Simons Center Talk
8-16-10

S. Gubser Princeton University

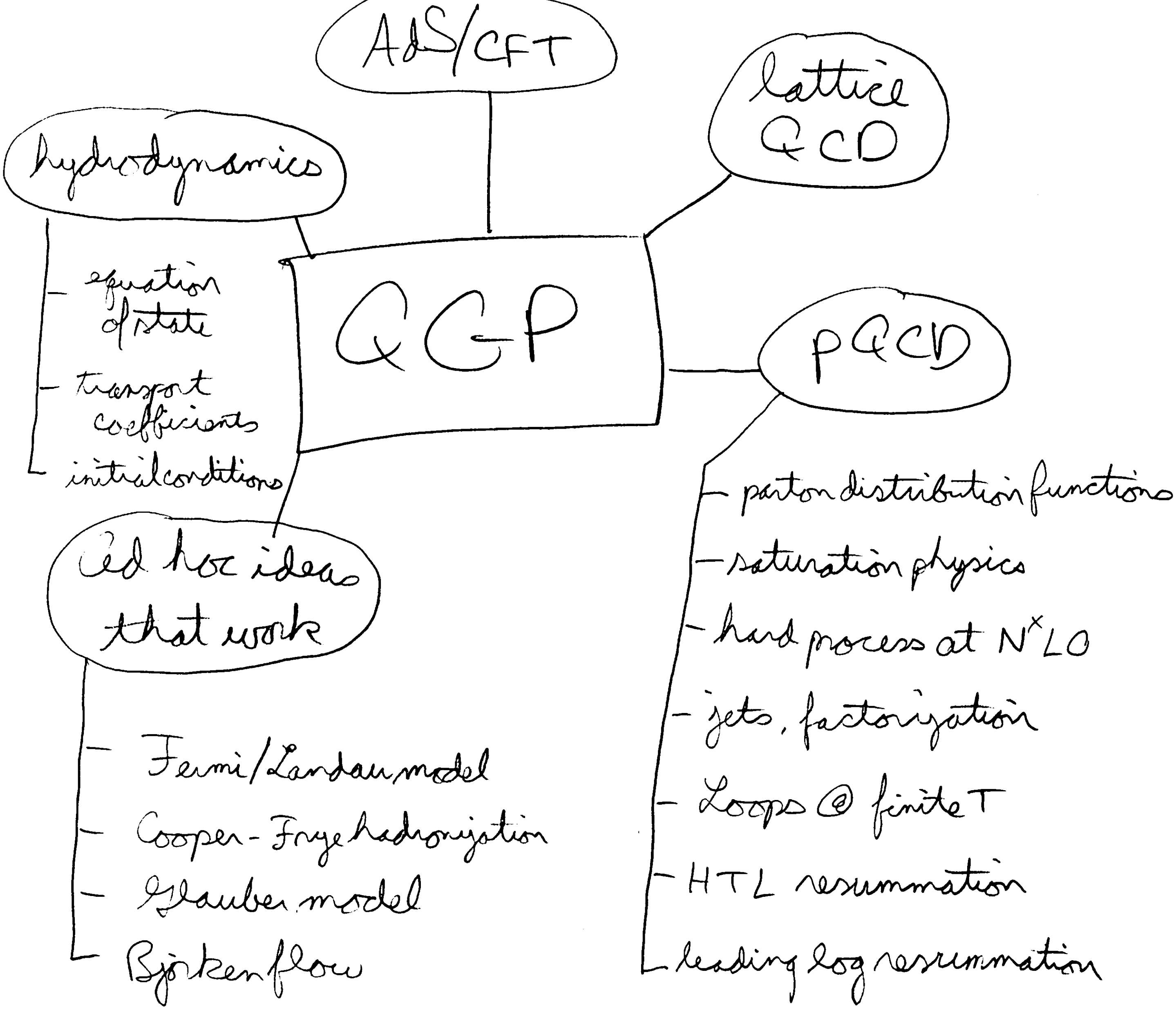
Most of the dynamics in a heavy-ion collision is ~~dictated~~ by the QCD lagrangian,

$$\mathcal{L} = -\frac{1}{4}\sum_{a=1}^{8}\left(F^a_{\mu\nu}\right)^2 + \sum_{i=1}^{3}\sum_{f=1}^{6}\bar{\psi}^i_f\left(i\not{D}-m_f\right)\psi^f_i$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^b_\mu + gf^{abc}A^b_\mu A^c_\nu$$

$$D_\mu\psi_i = \partial_\mu\psi_i - ig\sum_{a=1}^{\infty}A^a_\mu\left(t^a\right)^j_i\psi_j$$

But this dynamics is complicated. So there are various approximation schemes




Naturally I plan to talk mostly about what I know best, namely AdS/CFT — but with emphasis on relevant results from hydro & l QCD.

