Generalized Hodge structures and mirror symmetry

- joint with L. Katzarkov and M. Kontsevich
- will describe a proposal for the Hodge theory of nc spaces of Kähler type
- will discuss the Hodge theoretic implications of homological mirror symmetry.

1. nc Hodge structures.

$\mathbb{A}^1_{\mathbb{C}}$ - copy of the complex affine line,

u - affine coordinate on $\mathbb{A}^1_{\mathbb{C}}$.

$\mathbb{C}\{u\}$ - complex power series in u with a positive radius of convergence.

Def: A rational pure nc Hodge structure is a triple

$$(H, \mathcal{E}_B, \underline{iso})$$

where:

- H is a $\mathbb{Z}/2$-graded algebraic vector bundle on $\mathbb{A}^1_{\mathbb{C}}$ of finite rank.
- $\mathcal{E}_B$ is a $\mathbb{Z}/2$-graded local system of $\mathbb{Q}$-vector spaces on $\mathbb{A}^1_{\mathbb{C}} - \{0\}$.
- $\underline{iso} : \mathcal{E}_B \otimes \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}} - \{0\}} \to H^{alg}_{|\mathbb{A}^1_{\mathbb{C}} - \{0\}}$

is an analytic isomorphism of holomorphic vector bundles on $\mathbb{A}^1_{\mathbb{C}} - \{0\}$.

Note: iso gives rise to a natural holomorphic connection

$$\nabla : H_{|\mathbb{A}^1_{\mathbb{C}} - \{0\}} \to H_{|\mathbb{A}^1_{\mathbb{C}} - \{0\}} \otimes \Omega^1$$

These data should satisfy the following axioms

(nc filtration axiom) ∇ is a meromorphic connection on H with poles of order

≤ 2 at 0
≤ 1 at ∞

i.e. there exist:

- a holomorphic frame of H near $0 \in \mathbb{A}^1_{\mathbb{C}}$ so that in this frame

$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k \right) du$$

and $A_k \in \mathrm{Mat}_{r \times r}(\mathbb{C})$.

- a holomorphic frame of H near $\infty \in \mathbb{A}^1_{\mathbb{C}}$ so that in this frame

$$\nabla = d + \left(\sum_{k \geq -1} B_k u^{-k} \right) d(u^{-1})$$

and $B_k \in \mathrm{Mat}_{r \times r}(\mathbb{C})$.

(**$\mathbb{Q}$-structure axiom**)

The $\mathbb{Q}$-structure $\mathcal{E}_B$ on (H, ∇) is compatible with Stokes data:

- The connection (H, ∇) on $\mathbb{A}^1_{\mathbb{C}}$ gives a local system $\mathcal{S}$ of $\mathbb{Z}/2$-graded $\mathbb{C}$-vector spaces on the circle

$$S^1 := \mathbb{C}^\times / \mathbb{R}^\times_{>0}$$

(by contractibility of $\mathbb{R}^\times_{>0}$), and the isomorphism <u>iso</u> equips $\mathcal{S}$ with a $\mathbb{Q}$-structure

$$\mathcal{S}_B \subset \mathcal{S}.$$

- $\mathcal{S}$ comes equpped with a natural Deligne-Malgrange-Stokes filtration by subsheaves

$$\left\{ \mathcal{S}_{\leq \omega} \right\}_{\omega \in \mathbb{D}\mathrm{el}}$$

where

(i) $\mathbb{D}\mathrm{el}$ is the local system on S^1, s.t. for every $U \subset S^1$ open we have

$\mathrm{Del}(\mho) =$ $\mathbb{C}$-vector space of all holomorphic 1-forms ω on the sector

$$\mathrm{Sec}(\mho) := \{ r e^{i\varphi} \mid r>0,\ \varphi \in \mho \}$$

of the form

$$\omega = \Big(\sum_{\substack{a \in \mathbb{Q} \\ a < -1}} c_a u^a \Big) du$$

where at most finitely many $c_a \neq 0$ and the branches u^a are chosen arbitrarily.

