrt{2}\overline{\phi}\lambda_+\psi_- - i\sqrt{2}\overline{\phi}\lambda_-\psi_+ \Big).
$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Superpotential terms

$$L_W = -\int \mathrm{d}^2y \mathrm{d}\theta^+ \mathrm{d}\theta^- \; W(\Phi_i)|_{\overline{\theta}^+=\overline{\theta}^-=0} - h.c.$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Superpotential terms

$$L_W = -\int \mathrm{d}^2 y \left(F_i \frac{\partial W}{\partial \phi_i} + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_{-,i} \psi_{+,j} \right) - h.c.$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted superpotential terms

$$\Delta L = \int \mathrm{d}^2 y \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^- \ \widetilde{W}(\Sigma)\Big|_{\theta^- = \overline{\theta}^+ = 0} + h.c.$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted superpotential terms

$$= \int \mathrm{d}^2 y \left(\sqrt{2}\widetilde{W}'(\sigma)(D - i v_{01}) + 2\widetilde{W}''(\sigma)\overline{\lambda}_+ \lambda_- \right) + h.c.$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted mass terms

$$\mathcal{L}_{\widetilde{\text{mass}}} = \int \mathrm{d}^4\theta \ \mathrm{tr}\left(\mathbf{\Phi}^\dagger e^{\widetilde{V}} \otimes \mathrm{Id}_{\text{color space}} \mathbf{\Phi}\right)$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted mass terms

$$\mathcal{L}_{\widetilde{\text{mass}}} = \int \mathrm{d}^4\theta \;\; \mathrm{tr}\left(\mathbf{\Phi}^\dagger e^{\widetilde{V}} \otimes \mathrm{Id}_{\text{color space}} \mathbf{\Phi}\right)$$

where $\widetilde{V} = \widetilde{m}\,\theta_+\overline{\theta}_-$, $\widetilde{m}$ acts in a flavour space, and, to preserve susy:

$$[\widetilde{m}, \widetilde{m}^*] = 0$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted mass terms

$$\mathcal{L}_{\widetilde{\text{mass}}} = \int \mathrm{d}^4\theta \;\; \mathrm{tr}\left(\Phi^\dagger e^{\widetilde{V}} \otimes \mathrm{Id}_{\text{color space}} \Phi\right)$$

$$\widetilde{V} = \widetilde{m}\,\theta_+\overline{\theta}_-$$

- Background vector field for global symmetry

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Cf. the ordinary mass terms

$$\mathcal{L}_{mass} = \sum_{i,\tilde{j}} \int \mathrm{d}^2\theta \; m_i^{\tilde{j}} \widetilde{Q}_{\tilde{j}} Q^i + h.c.\,,$$

Which are just the superpotential terms

General strategy

Take an $\mathcal{N}$=2 d=2 gauge theory with matter,
in some representations $\mathbf{R_f}$
of the gauge group $\mathbf{G}$
<u>integrate</u> <u>out</u> the massive matter fields,
compute
the effective twisted super-potential
on the Coulomb branch

$$W_{\text{eff}} = \sum_f \text{Tr}_{R_f}\, (\sigma + m_f)\left(\log(\sigma + m_f) - 1\right) + \; 2\pi i \tau\, \text{Tr}\sigma$$

Vacua of the gauge theory

For G = U(N)

$$\sigma \to \mathrm{diag}\,(\sigma_1, \ldots, \sigma_N)$$

$$\frac{\partial W_{\mathrm{eff}}}{\partial \sigma_i} = \lambda_i - i + \tfrac{1}{2}\,(N+1)$$

Due to quantization of the gauge flux

$$\lambda_i \in \mathbf{Z}$$

Familiar example:
CP^N *model*

Field content:

$(\mathbf{N}+1)$ *chiral multiplet of charge* $+1$

Q^i $i=1, \ldots, \mathbf{N}+1$

$U(1)$ *gauge group*

σ *is a scalar*

Familiar example:
CP^N model

Effective twisted superpotential

(d'Adda, A.Luscher, di Vecchia)

$$(\mathbf{N}+1)\sigma(\log\sigma - 1) + 2\pi i\tau\,\sigma$$

N+1 vacuum

$$\sigma^{\mathbf{N}+1} = e^{2\pi i\tau}$$

Quantum cohomology

More interesting example

Field content

Gauge group: **G=U(N)** $\sigma \to \mathrm{diag}\,(\sigma_1, \ldots, \sigma_N)$

Matter chiral multiplets:

1	<u>Adjoint</u>	twisted mass	M
N_f	<u>fundamentals</u>	... mass	m_f
$N_{\overline{f}}$	<u>anti-fundamentals</u>	... mass	$m_{\overline{f}}$

More interesting example

Effective superpotential:

$$N_f\left(\sigma_i+m_f\right)\left(\log\left(\sigma_i+m_f\right)-1\right)\ +$$

$$N_{\bar{f}}(-\sigma_i+m_{\bar{f}})\left(\log(-\sigma_i+m_{\bar{f}})-1\right)\ +$$

$$\sum_{i,j}\left(\sigma_i-\sigma_j+M\right)\left(\log\left(\sigma_i-\sigma_j+M\right)-1\right)$$

$$+(N_{\bar{f}}-N_f)\log\Lambda\sum_i\sigma_i$$

More interesting example

Equations for vacua:

$$\Lambda^{N_{\bar{f}}-N_f} \frac{\left(\sigma_i + m_f\right)^{N_f}}{\left(-\sigma_i + m_{\bar{f}}\right)^{N_{\bar{f}}}} = \prod_{j \neq i} \frac{\sigma_i - \sigma_j + M}{\sigma_i - \sigma_j + M}$$

More interesting example

Non-anomalous, UV finite case.

$$N_f = N_{\overline{f}} = L$$

More interesting example

Non-anomalous, UV finite case.

$$N_f = N_{\bar{f}} = L$$

Redefine:

$$\sigma_j = \tfrac{1}{2}\left(m_{\bar{f}} - m_f\right) - iM\lambda_j$$

$$\frac{1}{2}\left(m_{\bar{f}} + m_f\right) = Ms$$

Vacua of gauge theory

$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$

Vacua of gauge theory

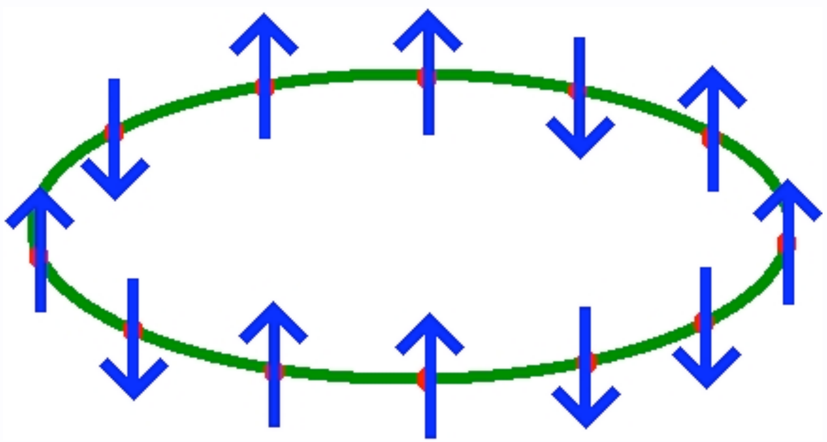
$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \times e^t$$