## Plan of the talk:

1) Reasons to do lots of calculations;
Reasons to distrust the answers

2) Two calculations that have stood the test of time

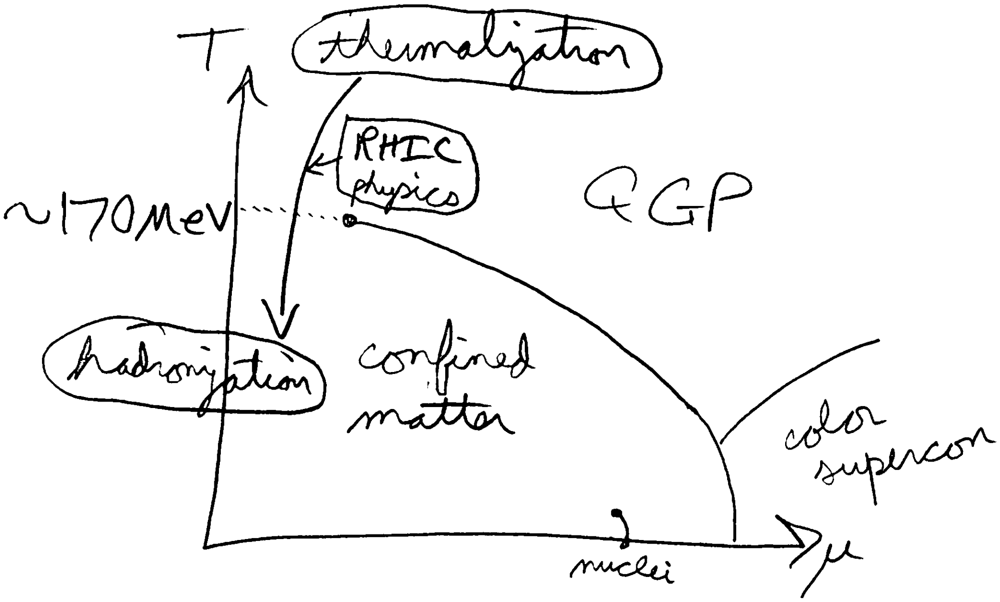
a) $\frac{\eta}{s} = \frac{1}{4\pi}$

b) $F_{drag} = -\frac{\pi\sqrt{\lambda}}{2} T^2 \frac{v}{\sqrt{1-v^2}}$ where $\lambda = g_{YM}^2 N$

3) Advanced topics, probably to include thermalization, $O(3)$-symmetric Bjorken flow, and/or QCD phase diagram

Two introductory accounts:
0704.0240 by Son and Starinets
0901.0935 by Gubser and Karch

## 1) Reasons to do lots of calculations




a) $\alpha_s$ is not small at RHIC.

$$\alpha_s \sim 0.5 \rightarrow g_{YM}^2 N = (4\pi\alpha_s)(3) \approx 19 \gg 1$$

so pQCD could at least benefit from a complementary large $g_{YM}^2 N$ expansion

b) Real-time dynamics is hard to access via lattice QCD.

EOS and static $q\bar{q}$ potential are easier.

c) There's <u>data</u>.

Reasons to distrust the answers:

a) We don't have a gravity dual to QCD.
$\mathcal{N}=4$ SYM $\neq$ QCD.

b) Calculations are typically lowest order in $\frac{1}{g_{YM}^2 N}$, and corrections could be significant.

## 2) Two results that have stood the test of time

a) $\frac{\eta}{s} = \frac{1}{4\pi}$.

A nice way to "sneak up" on this from a heavy ion perspective is to review Bjorken flow.

$$\mathbb{R}^{3,1}: \quad ds^2 = -dt^2 + dx_3^2 + dx_1^2 + dx_2^2$$

$$= -d\tau^2 + \tau^2 dy^2 + dx_\perp^2 + x_\perp^2 d\phi^2$$

$$t = \tau \cosh y \qquad x_3 = \tau \sinh y$$

$$x_1 = x_\perp \cos\phi \qquad x_2 = x_\perp \sin\phi$$

Relativistic hydro consists of the ansatz

$$T_{mn} = \varepsilon U_m U_n + p P_{mn} - 2\eta\sigma_{mn} - \zeta(\nabla_\ell U^\ell) P_{mn}$$

$$P_{mn} = U_m U_n + g_{mn} \qquad U_m U^m = -1 \quad U_\tau < 0$$

$$\sigma_{mn} = P_m^a P_n^b \left( \frac{\nabla_a U_b + \nabla_b U_a}{2} - \frac{g_{ab}}{3} \nabla_\ell U^\ell \right)$$

and the equation $\nabla^m T_{mn} = 0$.

Bjorken flow is the solution respecting

$\mathcal{L}_\xi U_m = 0$ for $\xi \in ISO(2)$ of $x^1-x^2$ plane or $SO(1,1)$ $t-x^3$ boosts

as well as the $\mathbb{Z}_2$ symmetry $y \to -y$.

$U^m \partial_m = \partial_\tau$ is fixed by $ISO(2) \times SO(1,1) \times \mathbb{Z}_2$.

To determine $\mathcal{E}(\tau)$ we have to use more than symmetry, namely:

$\longrightarrow$ $P = \frac{\mathcal{E}}{3}$ for a CFT

$\longrightarrow$ $\zeta = 0$ " " "

$\longrightarrow$ $H_0 \equiv \frac{\eta}{\mathcal{E}^{3/4}} =$ constant for a CFT

Then one finds

$$\mathcal{E}(\tau) = \left(\frac{e_0}{\tau^{1/3}} - \frac{H_0}{2\tau}\right)^4$$

(an exact soln to hydro!)

$e_0$ is an integration constant.

Note $\mathcal{E}(\tau) = 0$ @ $\tau = \tau_0 \equiv \left(\frac{H_0}{2e_0}\right)^{3/2}$.
Hydro is only justified for $\tau \gg \tau_0$.
Numerically, $\tau_0 \approx 0.01$ fm/c for central Au-Au @ RHIC. Standard practice is to initialize hydro codes at $\tau \sim 0.6 - 1$ fm/c, using Bjorken flow as an approximate initial condition.