Note: The germs of sections in Del are naturally ordered. If

$$\omega', \omega'' \in \mathrm{Del}(\mho)$$
$$\varphi \in \mho$$

and if

$$\omega' - \omega'' = c_a u^a + \begin{pmatrix} \text{higher order} \\ \text{terms} \end{pmatrix},$$

then

$$\omega' <_\varphi \omega'' \iff \mathrm{Re}\left(\frac{c_a e^{i\varphi(a+1)}}{a+1} \right) < 0 .$$

(ii) For every $\varphi \in S^1$ and every $\omega \in \mathrm{Del}_\varphi$ define the stalk

$$(\mathcal{S}_{\leq \omega})_\varphi \subset \mathcal{S}_\varphi$$

as the subspace

$$(\mathcal{S}_{\leq \omega})_\varphi := \left\{ s \in \Gamma(re^{i\varphi}, H)^\nabla \;\middle|\; \begin{array}{l} e^{-\int \omega} s \text{ has moderate} \\ \text{growth in the direction } \varphi \\ \text{i.e.} \\ \left\| e^{-\int \omega} s \right\|_{|re^{i\varphi}} \sim \underline{O}(|u|^N) \\ \text{for some } N \geq 0 \\ \text{when } r \to 0. \end{array} \right\}$$

Here $\|\cdot\|$ is the norm of a section of H computed in some (any) meromorphic trivialization of H near $u=0$.

<u>Note:</u> A fundamental theorem of Deligne - Malgrange asserts that the functor

$$(\mathcal{H}, \nabla) \longmapsto (\mathcal{S}, \{\mathcal{S}_{\leq \omega}\}_{\omega \in \mathrm{Del}})$$

is an equivalence between the category of germs of meromorphic connections at 0 and the category of Del-filtered local systems on S_1.

We are now ready to state the $\mathbb{Q}$-structure axiom:

• The Deligne-Malgrange-Stokes filtration on $\mathcal{S}$ is compatible with the rational structure i.e.

$$(\mathcal{S}_{\leq w} \cap \mathcal{S}_B) \otimes \mathbb{C} = \mathcal{S}_{\leq w}.$$

(Oppposednes axiom) The $\mathbb{Q}$ structure $\mathcal{S}_B$ induces a real structure on $\mathcal{S}$ and hence a complex conjugation

$$\tau : \mathcal{S} \longrightarrow \mathcal{S}$$

Consider $\hat{H}$ - the holomorphic bundle on $\mathbb{P}^1$ obtained as the gluing of

$$H_{|\{|u|\leq 1\}} \quad \text{and} \quad \gamma^*\left(\overline{H}_{|\{|u|\leq 1\}}\right)$$

via $\tilde{\iota}$. Here γ is the real structure on $\mathbb{P}^1$ given by

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1, \quad u \to \frac{1}{\bar{u}}.$$

The opposedness axiom requires that $\hat{H}$ is holomorphically trivializable on $\mathbb{P}^1$.

Note: There is an obvious notion of a morphism of nc Hodge structures, so we get a category

$$(\mathbb{Q}\text{-nc HS})$$

This category has some interesting subcategories. First we can look at the subcategory

$$(\mathbb{Q}\text{-nc HS})^{\text{reg}} \subset (\mathbb{Q}\text{-nc HS})$$

of all nc Hodge structures for which

$$(H, \nabla)$$

has a regular singularity at $u=0$.

Similarly we can look at nc Hodge structures for which (H, ∇) has some a special property. In particular we have the following notion:

Def: We say that a nc Hodge structure is of exponential type if we can find a formal isomorphism

$$(\mathcal{H}[u^{-1}], \nabla) \cong \bigoplus_{i=1}^{m} \mathcal{E}^{\frac{c_i}{u}} \otimes (\mathcal{R}_i, \nabla_i)$$

where

- $(\mathcal{R}_i, \nabla_i)$ meromorphic bundles over $\mathbb{C}\{u\}[u^{-1}]$ with a connection with regular singularity
- $c_1, \dots, c_m$ are the distinct eigenvalues of A_{-2}
- $\mathcal{E}^{\frac{c_i}{u}} = \left(\mathbb{C}\{u\}, d - d\left(\frac{c_i}{u}\right)\right)$ is a rank one irregular connection.

Note: By the Turrittin - Levelt formal decomposition theorem we know that a formal isomorphism of the above type exists always possibly after making a finite cyclic base change in u. So the notion of exponential type guarantees that a base change is not needed.

We now have a full subcategory of nc Hodge structures of exponential type:

$$\left(\mathbb{Q}\text{-ncHS}\right)^{reg} \subset \left(\mathbb{Q}\text{-ncHS}\right)^{exp} \subset \left(\mathbb{Q}\text{-ncHS}\right)$$

In fact in the exponentia type case one can state the nc Hodge structure axioms in an easier way. The simplification comes from the fact that in this case the Deligne - Malgrange - Stokes filtration is labelled by $\lambda \in \mathbb{R}$ and is easier to describe.