$$t = r + i\vartheta$$

Gauge theory - spin chain

Identical to the Bethe equations for spin s XXX magnet

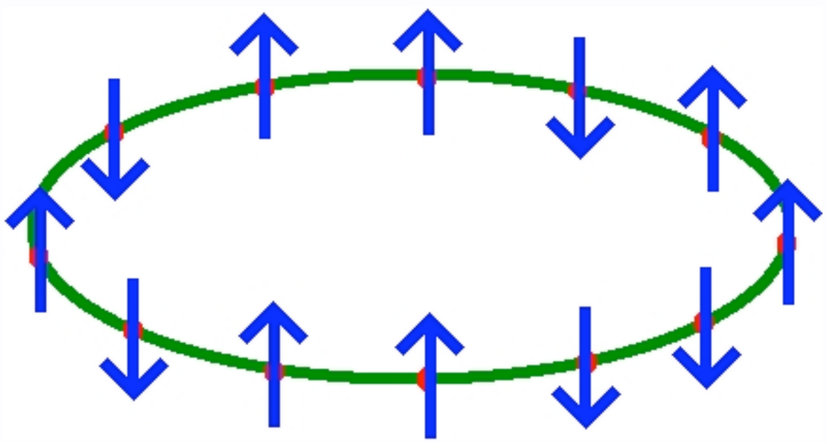
$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$



Gauge theory - spin chain

$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j\neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \times e^t$$

Identical to the Bethe equations for spin s XXX magnet
With twisted boundary conditions



Gauge theory - spin chain

Gauge theory vacua - eigenstates of the spin Hamiltonian (transfer-matrix)

$$H = \sum_{n=1}^{L} \sigma_n^a \otimes \sigma_{n+1}^a$$

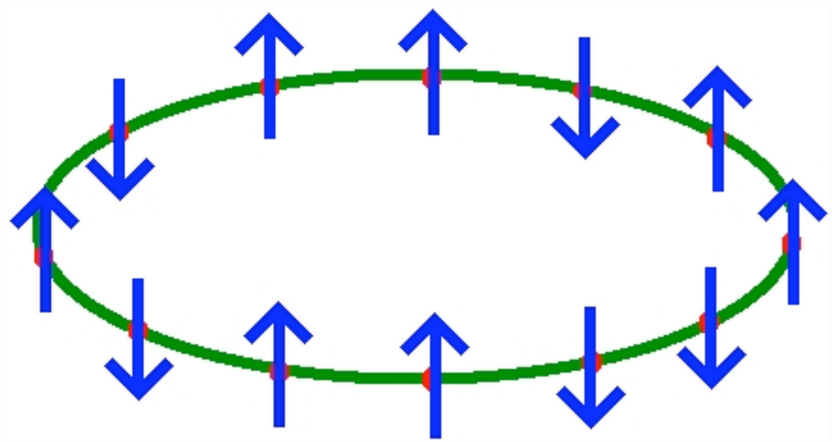


Table of dualities

XXX spin chain
SU(2)
L spins
N excitations

$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j\neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$

U(N) d=2 $\mathcal{N}$=2
Chiral
multiplets:
1 adjoint
L fundamentals
L anti-fund.

NB: Special masses, to be explained

Table of dualities

XXZ spin chain

SU(2)

L spins

N excitations

$$\left(\frac{\sinh(\lambda_i + is\gamma)}{\sinh(\lambda_i - is\gamma)}\right)^L = \prod_{j\neq i} \frac{\sinh(\lambda_i - \lambda_j + i\gamma)}{\sinh(\lambda_i - \lambda_j - i\gamma)}$$

U(N) d=3 N=2

Compactified on a circle

Chiral multiplets:

1 adjoint

L fundamentals

L anti-fund.

Special masses again

Table of dualities

XYZ spin chain

SU(2), L = 2N spins

N excitations

$$H = \sum_{n=1}^{L} J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

U(N) d=4 $\mathcal{N}$=1

Compactified on a 2-torus = elliptic curve E

Chiral multiplets:

1 adjoint

L = 2N fundamentals

L = 2N anti-fund.

Masses = wilson loops of the flavour group

Table of dualities

XYZ spin chain

SU(2), L = 2N spins

N excitations

$$H = \sum_{n=1}^{L} J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

U(N) d=4 N=2

Compactified on a 2-torus = elliptic curve E

L = 2N fundamental hypermultiplets

Softly broken down to N=1 by the wilson loops of the global symmetry group = flavour group U(L) X U(1) = points on the Jacobian of E

Table of dualities

It is remarkable that the spin chain has
precisely those generalizations:
rational (XXX), trigonometric (XXZ) and elliptic (XYZ)
that can be matched to the 2, 3, and 4 dim cases.

$$H = \sum_{n=1}^{L} J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

$$J_x = 1 + k\, \mathrm{sn}^2\, 2\eta, \quad J_y = 1 - k\, \mathrm{sn}^2\, 2\eta, \quad J_z = \mathrm{cn}\, 2\eta\, \mathrm{dn}\, 2\eta$$

Table of dualities

The L fundamentals and L anti-fundamentals can have different twisted masses

Field	Twisted mass
$\widetilde{Q}_a$	$\mu_a + i s_a M$
Q^a	$-\mu_a + i s_a M$
Φ	$-iM$

This theory maps to inhomogeneous spin chain with different spins at different sites

Table of dualities

Yang-Yang counting function =
effective twisted superpotential

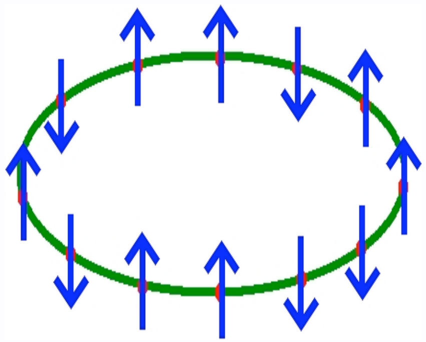
Table of dualities

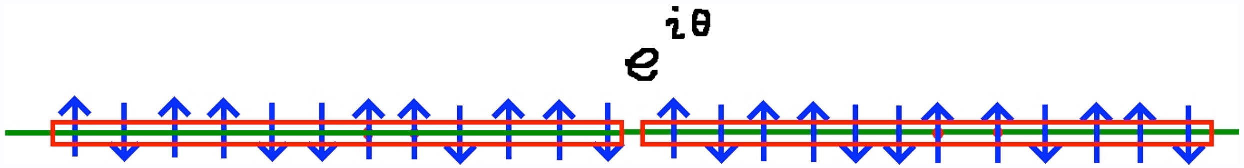
Commuting hamiltonians (expansion of transfer matrix) =
the chiral ring generators, like

$$\mathrm{Tr}\,\sigma^{M}$$

Table of dualities

Gauge theory theta angle (complexified)
is mapped to the spin chain theta angle
(twisted boundary conditions)





Algebraic Bethe Ansatz

Faddeev et al.