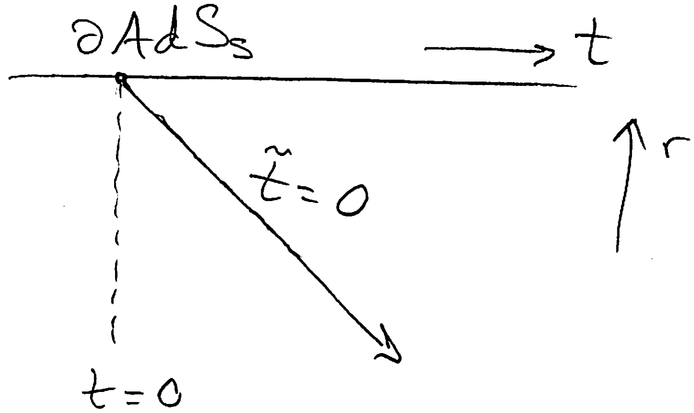
Thermal state of a 3+1-dim'l CFT is described by the dual metric

$$ds^2 = -\frac{r^2}{L^2} h\, dt^2 + \frac{L^2}{r^2}\frac{dr^2}{h} + \frac{r^2}{L^2} d\vec{x}^{\,2} \qquad \vec{x} = (x_1, x_2, x_3)$$

$h = 1 - \frac{r_H^4}{r^4}$, horizon is at $r = r_H = \pi T L^2$

<u>First trick</u>: Define $\tilde{t} = t - \int_r^\infty \frac{dr}{r^2 h}$

$\tilde{t} = 0$ is the trajectory of a light ray sent in from bdy @ $t = 0$.




$$ds^2 = -\frac{r^2}{L^2} h \, d\tilde{t}^2 + 2 d\tilde{t} \, dr + \frac{r^2}{L^2} d\vec{x}^2$$

<u>Second trick</u>: if $U_m(\tilde{t}, \vec{x})$ is slowly varying, then we can get an <u>approximate</u> soln to

$$R_{\mu\nu} = -\frac{4}{L^2} g_{\mu\nu}$$

by replacing $d\tilde{t} \longrightarrow -U_m dx^m$ and

$d\vec{x}^2 \longrightarrow P_{mn} dx^m dx^n \qquad dx^m = (d\tilde{t}, d\vec{x})$

$$ds_0^2 = -\frac{r^2}{L^2}\left(1 - \frac{r_H(\tilde{t},\vec{x})^4}{r^4}\right) u_m u_n dx^m dx^n +$$

$$2d\tilde{t}dr + \frac{r^2}{L^2} P_{mn} dx^m dx^n \qquad (\star)$$

where $r_H(\tilde{t},\vec{x}) = \pi T(\tilde{t},\vec{x}) L^2$ and $T(\tilde{t},\vec{x})$ is the local temperature — $\mathcal{E}(\tilde{t},\vec{x})^{1/4}$ up to a constant prefactor.

For inviscid ($H_0 = 0$) Bjorken flow, $(\star)$ boils down to

$$ds_0^2 = -\frac{r^2}{L^2}\left(1 - \frac{\overbrace{K\tau^{-4/3}}^{r_H^4}}{r^4}\right) d\tau^2 +$$

$$2d\tau dr + \frac{r^2}{L^2}\tau^2 dy^2 + \frac{r^2}{L^2} dx_\perp^2$$

<u>Third trick</u>: Just as viscous corrections to hydro are at relative order $\frac{1}{\tau^{2/3}}$, corrections to $ds_0^2$ are $\frac{1}{\tau^{2/3}}$ times functions of $v = r\tau^{1/3}$:

$$ds_0^2 = -\frac{r^2}{L^2}\left(1 - \frac{v_H^4}{v^4}\right) e^{\frac{a_1(v)}{\tau^{2/3}} + \frac{a_2(v)}{\tau^{4/3}} + \cdots} d\tau^2 +$$

$$2\,d\tau\, dr +$$

$$\frac{r^2}{L^2}\tau^2 e^{\frac{b_1(v)}{\tau^{2/3}} + \frac{b_2(v)}{\tau^{4/3}} + \cdots} dy^2 +$$

$$\frac{r^2}{L^2} e^{\frac{c_1(v)}{\tau^{2/3}} + \frac{c_2(v)}{\tau^{4/3}} + \cdots} dx_\perp^2$$

Now we can solve first for $a_1, b_1, c_1$, then $a_2, b_2, c_2, \ldots$

$$R_{rr} = -\frac{4}{L^2} g_{rr} \implies c_1 = -\frac{1}{2} b_1$$

No surprise, this is a pure shear deformation.

$$a_1' + \frac{4v^3 a_1}{v^4 - v_H^4} + \frac{L^2 v^2}{v^4 - v_H^4} = 0$$

$$\hookrightarrow a_1 = -\frac{L^2}{3}\frac{v^3 - v_H^3}{v^4 - v_H^4}$$

(an integration constant is fixed by $a_1 \to$ finite @ $v = v_H$ )

$$b_1'' + \frac{1}{v}\frac{5v^4 - v_H^4}{v^4 - v_H^4} b_1' + \frac{2L^2 v}{v^4 - v_H^4} = 0$$

$$\hookrightarrow b_1' = -\frac{2L^2}{3v}\frac{v^3 - v_H^3}{v^4 - v_H^4}$$

(ditto)