Def: A rational pure nc Hodge structure of exponential type is a triple

$$(H, \mathcal{E}_B, \underline{iso})$$

where H, $\mathcal{E}_B$, $\underline{iso}$ are as above and satisfy the following axioms

$(\text{nc filtration axiom})^{\exp}$ There exists a holomorphic frame of H near 0 so that in this frame

$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k \right) du$$

$$A_k \in \mathrm{Mat}_{r \times r}(\mathbb{C})$$

and there is a formal isomorphism

$$(\mathcal{H}[u^{-1}], \nabla) \cong \bigoplus_{i=1}^{m} \mathcal{E}^{\frac{c_i}{u}} \otimes (\mathcal{R}_i, \nabla_i)$$

with $(\mathcal{R}_i, \nabla_i)$ regular and $c_1, \ldots, c_m$ = distinct eigenvalues of A_{-2}

$(\mathbb{Q}\text{-structure axiom})^{\exp}$ The $\mathbb{Q}$-structure $\mathcal{S}_B$ on $\mathcal{S}$ induced by $\mathcal{E}_B$ and $\underline{iso}$ is compatible with the Stokes data in the following sense:

The filtration $(\mathcal{S}_{\leq\lambda})_{\lambda\in\mathbb{R}}$ of $\mathcal{S}$ by the subsheaves $\mathcal{S}_{\leq\lambda}$ whose stalk at $\varphi \in S^1$ is given by

$$(\mathcal{S}_{\leq\lambda})_\varphi := \left\{ s \;\middle|\; \begin{array}{l} s \in \Gamma(re^{i\varphi}, H)^\nabla \\ \| s(re^{i\varphi}) \| = \underline{O}\left(e^{\frac{\lambda + O(1)}{r}}\right) \\ a \quad r \longrightarrow 0 \end{array} \right\}$$

is defined over $\mathbb{Q}$, i.e.

$$(\mathcal{S}_{\leq\lambda} \cap \mathcal{S}_B) \otimes \mathbb{C} = \mathcal{S}_{\leq\lambda}$$

for all $\lambda \in \mathbb{R}$.

$(\text{opposedness axiom})^{\exp} = (\text{opposedness axiom})$

Comparison with classical Hodge theory:

Recall: a pure rational Hodge structure of weight w is the data

$$(V, F^\bullet V, V_\mathbb{Q})$$

where

- V is a $\mathbb{C}$-vector space
- $V_{\mathbb{Q}} \subset V$ - $\mathbb{Q}$-subspace
- $F^{\bullet}V$ - decreasing filtration on V by complex subspaces.

and these data should satisfy the following

($\mathbb{Q}$-structure axiom) $V_{\mathbb{Q}} \otimes \mathbb{C} = V$

(opposedness axiom) $gr^p_{F^\bullet}\, gr^q_{\bar{F}^\bullet} V = 0$ for all $p+q \neq w$

($\Leftrightarrow$ $F^p \cap \overline{F}^{w+1-p} = V$ $\forall p$).

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pure Hodge structure = $\oplus$ of pure HS of various weights.

morphism of pure HS = $\mathbb{C}$-linear maps respecting $V_{\mathbb{Q}}$ s.t. $f(F^p) = F^p \cap f(V)$.

Properties:

(i) The category $(\mathbb{Q}\text{-}HS)$ is a $\mathbb{Q}$-linear abelian category.

Note: The opposedness axiom guarantees that this category is abelian.

(ii) Pure rational HS are special cases of pure rational nc HS. Indeed, let

$$(V, F^\bullet V, V_{\mathbb{Q}})$$

be a pure HS of weight w.

Consider

- $$\mathcal{J}_{\frac{w}{2}} := \left(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}, \; d - \frac{w}{2}\frac{du}{u}\right)$$

- $H \longrightarrow \mathbb{A}^1_{\mathbb{C}}$ - the algebraic bundle corresponding to the Rees module

$$\xi(V, F^\bullet) = \sum_p u^{-p}\,\mathbb{C}[u]\,F^p$$
$$\cap$$
$$V \underset{\mathbb{C}}{\otimes} \mathbb{C}[u]$$

Note: • The Rees module $\xi(V, F^\bullet)$ is preserved by the connection

$$\nabla := \left(d - \frac{w}{2}\frac{du}{u}\right) \otimes id_V$$

i.e.

$$(H, \nabla)$$

is a logarithmic lattice on the holomorphic bundle with connection

$$\mathcal{J}_{\frac{w}{2}} \underset{\mathbb{C}}{\otimes} V$$

• H_1 = fiber of H at $1 = V$

∇ has monodromy $(-1)^w id_V$ and so ∇ preserves any rational structure on H.