The spin chain is solved algebraically using certain operators,

$$A(\lambda), B(\lambda), C(\lambda), D(\lambda)$$

obeying exchange commutation relations

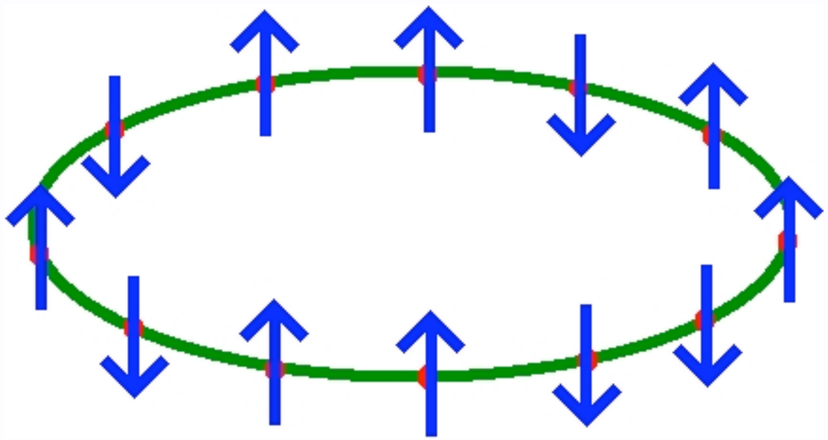
Algebraic Bethe Ansatz

The eigenvectors, Bethe vectors, are obtained by applying these operators to the (pseudo)vacuum.

$$\Psi_{\vec{\lambda}} = B(\lambda_1)B(\lambda_2)\dots B(\lambda_N)\Omega$$

Algebraic Bethe Ansatz vs GAUGE THEORY

For the spin chain it is natural to fix L = total number of spins
and consider various N = excitation levels



In the gauge theory context N is fixed.

Algebraic Bethe Ansatz vs STRING THEORY

However, if the theory is embedded into string theory via brane realization

then changing N is easy: bring in an extra brane.

One might use the constructions of Witten'96, Hanany-Hori'02

Algebraic Bethe Ansatz
vs STRING THEORY

THUS:

$B(\lambda)$ is for BRANE!

λ is for location!

Are these models too special, or the gauge theory/integable lattice model correspondence is more general?

Actually, virtually any Bethe ansatz soluble system can be mapped to a <u>$\mathcal{N}=2$ $d=2$ gauge theory</u>:

General spin group $\mathcal{H}$,

8-vertex model,

Hubbard model, ...

More general spin chains

The SU(2) spin chain
has generalizations to
other groups and representations.

Quoting the (nested) Bethe ansatz
equations from N.Reshetikhin

General groups/reps

For simply-laced group $\mathcal{H}$ of rank r

$$N \longrightarrow \sum_{\mathbf{i}=1}^{r} N_{\mathbf{i}} \ , \ L \longrightarrow \sum_{\mathbf{i}=1}^{r} L_{\mathbf{i}}$$

$$i \longrightarrow (\alpha, \mathbf{i}) \, ; \, \mathbf{i} = 1, \ldots, r \ , \ \alpha = 1, \ldots, N_{\mathbf{i}}$$

General groups/reps

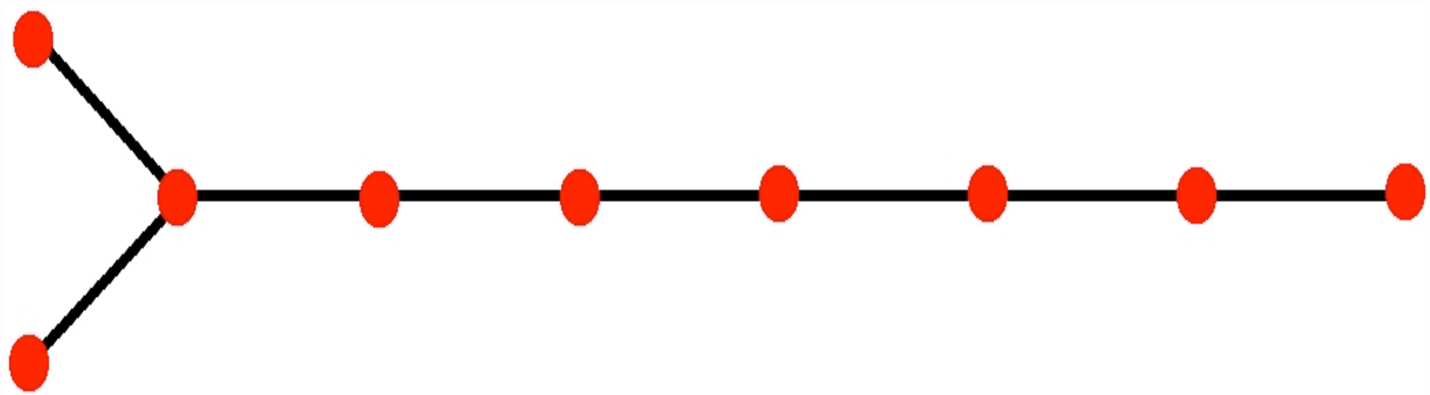
For simply-laced group $\mathcal{H}$ of rank r

$$\prod_{a=1}^{L_{\mathbf{i}}} \frac{\lambda_\alpha^{(\mathbf{i})} - \mu_a^{(\mathbf{i})} + s_a^{(\mathbf{i})}}{\lambda_\alpha^{(\mathbf{i})} - \mu_a^{(\mathbf{i})} - s_a^{(\mathbf{i})}} = \prod_{\mathbf{j}=1}^{r} \prod_{\beta}{}' \frac{\lambda_\alpha^{(\mathbf{i})} - \lambda_\beta^{(\mathbf{j})} + C_{\mathbf{ij}}}{\lambda_\alpha^{(\mathbf{i})} - \lambda_\beta^{(\mathbf{j})} - C_{\mathbf{ij}}}$$

$(\mu_a^{\mathbf{i}}, s_a^{\mathbf{i}})$ Label representations of the Yangian of $\mathcal{H}$: Kirillov-Reshetikhin modules

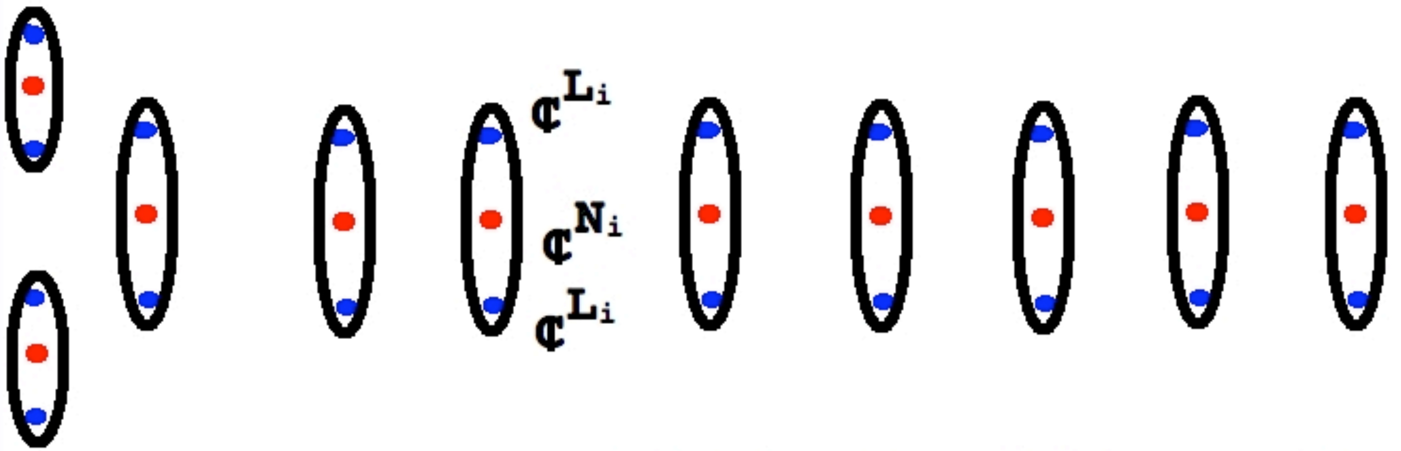
$C_{\mathbf{ij}}$ Cartan matrix of $\mathcal{H}$