With metric known, one extracts holographic stress tensor as

$$T_{mn} = \frac{1}{8\pi G_5}\left[\Theta_{mn} - \Theta g_{mn}^{bdy} - \frac{3}{L} g_{mn}^{bdy}\right]$$

$$T_{\tau\tau} = \frac{3v_H^4/L^5}{16\pi G_5}\Big(\underset{\substack{\uparrow \\ \text{inviscid}}}{\tau^{-4/3}} - \underset{\substack{\uparrow \\ a_1 b_1 c_1 \text{ term}}}{\frac{L^2}{3v_H}\tau^{-2}} + \dots\Big)$$

$$= \left(\frac{e_0}{\tau^{1/3}} - \frac{H_0}{2\tau}\right)^4$$

gives $e_0^4 = \frac{3v_H^4/L^5}{16\pi G_5}$ and $\frac{H_0}{e_0} = \frac{L^2}{3v_H}$.

On the other hand, $S = \frac{A}{4G_5}$ ← proper volume of horizon

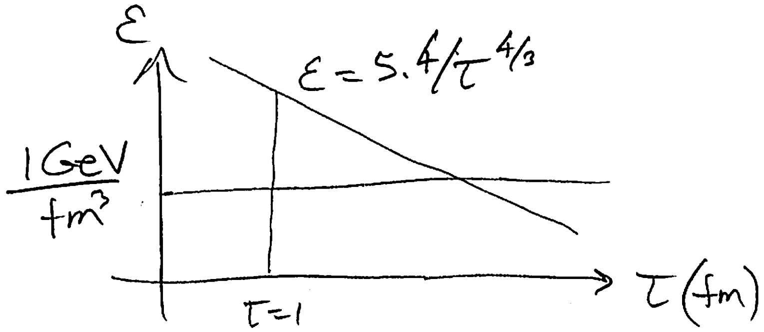
gives $s = \frac{S}{V_{coord}}$ ← coordinate volume wrt $\mathbb{R}^{3,1}$ bdy metric

$$= \frac{r_H^3/L^3}{4G_5} = \frac{v_H^3/L^3\tau}{4G_5} = \frac{4\pi}{3v_H}\frac{L^2}{\tau}e_0^4$$

cf $\eta = H_0 \varepsilon^{3/4} = \frac{L^2}{3v_H} e_0 \frac{e_0^3}{\tau} = \frac{1}{3v_H}\frac{L^2}{\tau} e_0^4 = \frac{s}{4\pi}$,

QED.

n.b. Why $\tau_0 = 0.01$ fm/c ?




$$T = \underset{\displaystyle 1.25/\text{fm}}{\overset{\displaystyle 1.25/\text{fm}}{}}$$

$$\begin{array}{c} 1.25/\text{fm} \\ \parallel \\ T = 250 \text{ MeV @} \\ \tau = 1 \text{fm} \end{array}$$

$$\tau_0^{2/3} = \frac{H_0}{2e_0} = \frac{\eta/\varepsilon^{3/4}}{2\varepsilon^{1/4}\tau^{1/3}} = \frac{\eta/\varepsilon}{2\tau^{1/3}} = \frac{\eta/s}{2\tau^{1/3}}\frac{s}{\varepsilon}$$

$$= \frac{1}{8\pi\tau^{1/3}}\,\frac{4}{3}\,\frac{1}{T} = \frac{1}{6\pi T \tau^{1/3}}$$

In units of fm everywhere,

$$\tau_0 = \frac{1}{(6\pi)^{3/2}}\,\frac{1}{(1.25)^{3/2}} \approx \frac{1}{114}$$

Such short times have little phenomenological significance, but it's nice to know that $\tau \gg \tau_0$ holds nicely in the actual practice of hydro by RHIC phenomenologists.

b) $$F_{drag} = -\frac{\pi\sqrt{\lambda}}{2} T^2 \frac{v}{\sqrt{1-v^2}}$$

A well-studied quantity in lQCD is the potential between two static quarks (i.e. infinitely heavy sources of fundamental / anti-fundamental color).

A good fit to lattice data @ $T=0$ between

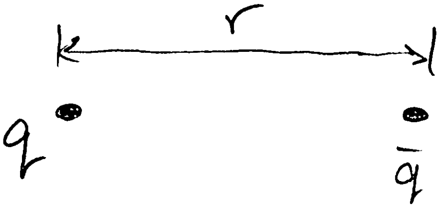
$0.1\,\text{fm} \lesssim r \lesssim 1.2\,\text{fm}$ is