Set $\mathcal{E}_B \subset H_{|\mathbb{A}^1_{\mathbb{C}} - \{0\}}$ to be the local system of locally flat sections of H whose value at $1 \in \mathbb{A}^1_{\mathbb{C}}$ is in $V_{\mathbb{Q}} \subset V$.

Let iso correspond to $\mathcal{E}_B \subset H_{|\mathbb{A}^1 - \{0\}}$.

The data $(H, \mathcal{E}_B, \underline{iso})$ satisfies tautologically the axioms for a nc Hodge structure of exponential type:

$(\text{nc filtration})^{exp}$ + $(\mathbb{Q}\text{-structure})^{exp}$ hold since ∇ has a regular singularity at 0 and so $S_{\leq\lambda} = S$ for all $\lambda \in \mathbb{R}$.

(opposedness) follows from the fact that $F^\bullet$ and $\bar{F}^\bullet$ are w-opposed.

Thus we get a functor

$$(\mathbb{Q}-HS) \longrightarrow (\mathbb{Q}-nc\ HS)^{exp}$$

which by construction factors through the orbit category

$$(\mathbb{Q}-HS) / (\bullet \otimes \mathbb{Q}(1))$$

That is we get a functor

$$(\mathbb{Q}\text{-}HS) \xrightarrow{\mathcal{N}} (\mathbb{Q}\text{-}ncHS)^{exp}.$$

One now has the following simple

Lemma: The functor $\mathcal{N}$ is fully faithful with essential image consisting of all nc Hodge structures having a regular singularity and monodromy

- id on $H^0 := V^{even}$
- $-$id on $H^1 := V^{odd}$

That is: Usual Hodge theory embeds in nc Hodge theory of exponential type modulo Tate twists.

This suggests that nc Hodge structures behave similarly to ordinary Hodge structures. In fact we have the following

Proposition: The categories

$$(\mathbb{Q}\text{-nc }HS)^{reg} \subset (\mathbb{Q}\text{-nc}HS)^{exp} \subset (\mathbb{Q}\text{-nc}HS)$$

are $\mathbb{Q}$-linear abelian categories.

The respective subcategories of polarizable nc Hodge structures are all semi-simple.

Historical Comments: Various partial versions and special cases of our notion of a nc Hodge structures have been studied before in different but related setups:

- in singularity theory in the work of Kyoji Saito (weight systems)
- in Conformal Field Theory in the work of Cecotti-Vafa (tt^* geometries)
- in mirror symmetry in the work of Barannikov and Kontsevich and (more recently) Iritani (semi infinite Hodge structures)

- in singularity theory in the work of Hertling (TER structures)
- in algebraic geometry in the work of Sabbah (pure twistor D-modules)

2. The main conjecture

The significance of the nc Hodge structures comes from the following:

Conjecture: The periodic cyclic homology of any smooth and compact $\mathbb{Z}/2$-graded dg category carries a natural functorial pure rational nc Hodge structure.

If the $\mathbb{Z}/2$-grading can be refined to a $\mathbb{Z}$-grading, then the nc Hodge structure is an ordinary Hodge structure.

Remarks: There are some natural candidates for the various ingredients of the nc Hodge structure that should be associated to a category.

(i) For simplicity look at a $\mathbb{Z}/2$-graded dg-category C which is affine, i.e.

$$C = (A\text{-mod})$$

for some $\mathbb{Z}/2$-graded dg algebra.

Recall: A perfect object of C is an object $E \in Ob\, C$, s.t. the functor

$$\mathrm{Hom}_C(E, \bullet) : C \to (\mathbb{C}\text{-Vect})$$

preserves all small colimits in C.

The category C is called

Smooth: if $A \in \mathrm{Perf}_{A \otimes A^{op}}$

Compact: if $\dim_{\mathbb{C}} H^{\bullet}(A, d) < +\infty$

Note: The smooth + compact properties of C do not depend on the choice of the algebra A.

In the case of an affine C one can compute the periodic cyclic homology of C as the periodic cyclic homology of A.