General groups/reps from GAUGE THEORY



QUIVER GAUGE THEORY

Symmetries

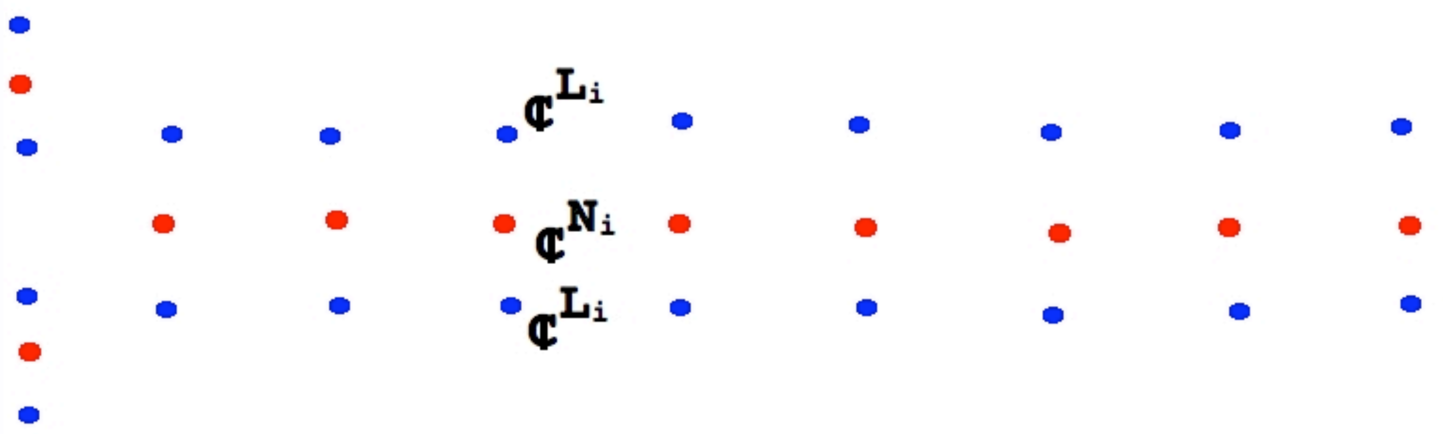


gauge group: $U(N_1) \times U(N_2) \times \ldots. \times U(N_r)$

flavor group: $U(L_1)^2 \times U(L_2)^2 \times \ldots. \times U(L_r)^2$

QUIVER GAUGE THEORY

Symmetries



gauge group: $U(N_1) \times U(N_2) \times \ldots \times U(N_r)$

flavor group: $U(L_1)^2 \times U(L_2)^2 \times \ldots \times U(L_r)^2$

QUIVER GAUGE THEORY
Charged matter