$$V(r) = -\frac{4\alpha/3}{r} + \sigma r \qquad \text{(Cornell potential)}$$

Singlet gluon exchange in color singlet channel

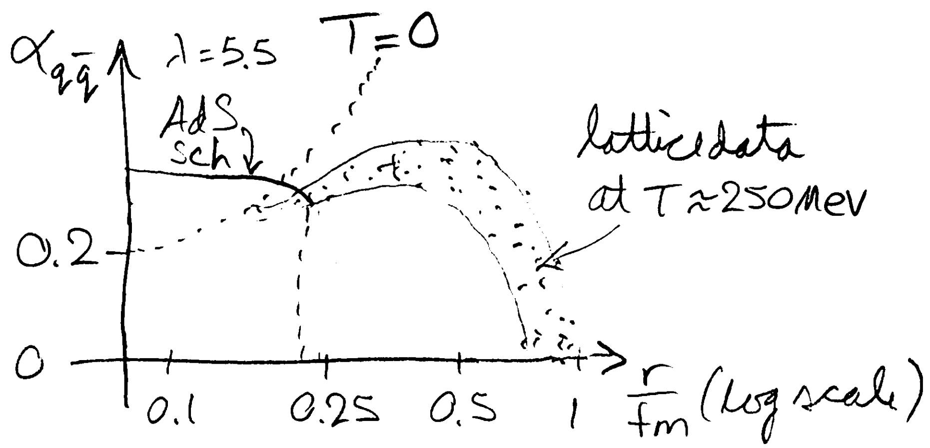
Ad hoc confinement term

$\alpha \simeq 0.2 \qquad \sigma \simeq (0.5\,\text{fm})^{-2}$



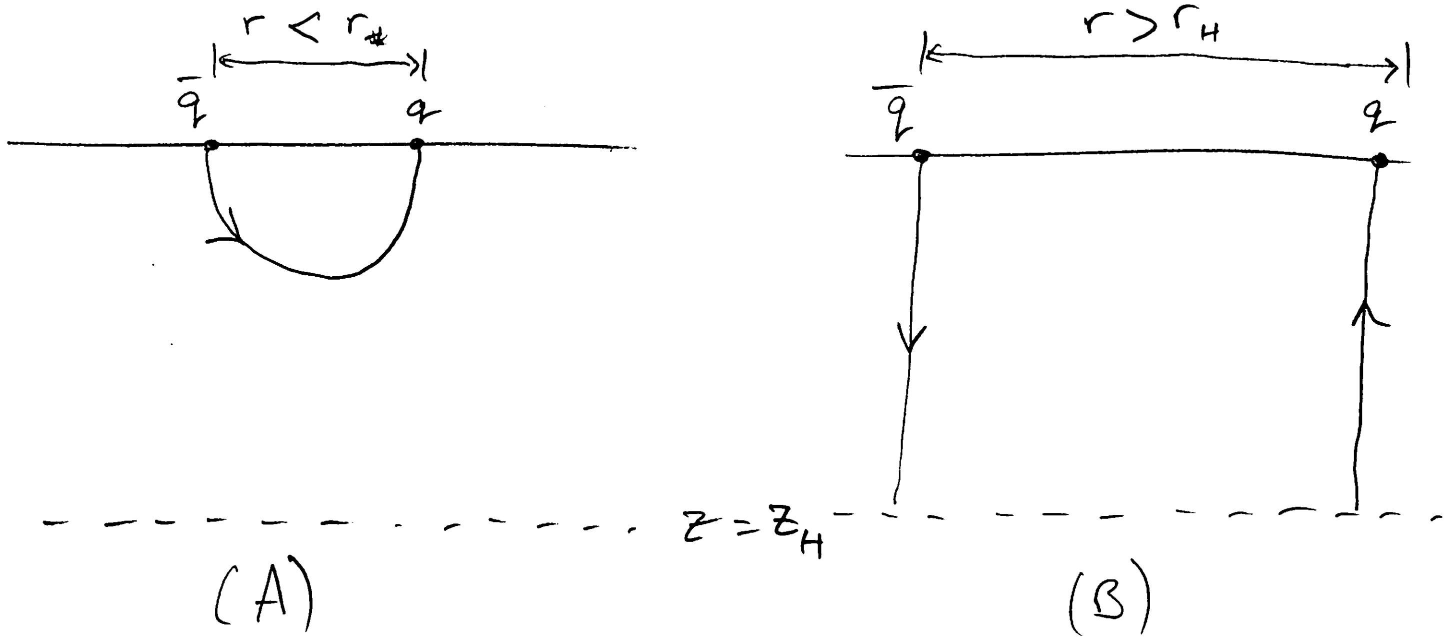
At $T>0$ you don't compute $V(r)$, but rather $F_{q\bar{q}}$, the excess free energy due to presence of $q\bar{q}$ pair. A common practice is to quote

$$\alpha_{q\bar{q}}(r,T) \equiv \frac{3}{4} r^2 \frac{\partial F_{q\bar{q}}}{\partial r}$$




In AdS/CFT, find $V = -\frac{4\pi^2}{\Gamma(1/4)^4}\frac{\sqrt{\lambda}}{r}$ @ $T=0$.

When $T>0$, main effect is $\alpha_{q\bar{q}} \approx$ const up to $r = r_* \approx \frac{1}{4T}$, then $\alpha_{q\bar{q}} = 0$ in naively dominant saddle-point approximation