In particular we have

$$HP_\bullet(C) := HP_\bullet(A)$$

$$= H^\bullet\left(C^{per}_\bullet(A,A)((u)),\ \partial + uB\right)$$

Where:

- u is a formal variable
- $C^{per}_{-k+1}(A,A)((u)) = A \otimes \left(A/\mathbb{C}\cdot 1\right)^{\otimes k}$

  for all $k \geq 0$

- $\partial = b + \delta$, where

$b(a_0 \otimes \cdots \otimes a_n) :=$

$$\sum_{i=0}^{n-1} (-1)^{\deg(a_0 \otimes \cdots \otimes a_i)} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

$$+ (-1)^{\deg(a_0 \otimes \cdots \otimes a_n)(\deg a_n + 1) + 1} a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

is the Hochschild differential

(Here
$$\deg(a_1 \otimes \cdots \otimes a_n) = 1 - n + \sum_{i=1}^{n} \deg a_i \quad),$$

and

$\delta(a_0 \otimes \cdots \otimes a_n) :=$

$$\sum_{i=0}^{n} (-1)^{\deg(a_0 \otimes \cdots \otimes a_{i-1})} a_0 \otimes \cdots \otimes d_A a_i \otimes \cdots \otimes a_n$$

is the differential induced from d_A by the Leibnitz rule.

- $B(a_0 \otimes \cdots \otimes a_n) :=$
$$\sum_{i=0}^{n} (-1)^{(\deg(a_0 \otimes \cdots \otimes a_i)-1)(\deg(a_i \otimes \cdots \otimes a_n)-1)}$$
$$\cdot\ 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_i$$

is Connes cyclic differential.

Note that by definition $HP_\bullet(C)$ is a module over $\mathbb{C}((u))$ and hence a module over $\mathbb{C}\{u\}[u^{-1}]$.

Furthermore if we look at the family

$$(A((u)),\ u \cdot d_A)$$

of dg algebras parameterized by the u-line we get a natural Getzler-Gauss-Manin connection on the $\mathbb{C}\{u\}[u^{-1}]$ module $HP_\bullet(C)$

Now we can take (H, ∇) to be the Birkhoff extension of $(HP_\bullet(C) + \text{Gauss-Manin})$ to a meromorphic bundle on $\mathbb{A}^1$ with poles at $0 + \infty$ and with a regular singularity at ∞.

Conjecturally the specialization

$$HP_\bullet(C) \longmapsto HH_\bullet(C)$$

corresponding to $u \to 0$ is flat and so $\mathcal{H}$ is a well defined vector bundle on $\mathbb{A}^1_{\mathbb{C}}$.

This gives the data $(\mathcal{H}, \nabla)$ appearing in the definition of a nc Hodge structure.

(ii) The categorical origin of the rational (or integral) structure on $\mathcal{H}$ is much more mysterious.

There are two natural approaches to constructing $E_B \subset HP_\bullet(C)$.

(a) The soft algebra approach (Kontsevich)

Suppose again A is a $\mathbb{Z}/2$-graded dg algebra over $\mathbb{C}$ s.t.

$$C = (A\text{-mod})$$

Assume C is compact. Then there should exist a nuclear Fréchét algebra A_{C^∞} s.t.

- A_{C^∞} satisfies Bott periodicity, i.e.

$$K_i(A_{C^\infty}) \cong K_{i+2}(A_{C^\infty})$$

for all $i \geq 0$.

- There is an algebra homomorphism

$$A \xrightarrow{\varphi} A_{C^\infty}$$

for which:

$$\varphi_* : HP_\bullet(A) \xrightarrow{\sim} HP_\bullet(A_{C^\infty})$$

is an isomorphism

$$ch : K_{ev/odd}(A_{C^\infty}) \to HP_{ev/odd}(A_{C^\infty})$$

is an integral lattice.

(b) The semi-topological K-theory approach (Toën)

Consider again $C = (A\text{-mod})$

and let $\mathcal{M}_C := \left(\begin{array}{l} \text{the stack of} \\ \text{objects in Perf}(C) \end{array} \right)$

∞-stack defined by Toën-Vaquié

Consider the topological realization functor

$$|\bullet| : Ho(Stacks/\mathbb{C}) \longrightarrow Ho(Top)$$

from the homotopy category of stacks to the homotopy category of topo spaces.

Define the semi-topological K-theory of C to be

$$K_0^{st}(C) := \pi_0(|\mathcal{M}_C|)$$

This is a group since the direct sum $\oplus$ of A-modules induces a group structure on $|\mathcal{M}_C|$.

Note: More generally $\oplus$ makes $|\mathcal{M}_C|$ into a topological monoid whose (derived) group completion is a space

$$K^{st}(C)$$

which can be viewed as the degree 0 part of a natural spectrum.