Adjoint chiral multiplet

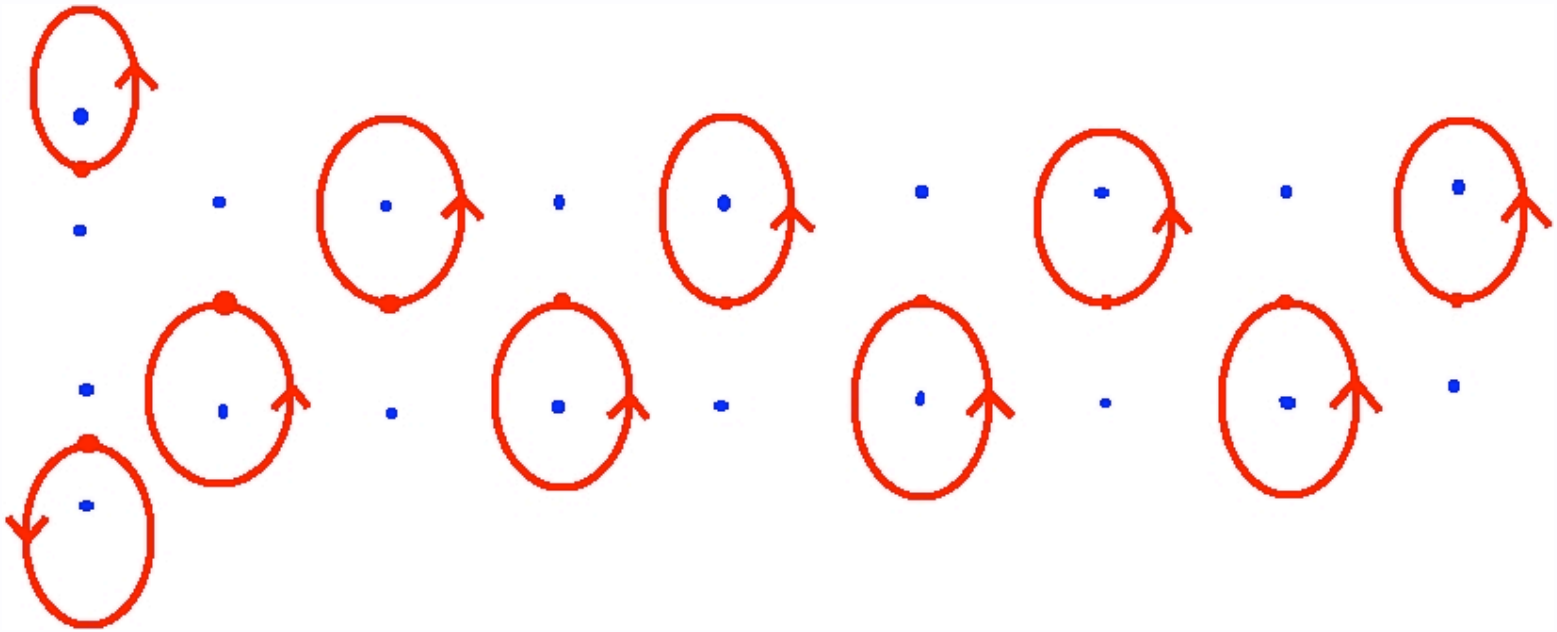
Fundamental chiral multiplet

Anti-fundamental chiral multiplet

Bi-fundamental chiral multiplet

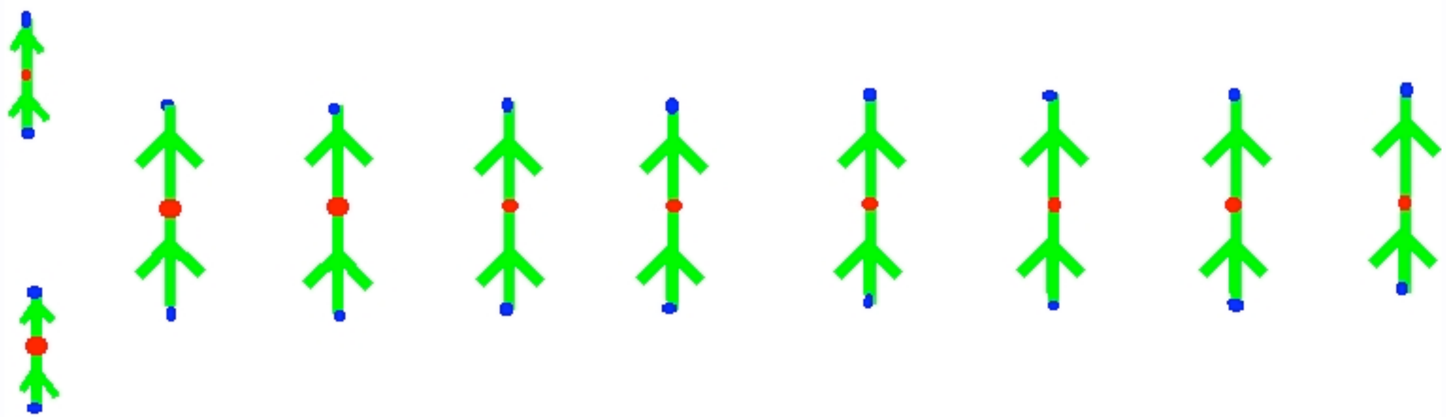
QUIVER GAUGE THEORY

Matter fields: adjoints



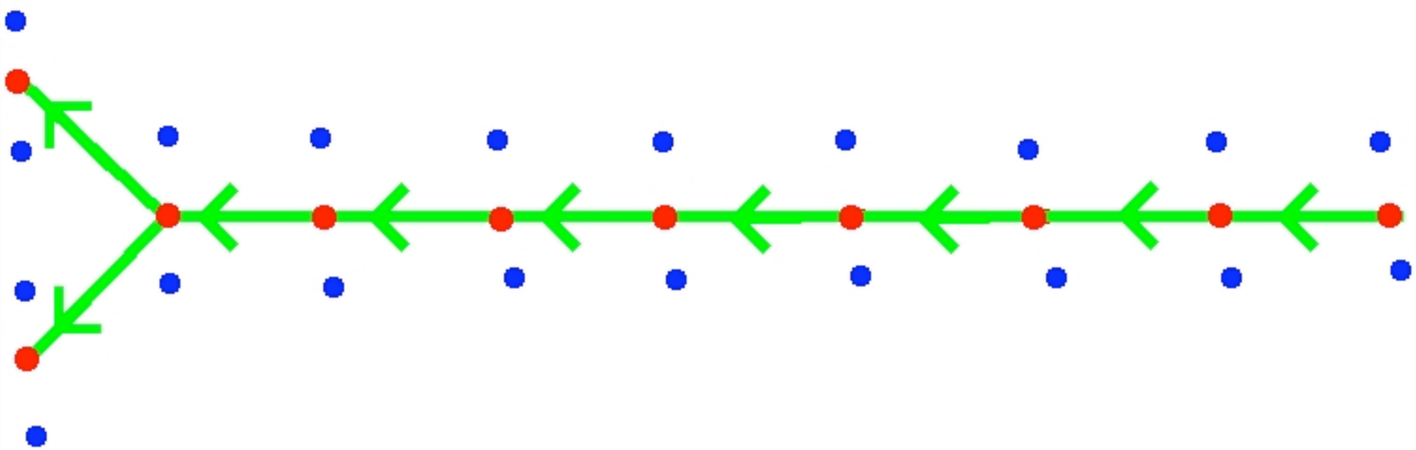
QUIVER GAUGE THEORY

Matter fields: fundamentals +
antifundamentals



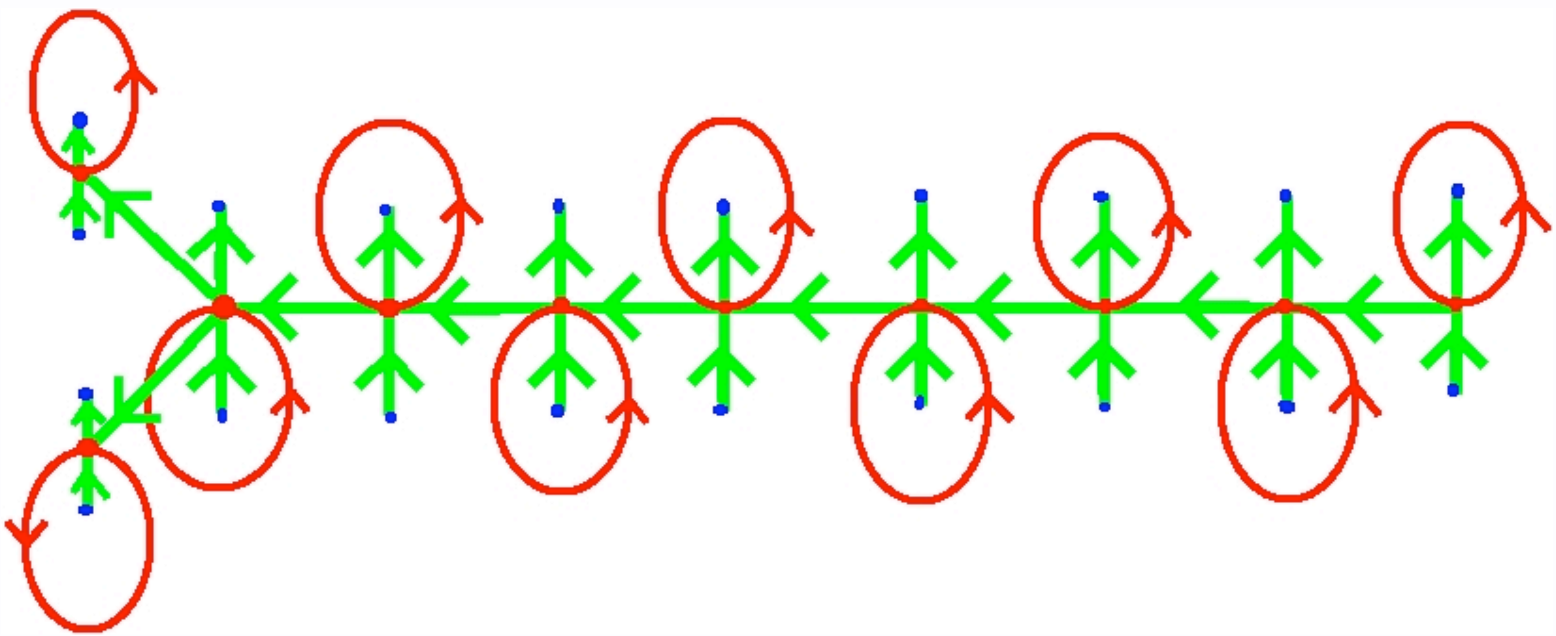
QUIVER GAUGE THEORY

Matter fields: bi-fundamentals



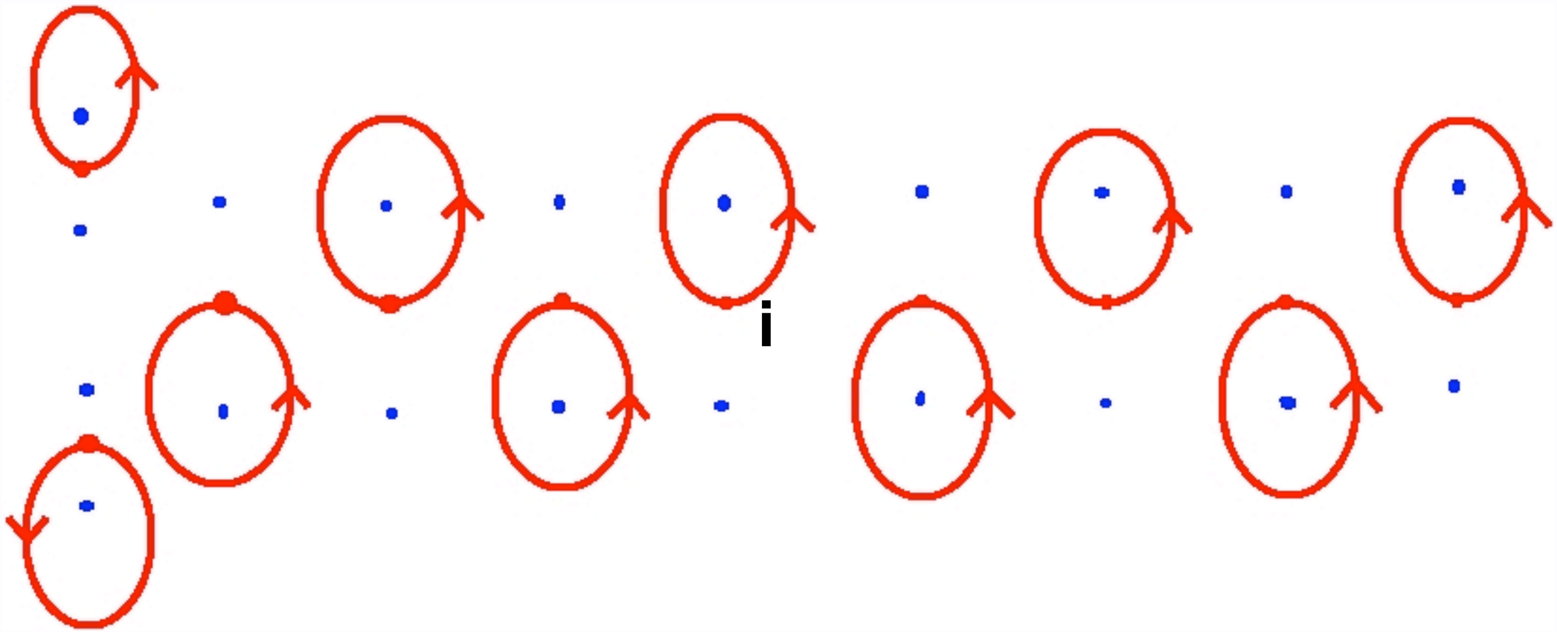