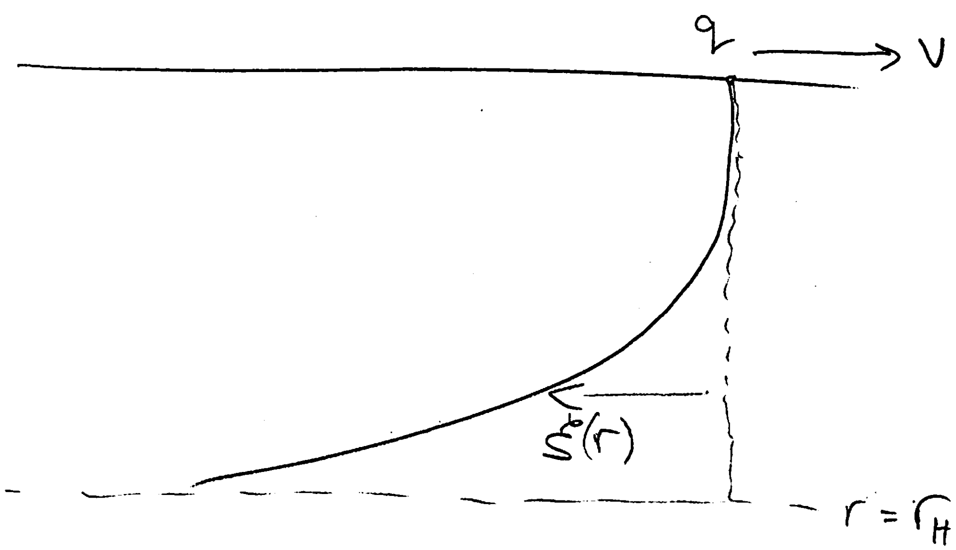



But graviton exchange in (B) is ~~potentially~~ an important effect.

Based just on (A), get the best match with lQCD @ $r \simeq 0.2$ fm with $\lambda \simeq 5.5$.

[Also it helps to use $\mathcal{E}_{lQCD} = \mathcal{E}_{\substack{N=4 \\ SYM}}$ rather than $T_{lQCD} = T_{\substack{N=4 \\ SYM}}$ ]

What's hard to do on the lattice is to find the force on a moving quark. This is where AdS/CFT can provide some interesting information.




Consider a fixed velocity $v$ & fixed temperature $T$.

Shape of trailing string is

$$\begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \\ r \end{pmatrix} = \begin{pmatrix} t \\ vt + \xi(r) \\ 0 \\ 0 \\ r \end{pmatrix}$$

using $t$ & $r$ to parametrize worldsheet

Geometry is same AdS-Sch as before:

$$ds^2 = -\frac{r^2}{L^2} h\,dt^2 + \frac{L^2}{r^2}\frac{dr^2}{h} + r^2 d\vec{x}^2$$

$$h = 1 - \frac{r_H^4}{r^4}.$$

To determine $\xi(r)$ we must extremize classical action for the string:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det \gamma_{\alpha\beta}}$$

where $\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$ is induced metric on string worldsheet.

Find $\frac{d\xi}{dr} = v \frac{r_H^2 L^2}{r^4 - r_H^4}$ after fixing an integration constant to avoid a pathology at $h = v^2$

(more generally, $\xi' \propto \sqrt{\frac{h - v^2}{h - \pi_\xi^2 \frac{L^4}{r^4}}}$ which must be everywhere real)

Equations of motion from $S_{NG}$ can be phrased as

$$\nabla_\alpha P^\alpha_\mu = 0 \qquad P^\alpha_\mu \equiv -\frac{1}{2\pi\alpha'} G_{\mu\nu} \gamma^{\alpha\beta} \frac{\partial X^\nu}{\partial \sigma^\beta}$$

$\nabla_\alpha$ is covariant wrt $\gamma_{\alpha\beta}$.

$P^\alpha_\mu$ is worldsheet current of spacetime energy-momentum, so in particular

$$F_{drag} = \frac{dp_1}{dt} = \sqrt{-\gamma}\, P^r_{x^1} = \ldots = -\frac{r_H^2/L^2}{2\pi\alpha'}\frac{v}{\sqrt{1-v^2}}$$

recalling $r_H = \pi T L^2$ and $\frac{L^2}{\alpha'} = \sqrt{\lambda}$, get

$$F_{drag} = -\frac{\pi\sqrt{\lambda}}{2} T^2 \frac{v}{\sqrt{1-v^2}} = -\frac{\pi\sqrt{\lambda}}{2} T^2 \frac{p_1}{m}$$

in limit where $m \to \infty$, with $p_1 = \frac{mv}{\sqrt{1-v^2}}$

Note $\frac{dp_1}{dt} = -\frac{\pi\sqrt{\lambda}}{2} T^2 \frac{p_1}{m}$ means $p_1 \propto e^{-t/t_q}$

where $t_q = \frac{2m}{\pi\sqrt{\lambda}\, T^2}$

To compare to data, use $\lambda \approx 5.5$ as per $\alpha_{q\bar{q}}$ discussion; also use $\mathcal{E}_{QCD} = \mathcal{E}_{\substack{N=4 \\ SYM}}$ rather than $T_{QCD} = T_{\substack{\mathcal{N}=4 \\ SYM}}$ to adjust for differing numbers of degrees of freedom:

$$\varepsilon \approx 11 T^4 \text{ for QCD at } T \approx 250 \text{ MeV}$$

$$\varepsilon \approx 30 T^4 \text{ for } \mathcal{N}=4 \text{ SYM with group } SU(3)$$