Next note that since C is triangulated, we can view C as a module over the dg category Perf_{pt} of complexes of $\mathbb{C}$-vector spaces with finite dimensional total cohomology

In particular $K^{st}_\bullet(C)$ is in a natural way a graded module over $K^{st}_\bullet(\mathrm{Perf}_{pt})$.

Now one checks that

$$K^{st}(\mathrm{Perf}_{pt}) = BU = K^{top}(pt)$$

and so $K^{st}_\bullet(C)$ is a graded $\mathbb{Z}[u]$-module ($\deg u = 2$)

We set

$$K_\bullet^{top}(C) := K_\bullet^{st}(C)[u^{-1}]$$
$$= K_\bullet^{st}(C) \otimes_{\mathbb{Z}[u]} \mathbb{Z}[u, u^{-1}]$$

Conjecturally there is a Chern character map

$$K_\bullet^{top}(C) \xrightarrow{ch} HP_\bullet(C)$$

whose image gives a rational structure on $HP_\bullet(C)$.

Evidence: The semi-topological K-theory of schemes was introduced by Friedlander - Walker who also showed that if X is smooth and proper variety $/\mathbb{C}$, then

$$K^{top}(D(Qcoh(X))) = K^{top}(X^{top})$$

where X^{top} is the topological

space underlying X (taken in the classical topology).

It is useful to separate the de Rham and Betti ingredients in the definition of a nc Hodge structure and to study them separately.

3. Dual description of nc Hodge structures and gluing

Suppose we are given a nc Hodge structure $(H, \mathcal{E}_B, \underline{iso})$. We get

$$(H, \mathcal{E}_B, \underline{iso}) \rightsquigarrow (H, \nabla) \quad \text{de Rham}$$

$$(H, \mathcal{E}_B, \underline{iso}) \rightsquigarrow \mathcal{E}_B \quad \text{Betti}$$

We have the following

Theorem: There is a natural equivalence of categories

$$\left(\begin{array}{l} \text{triples} \quad (H, \mathcal{E}_B, \underline{\text{iso}}) \\ \text{satisfying} \quad (\text{nc filtration})^{\exp} \\ \qquad\qquad\quad (\mathbb{Q}\text{-structure})^{\exp} \end{array} \right)$$

$$\Updownarrow$$

$$\left(\begin{array}{l} \text{quadruples} \quad ((H, \nabla), \mathcal{F}_B, f) \\ \text{where} \\ \bullet\ (H, \nabla) \text{ satisfies } (\text{nc filtration})^{\exp} \\ \bullet\ \mathcal{F}_B \in \mathrm{Constr}(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Q}) \\ \text{with} \\ \qquad R\Gamma(\mathbb{A}^1_{\mathbb{C}}, \mathcal{F}_B) = 0 \\ \bullet\ \mathcal{F}_B \otimes \mathbb{C} \xrightarrow[\sim]{f} DR\left(\widehat{\imath_*(H_{|\mathbb{A}^1 - \{0\}}, \nabla)}\right) \end{array} \right)$$

Note: Here $\imath : \mathbb{A}^1 - \{0\} \to \mathbb{A}^1$

$$u \longrightarrow u^{-1}$$

$\imath_*$ is the pushforward of D-modules

$\widehat{\ }$ is the Fourier transform of D-modules on $\mathbb{A}^1_{\mathbb{C}}$.

Note: It is also possible to formulate the (opposedness) axiom in these terms.

Indeed, given

$$((H, \nabla), \mathcal{F}_B, f)$$

as above, we can reconstruct the rational structure on the local system $\mathbb{S}$ by setting

$$(\mathbb{S}_B)_{e} := \lim_{r \to +\infty} (\mathcal{F}_B)_{re^{ie}}$$

Using this $\mathbb{S}_B$ we can now formulate the (opposedness) axiom as before

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The above theorem allows us to glue nc Hodge structures out of their regular pieces. More precisely we have the following gluing theorem:

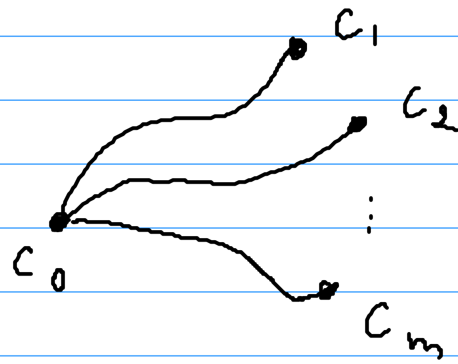
Theorem: Let $(H, \mathcal{E}_B, \underline{iso})$ be a nc Hodge structure of exponential type.
Then specifying $(H, \mathcal{E}_B, \underline{iso})$ is equivalent to specifying the following data:

(regular type) a finite set $S = \{c_1, c_2, \ldots, c_m\} \subset \mathbb{C}$ and a collection

$$\{(R_i, \mathcal{E}_{B,i}, \underline{iso}_i)\}_{i=1}^m$$

of nc Hodge structures with regular singularities.