QUIVER GAUGE THEORY

Full assembly in the N=2 d=2 language



QUIVER GAUGE THEORY: twisted masses

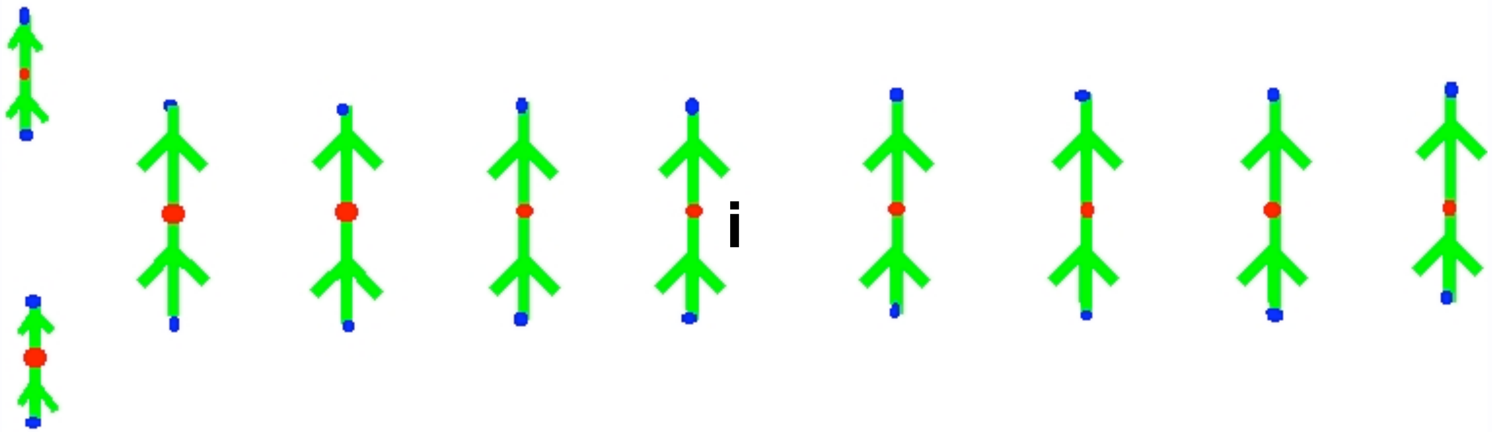
Adjoints MC_{ii}



QUIVER GAUGE THEORY: twisted masses

fundamentals
anti-fundamentals

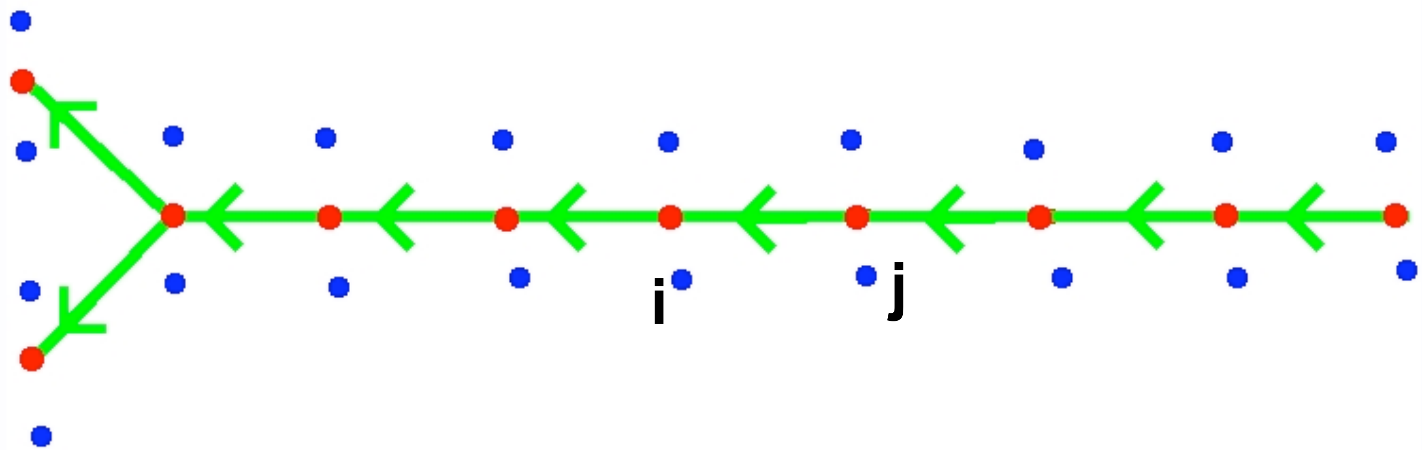
$$M(\mu_a^{(\mathbf{i})} \pm s_a^{(\mathbf{i})})$$



$a = 1, \ldots., L_i$

QUIVER GAUGE THEORY: twisted masses

Bi-fundamentals MC_{ij}



What is so special about all these masses?