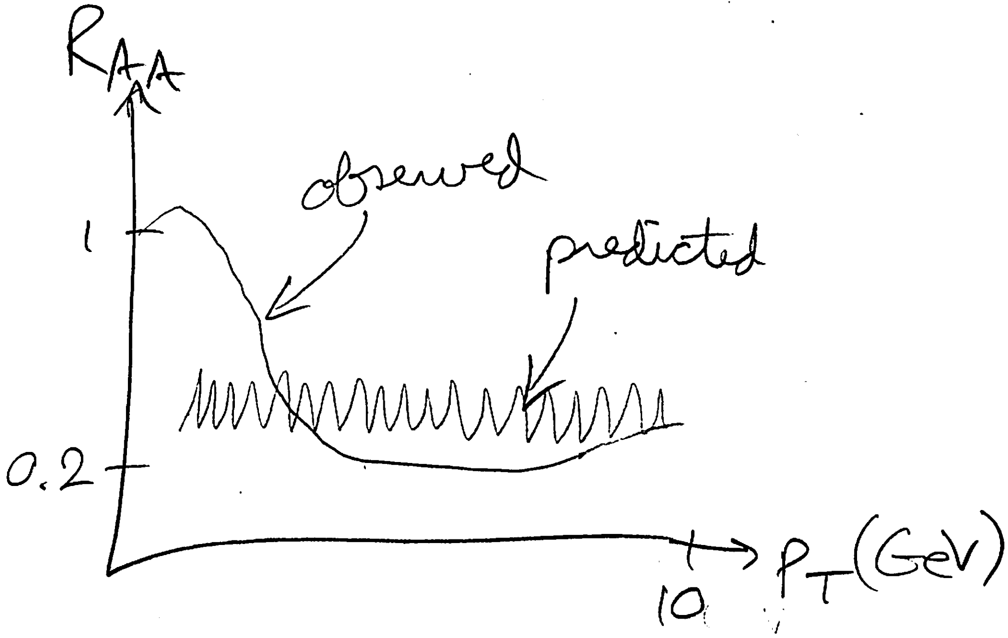
$$\text{SYM:} \quad t_q = \frac{2m}{\pi\sqrt{\lambda}\sqrt{\frac{\varepsilon}{30}}} \approx 1.5 \frac{m}{\sqrt{\varepsilon}}$$

$$\text{QCD:} \quad \varepsilon \approx 5.4 \frac{\text{GeV}}{\text{fm}^3} \approx \frac{27}{\text{fm}^4}$$

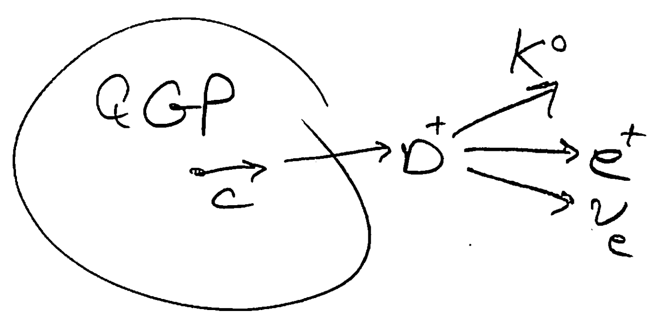
$$m \approx 1.4 \text{ GeV} \approx \frac{7}{\text{fm}} \quad \text{for charm}$$

gives $t_q \approx 2$ fm

Including stochastic forces & b quark contribution, Akamatsu, Hatsuda, & Hirano found reasonable agreement with data on non-photonic electrons: 0809.1499




Here's a standard way to get a non-photonic ~~electron~~ positron:



# 3) <u>$O(3)$-symmetric Bjorken flow</u>

Recall that Bj didn't solve any equations of hydro to get $u^\mu \partial_\mu = \frac{\partial}{\partial \tau}$ in

$$ds^2_{\mathbb{R}^{3,1}} = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2 + x_\perp^2 d\phi^2$$

All he did was to solve symmetry constraints

$$\mathcal{L}_\xi u_\mu = 0 \quad \text{for } \xi = \underbrace{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial \phi}}_{ISO(2)}, \text{ and } \underbrace{\frac{\partial}{\partial y}}_{SO(1,1)}$$

(solns are $u = \frac{\partial}{\partial \tau} + \alpha \frac{\partial}{\partial y}$)

and impose $\mathbb{Z}_2 : y \to -y$ symmetry to pick out $\alpha = 0$. (alternatively, pass to cm frame).

$$ISO(2) \times SO(1,1) \subset SO(4,2),$$

and a natural follow-on to Bjorken's work ('83) is to consider a deformation to

$$SO(3) \times SO(1,1) \subset SO(4,2)$$

($\frac{\partial}{\partial y}$, same as before)

SO(3) generators: $\frac{\partial}{\partial \phi}$ and

$$\xi_i = \frac{\partial}{\partial x^i} + q^2 \left( 2x^i x^m \frac{\partial}{\partial x^m} - x^m x_m \frac{\partial}{\partial x^i} \right) \qquad i = 1,2 \quad m = 0,1,2,3$$

special conformal$^{q}$ transformation

$q$ has dimensions of momentum.

say $\xi = \xi_1$.

$$\mathcal{L}_\xi U_\mu = 0 \qquad \text{and} \qquad \mathcal{L}_\xi \varepsilon \equiv \xi^m \frac{\partial \varepsilon}{\partial x^m} = 0$$

implies $\mathcal{L}_\xi T_{mn}^{hydro} = 0$ (obvious);

but situation is a bit more subtle for a <u>conformal</u> isometry like $\xi$, where

$$\mathcal{L}_\xi g_{mn} = \frac{1}{2}(\nabla_\ell \xi^\ell) g_{mn}$$

In order to get $P_{mn} = U_m U_n + g_{mn}$ to transform nicely, we should require

$$\mathcal{L}_\xi U_m = \frac{1}{4}(\nabla_\ell \xi^\ell) U_m \qquad (\star)$$

$\frac{\partial}{\partial y}$ & $\frac{\partial}{\partial \phi}$ symmetry (plus $\mathbb{Z}_2$ symmetry and $U^m U_m = -1$) restricts

$$U = \cosh\kappa \, \partial_\tau + \sinh\kappa \, \partial_{x_\perp}$$

$$\kappa = \kappa(\tau, x_\perp)$$

Then $\tanh \kappa = v_\perp = \dfrac{2q^2 \tau x_\perp}{1+q^2\tau^2+q^2 x_\perp^2}$

follows immediately from (✩).