(gluing data) a base point $c_0 \in \mathbb{C}$ and a collection of paths



together with linear maps of $\mathbb{Q}$-vector spaces

$$T_{ij} : (\mathcal{E}_{B,j})_\infty \xrightarrow{\sim} (\mathcal{E}_{B,i})_\infty$$

Note: The theorem is useful since often the structures arising in examples are given in this form.

4. Examples from mirror symmetry.

(A model) Suppose (X, ω) is a compact symplectic manifold with $\dim_{\mathbb{R}} X = 2d$.

Working near the large volume limit

$$\left(X, \frac{\omega}{\hbar}\right), \quad \hbar \to 0$$

we can use the 3-point genus 0 GW invariants to construct a $\mathbb{C}_q$-valued quantum deformation of the cup product on $H^\bullet(X, \mathbb{C})$:

$$*_q : H^\bullet(X, \mathbb{C})^{\otimes 2} \longrightarrow H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}_q$$

Here as usual q is a complex parameter whose real values near $+\infty$ are identified with $\exp(-1/\hbar)$, and $\mathbb{C}_q$ is the Novikov ring

$$\mathbb{C}_q = \left\{ \sum_{i=0}^{\infty} c_i q^{E_i} \;\middle|\; \begin{array}{l} E_i \in \mathbb{R} \\ \lim_{i\to\infty} E_i = +\infty \end{array} \right\}$$

We will make the following simplifying

(Assumption) $*_q$ is absolutely convergent near $q=0$

Now we define

$$\mathcal{H} := H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}\{u, q\}$$

$$\mathcal{H}^0 := \left(\bigoplus_{k = d \bmod 2} H^k \right) \otimes \mathbb{C}\{u, q\}$$

$$\mathcal{H}^1 := \left(\bigoplus_{k = (d+1) \bmod 2} H^k \right) \otimes \mathbb{C}\{u, q\}$$

Define a meromorphic connection ∇ on $\mathcal{H}$ by setting

$$\nabla_{\partial/\partial u} := \frac{\partial}{\partial u} + u^{-2}\left(\kappa_X *_q \bullet\right) + u^{-1}\,\mathrm{Gr}$$

$$\nabla_{\partial/\partial q} := \frac{\partial}{\partial q} - q^{-1}u^{-1}\left([\omega] *_q \bullet\right)$$

where

$\kappa_X \in H^2(X, \mathbb{Z})$ is the first Chern class of $T_X^\vee$ in any ω-compatible almost complex structure

$$\mathrm{Gr}|_{H^k(X,\mathbb{C})} = \frac{k-d}{2}\,\mathrm{id}_{H^k(X,\mathbb{C})}$$

This defines a (q-variation) of the de Rham part of nc Hodge structures. The issue with the definition of the $\mathbb{Q}$-structure is much more delicate.

An explicit iterated integrals calculation for the case of $X = \mathbb{P}^n$ suggests the following

Def: The rational structure

$$\mathcal{E}_B \subset H^\bullet(X, \mathbb{C})$$

is the image of the map

$$H^\bullet(X, \mathbb{Q}) \longrightarrow H^\bullet(X, \mathbb{C}) \longrightarrow H^\bullet(X, \mathbb{C})$$

multiplication by $(2\pi i)^{\frac{k}{2}}$ on $H^k(X, \mathbb{C})$

$(\bullet) \wedge \hat{\Gamma}(X)$

where $\hat{\Gamma}(X)$ is a new characteristic class (symplectic Mukai vector) of X defined as

$$\hat{\Gamma}(X) = \prod_{i=1}^{d} \Gamma(1 + \delta_i)$$

where:

- $\Gamma(t)$ is the standard gamma function.
- δ_i are the Chern roots of T_X (for any a. complex structure).

Remark: Aside from the explicit calculation this definition is supported by

- the appearance of $\chi(X)\zeta(3)$ in the mirror formula for CY_3.
- Golyshev's hypergeometric description of the nc motive associated with the LG mirror of a toric Fano.
- Iritani's description of the rational structure on the semi infinite Hodge structure for a toric Fano orbifold.