The twisted masses correspond to symmetries.

Symmetries are restricted by the e.g. superpotential deformations

The N=2 d=4 (N=4 d=2) superpotential

$$W_0 = \sum_{a=1}^{L} \operatorname{tr} \widetilde{Q}_a \Phi Q^a$$

Has a symmetry

$$Q^a \mapsto e^{i\beta_a - i\frac{1}{2}\gamma} Q^a \,, \ \widetilde{Q}_b \mapsto e^{-i\beta_b - i\frac{1}{2}\gamma} \,, \Phi \mapsto e^{i\gamma} \Phi$$

$$W_0 = \sum_{a=1}^{L} \mathrm{tr}\, \widetilde{Q}_a \Phi Q^a$$

It is this symmetry

$$Q^a \mapsto e^{i\beta_a - i\frac{1}{2}\gamma} Q^a \,, \; \widetilde{Q}_b \mapsto e^{-i\beta_b - i\frac{1}{2}\gamma} \,, \Phi \mapsto e^{i\gamma} \Phi$$

which
explains the ratio of
adjoint and
Fundamental masses

Similarly, we should ask:

Why choose s_a

Half-integral

in the table

Field	Twisted mass
$\widetilde{Q}_a$	$\mu_a + i s_a M$
Q^a	$-\mu_a + i s_a M$
Φ	$-iM$

?

The answer is that one can turn on more general superpotential

(ONLY N=2 D=2 IS PRESERVED)

$$W_{\widetilde{Q}\Phi Q} = \sum_{a=1}^{L} u_a \, \widetilde{Q}_a \Phi^{2s_a} Q^a$$

Which has a symmetry

$$Q^a \mapsto e^{i\beta_a - is_a\gamma} Q^a \, , \, \widetilde{Q}_b \mapsto e^{-i\beta_b - is_b\gamma} \, , \Phi \mapsto e^{i\gamma} \Phi$$

Hubbard model
Bethe ansatz

$$e^{ik_\alpha L} = \prod_{l=1}^{M} \frac{\lambda_l - \sin k_\alpha + iU/4}{\lambda_l - \sin k_\alpha - iU/4}, \; j = 1, \ldots, N$$

$$\prod_{\alpha=1}^{N} \frac{\lambda_l - \sin k_\alpha - iU/4}{\lambda_l - \sin k_\alpha + iU/4} = \prod_{m=1}^{M} \frac{\lambda_l - \lambda_m - iU/2}{\lambda_l - \lambda_m + iU/2}, \; l = 1, \ldots, M$$

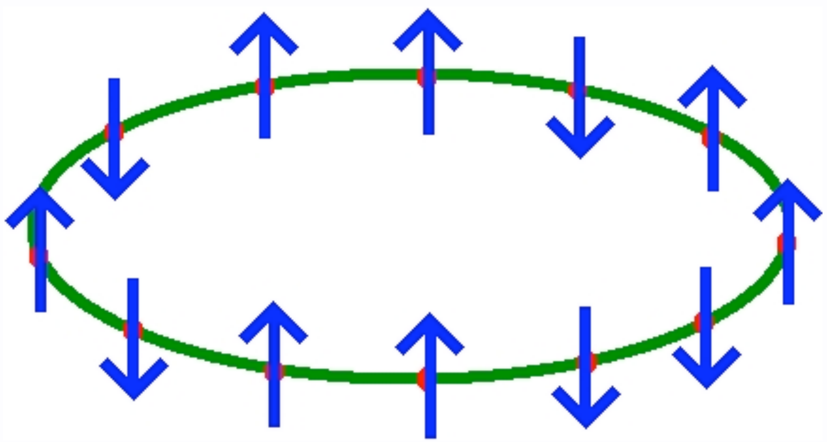
Lieb-Wu

The new ingredient is the coupling to a nonlinear sigma model

Mirrors to **CP**^{1}

$$
\begin{aligned}
-iW_{\vec{n},\vec{m}}(\vec{k},\vec{\mu},\vec{\sigma}) = \\
-i\ell\sum_{a=1}^{L}(k_a\mu_a + \cos k_a) \\
+\sum_{\alpha=1}^{N}\sum_{a=1}^{L}(\sigma_\alpha-\mu_a+u)\left[\log\left(\sigma_\alpha-\mu_a+u\right)-1\right]+ \\
+\sum_{\alpha=1}^{N}\sum_{a=1}^{L}(\mu_a-\sigma_\alpha+u)\left[\log\left(\mu_a-\sigma_\alpha+u\right)-1\right] \\
+\sum_{\alpha,\beta=1}^{N}(\sigma_\alpha-\sigma_\beta-2u)\left[\log\left(\sigma_\alpha-\sigma_\beta-2u\right)-1\right] \\
+2\pi i\sum_{a=1}^{L}n_a\mu_a+2\pi i\sum_{\alpha=1}^{N}m_\alpha\sigma_\alpha
\end{aligned}
$$

Finally, what is the meaning of the spins?



What is the meaning of Bethe wavefunction?

If time permits....

Further developments: Instanton corrected Bethe Ansatz equations

Instanton corrected Bethe Ansatz equations

Consider

$\mathcal{N}=2^*$ theory on $\mathbf{R}^2 \times \mathbf{S}^2$

With a partial twist along the two-sphere

One gets a deformation of the Yang-Mills-Hitchin theory (introduced in **Moore-NN-Shatashvili'97**)

(if $\mathbf{R}^2$ is replaced by a Riemann surface)

Twisted superpotential from prepotential

Tree level part

$$\widetilde{W}_{\mathrm{eff}}(a; m, \tau, \lambda) = \frac{\partial \mathcal{F}}{\partial m} + \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial \tau} + \sum_i \left(n_i a^i + m^i a_{D,i} \right)$$

Induced by twist

Flux superpotential (**Losev-NN-Shatashvili'97**)

Twisted superpotential from prepotential

Magnetic flux

$$m^i = \int_{\mathbf{S}^2} F^i$$

Electric flux

$$n_i = \lambda_i - i + \frac{1}{2}(N+1)$$

In the limit of vanishing S^2
the magnetic flux should vanish

Twisted superpotential from prepotential

The prepotential is known explicitly as a series in $q = e^{2\pi i \tau}$:

$$\mathcal{F} = \mathcal{F}^{\text{pert}} + q\mathcal{F}_1 + q^2\mathcal{F}_2 + \dots$$

$$\mathcal{F}^{\text{pert}} = \pi i \tau \sum_i a_i^2 + \frac{1}{2}\sum_{i\neq j} a_{ij}^2\left(\log(a_{ij}) - \frac{3}{2}\right) - (a_{ij}+m)^2\left(\log(a_{ij}+m) - \frac{3}{2}\right)$$

$$\mathcal{F}_1 = m^2 \sum_i T_i$$

$$\mathcal{F}_2 = \sum_i \left(-\frac{3m^2}{2}T_i^2 + \frac{m^4}{2}T_i T_i^{(2)}\right)$$

$$+ m^6 \sum_{i\neq j} \frac{T_i T_j}{a_{ij}^2}\frac{\left(m^2 - 3a_{ij}^2\right)}{(a_{ij}^2 - m^2)^2}$$

$$T_i(s) = \prod_{j\neq i}\left(1 - \frac{m^2}{(s+a_{ij})^2}\right) = T_i + \frac{s^2}{2}T_i^{(2)} + o(s^2)$$