Requiring $\mathcal{L}_\xi \mathcal{E} = -(\nabla_\lambda \xi^\lambda)\mathcal{E}$ for

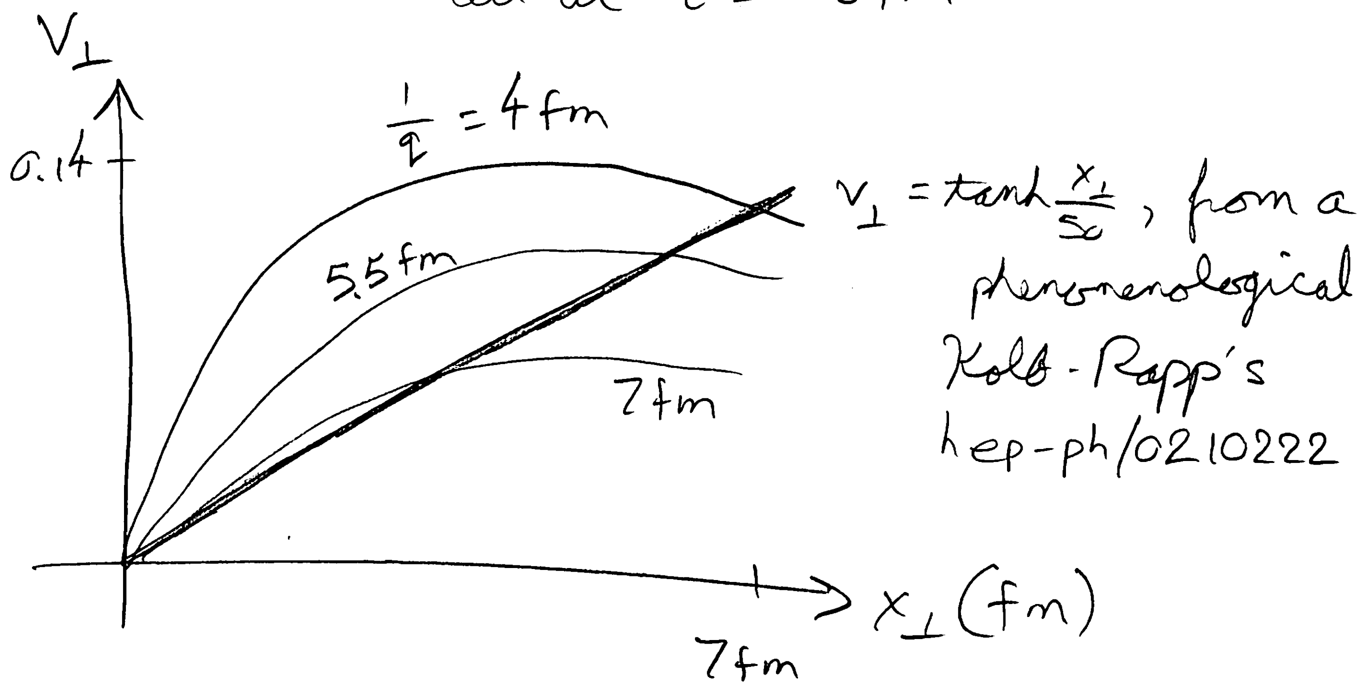
$\xi \in SO(3)\times SO(1,1)$ leads to a consistent soln to conformal relativistic hydro. For $\eta = 0$,

$$\mathcal{E} = \frac{\hat{\mathcal{E}}_0}{\tau^{4/3}} \frac{(2q)^{8/3}}{\left[1+2q^2(\tau^2+x_\perp^2)+q^4(\tau^2-x_\perp^2)^2\right]^{4/3}},$$

but explicit soln is available for any $\eta$.

Non-zero $v_\perp$ is missing (by assumption) from Bjorken flow, but obviously desirable phenomenologically:

all at $\tau = 0.6$ fm




spectrum of heavy hadrons like $\Omega$ (sss) suggests need for early non-zero $V_\perp$, and HBT may also be helped by $v_\perp \neq 0$ in initial conditions for hydro.

Incomplete set of references:

$\frac{\eta}{s}$: hep-th/0104066 (Policastro-Son-Starinets)

hep-th/0512162 (Janik-Peschanski)

hep-th/0607123 (Nakamura-Sin)

0805.3774 (Heller et al)

$F_{drag}$: hep-th/0605158 (Herzog et al)

hep-ph/0605199 (Casalderrey-Solana-Teaney)

hep-th/0605182 (Gubser)

hep-th/0611272 (Gubser)

$O(3)$: 1006.0006 (Gubser)