Conjecture: The triple

$$(\mathcal{H}, \mathcal{E}_B, \underline{iso})$$

is a variation of nc Hodge structures of exponential type.

Remark: (1) In general it is not clear if the $(\mathbb{Q}\text{-structure})^{exp}$ axiom holds in this case. It holds trivially in the graded case, e.g. when X is a CY complete intersection in a toric variety and ω is a restriction from the ambient toric variety.

(2) The "exponential type" part of the conjecture is not supported by any evidence. It is possible that for non-Kähler symplectic manifolds we do get a nc Hodge structure which is <u>not</u> of exponential type. A special case of this issue is the following:

Question: Suppose C is an affine smooth compact $\mathbb{Z}/2$-graded dg category. Is the connection ∇ on $HP_\bullet(C)$ with a regular singularity?

(B model) Suppose we have an algebraic map

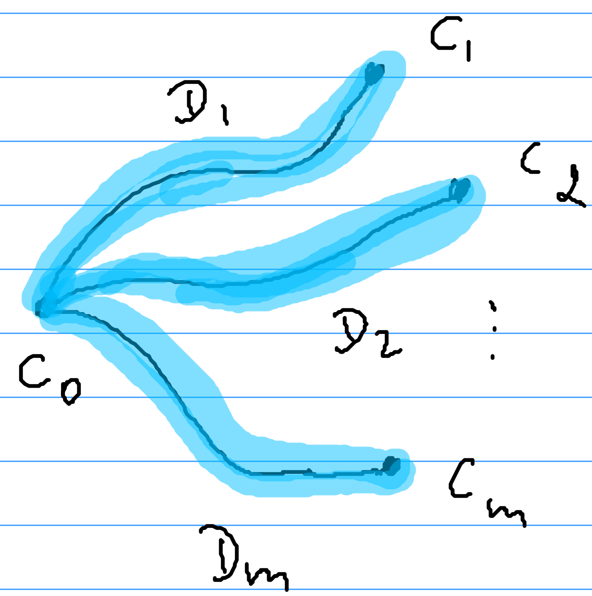
$$w : Y \longrightarrow \mathbb{C}$$

where

- Y is a quasi-projective manifold
- $\mathrm{crit}(w)$ is proper

Let $\$ = \{c_1, \dots, c_m\}$ be the critical values of w.

Fix $c_0 \in \mathbb{C} - \$$ - a base point and choose paths



and their thickenings D_i.

Fix $k \in \mathbb{N}$ and take

$$V_i := H^k(w^{-1}(D_i), w^{-1}(z_0); \mathbb{Q}).$$

$$\begin{aligned} V &:= \bigoplus_{i=1}^{m} V_i \\ &= H^k\Big(w^{-1}\Big(\bigcup_{i=1}^{m} D_i\Big), w^{-1}(z_0); \mathbb{Q}\Big) \\ &= H^k(Y, w^{-1}(z_0); \mathbb{Q}). \end{aligned}$$

Let $T_i : V_i \longrightarrow V_i$ be the monodromy around ∂D_i.

Define a constructible sheaf

$$(\mathcal{F}_B)_z := H^k(Y, w^{-1}(z); \mathbb{Q})$$

on $\mathbb{A}^1_{\mathbb{C}}$. We have

Theorem: $R\Gamma(\mathbb{A}^1_{\mathbb{C}}, \mathcal{F}_B) = 0.$

Using this theorem and the description of nc Hodge structures via gluing data we can now attach a formal nc Hodge structure to

$$w : Y \to \mathbb{C}.$$

Given a critical value $c_i \in S$ of w, consider the $\mathbb{Z}/2$-graded module over $\mathbb{C}[[u]]$:

$$\mathcal{H}^{\bullet} := \mathbb{H}^{\bullet \bmod 2}_{Zar}\left(Y, \left(\Omega^{\bullet}, u d_{dR} + dw\right)\right)$$

We have a de Rham $\leftrightarrow$ Betti isomorphism

$$\begin{array}{c} \mathbb{H}^{\bullet}_{Zar}\left(Y, \left(\Omega^{\bullet}\{u\}, u d + dw\right)\right) \\ \downarrow \cong \; \Phi \\ H^{\bullet}_{B}\left(Y, w^{-1}(c_0); \mathbb{C}\right) \end{array}$$

The isomorphism Φ can be constructed by using nearby cycles, the Jouanolou's trick, and integration along relative chains.

Now taking $f = \Phi^{-1}$ gives a nc Hodge structure described via gluing data.