Instanton corrected Bethe Ansatz equations

$$\exp\left\{2\pi i\frac{\partial^2\mathcal{F}}{\partial a_i\partial m}+\sum_k t_k\frac{du_k}{da_i}\right\}=1$$

We can read off an «S-matrix»
It contains 2-,3-, higher order interactions

Instanton corrected Bethe Ansatz equations

The prepotential of the low-energy effective theory is governed by a classical (holomorphic) integrable system

Donagi-Witten'95

Liouville tori = Jacobians of Seiberg-Witten curves

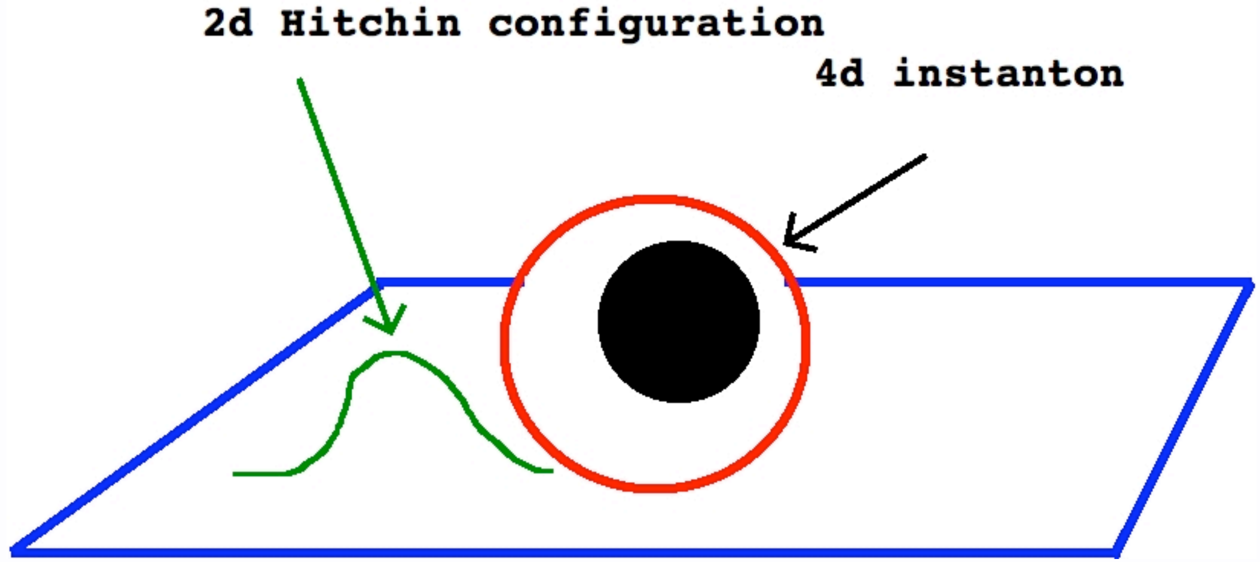
Classical integrable system
vs
Quantum integrable system

That system is quantized when
the gauge theory is subject to
the Omega-background

NN'02
NN-Okounkov'03
Braverman'03

Our quantum system is different!

Blowing up the two-sphere



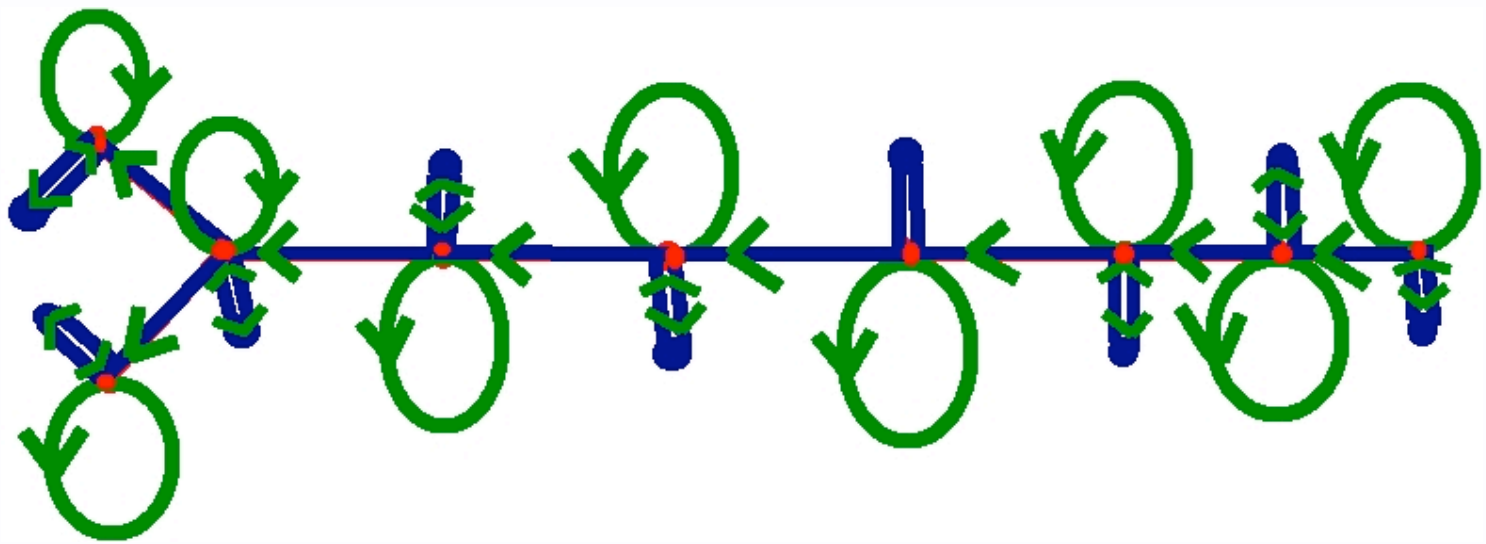
Wall-crossing phenomena
(new states, new solutions)

Something for the future

Remark

In the $s_a^{(i)} \to 0$ limit
the supersymmetry enhances

Cf. BA for QCD
Lipatov, Faddeev-Korchemsky'94



CONCLUSIONS

1. We found the Bethe Ansatz equations are the equations describing the vacuum configurations of certain quiver gauge theories in two dimensions
2. The duality to the spin chain requires certain relations between the masses of the matter fields to be obeyed. These masses follow naturally from the possibility to turn on the quasihomogeneous superpotentials (conformal fixed points)

CONCLUSIONS

3. The algebraic Bethe ansatz seems to provide a realization of the brane creation operators -- something of major importance both for topological and physical string theories

4. Obviously this is a beginning of a beautiful story....