PiTP Lectures on BPS States and Wall-Crossing in $d = 4$, $\mathcal{N} = 2$ Theories

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ABSTRACT: These are notes to accompany a set of three lectures at the July 2010 PiTP school. Lecture I reviews some aspects of $\mathcal{N} = 2$ $d = 4$ supersymmetry with an emphasis on the BPS spectrum of the theory. It concludes with the primitive wall-crossing formula. Lecture II gives a fairly elementary and physical derivation of the Kontsevich-Soibelman wall-crossing formula. Lecture III sketches applications to line operators and hyperkähler geometry, and introduces an interesting set of "Darboux functions" on Seiberg-Witten moduli spaces, which can be constructed from a version of Zamolodchikov's TBA. ♣♣♣ UNDER CONSTRUCTION!!! 7:30pm, JULY 29, 2010 ♣♣♣

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1. Lecture 0: Introduction and Overview

This minicourse is about four-dimensional field theories and string theories with $\mathcal{N} = 2$ supersymmetry.

It was realized as a result of several breakthroughs in the mid-1990's that the so-called BPS spectrum of these theories is an essential tool that can lead to exact nonperturbative results about these theories.

The BPS states are certain representations of the supersymmetry algebra which are "small" or "rigid." They are therefore generally invariant under deformations of parameters. Thus, one can hope to use information derived at weak coupling to obtain highly nontrivial results at strong coupling.

It turns out that the BPS spectrum is not completely independent of physical parameters. It is piecewise constant, but can undergo sudden changes. This is known as the "wall-crossing phenomenon." This phenomenon was recognized in the early 1990's in the context of two-dimensional theories with (2,2) supersymmetry in the work of Cecotti, Fendley, Intriligator and Vafa [5] and some explicit formulae for how the BPS spectrum changes in those 2d theories were derived by Cecotti and Vafa [6]. The wall-crossing phenomenon played an important role in verifying the consistency of the Seiberg-Witten solution of the vacuum structure of $d = 4$, $\mathcal{N} = 2$ SU(2) gauge theory. (See the final section of [25].) In the past four years there has been a great deal of progress in understanding *quantitatively* how the BPS spectrum changes in four-dimensional $\mathcal{N} = 2$ field theories and string theories. These lectures will sketch out the author's current (July, 2010) viewpoint on the BPS spectrum in these theories.

1.1 Why study BPS states?

The main motivation for the study of BPS states was sketched above. However, the subject has turned out to be an extremely rich source of beautiful mathematical physics.

A few closely related topics are the following

- As a result of the Strominger-Vafa insight, the study of BPS states is highly relevant to understanding the microscopic origin of the Beckenstein-Hawking entropy of certain supersymmetric black holes.

- The investigation of these susy black hole entropies has led to surprising connections to quantities in algebraic geometry known as (generalized) Donaldson-Thomas invariants. The physical conjecture of Ooguri, Strominger, and Vafa has suggested some far-reaching nontrivial relations between Donaldson-Thomas invariants and the Gromov-Witten invariants of enumerative geometry.

- Closely related is the theory of stability structures in a class of Categories satisfying a version of the Calabi-Yau property.

- Enumeration of BPS states has led to highly nontrivial interactions with the theory of automorphic forms and hence to certain aspects of analytic number theory.

- In some contexts the BPS spectrum appears to be closely related to interesting algebraic structures associated to Calabi-Yau varieties and quivers. A conjectural "algebra of BPS states" has recently been rigorously formalized as a "motivic Hall algebra" by Kontsevich and Soibelman. ♣ CITE ♣

- The study of BPS states has led to interesting connections to the work of Fock and Goncharov, et. al. on the quantization of Teichmuller spaces (and the quantization of Hitchin moduli spaces and "cluster varieties," more generally).

- A related development includes several new and interesting connections between four-dimensional gauge theories and two-dimensional integrable systems.

- The theory of BPS states appears to be an interesting way of understanding new developments in knot theory, such as knot homology theories and new knot invariants associated with noncompact Chern-Simons theories.

Some of these motivations are old, but some are less than one year old. The theory of BPS states continues to generate hot topics in mathematical physics.

1.2 Overview of the Lectures

2. Lecture I: BPS indices and the primitive wall crossing formula

2.1 The N=2 Supersymmetry Algebra

Let us begin by writing the $\mathcal{N} = 2$ superalgebra. We mostly follow the conventions of Bagger and Wess [3] for $d = 4, \mathcal{N} = 1$ supersymmetry. In particular $SU(2)$ indices are raised/lowered with $\epsilon^{12} = \epsilon_{21} = 1$. Components of tensors in the irreducible spin representations of $so(1,3)$ are denoted by $\alpha, \dot{\alpha}$ running over $1, 2$. The rules for conjugation are that $(\mathcal{O}_1\mathcal{O}_2)^\dagger = \mathcal{O}_2^\dagger \mathcal{O}_1^\dagger$ and $(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}}$.

The $d = 4, \mathcal{N} = 2$ supersymmetry algebra will be denoted by $\mathfrak{s}$. It has even and odd parts:

$$\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1 \tag{2.1}$$

where the even subalgebra is

$$\mathfrak{s}^0 = poin(1,3) \oplus su(2)_R \oplus u(1)_R \oplus \mathbb{C} \tag{2.2}$$

and the odd subalgebra, as a representation of $\mathfrak{s}^0$ is

$$\mathfrak{s}^1 = \left[(2,1;2)_{+1} \oplus (1,2;2)_{-1} \right] \tag{2.3}$$

A basis for the odd superalgebra is usually denoted:
Q_α^A, $\bar{Q}_{\dot{\alpha}}^A$,
The reality constraint is:

$$(Q_\alpha{}^A)^\dagger = \bar{Q}_{\dot{\alpha}A} := \varepsilon_{AB}\bar{Q}_{\dot{\alpha}}^B \tag{2.4}$$

The commutators of the odd generators are:

$$\begin{aligned}
\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\} &= 2\sigma^m_{\alpha\dot{\beta}}P_m\delta^A_B \\
\{Q_\alpha{}^A, Q_\beta{}^B\} &= 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \\
\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} &= -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB}Z
\end{aligned} \tag{2.5}$$

Remarks:

1. The last summand in $\mathfrak{s}^0$ is the central charge Z.

2. P_m is the Hermitian energy-momentum vector with $P^0 \geq 0$.

3. The commutators of the even generators with the odd generators are indicated by the indices. In particular, $SU(2)_R$ rotates the index A.

4. Under the $u(1)_R$ symmetry Q_α^A has charge $+1$ and hence $\bar{Q}_{\dot{\alpha}}^A$ has charge -1. The $u(1)_R$ symmetry can be broken explicitly by couplings in the Lagrangian, spontaneously by vevs, or it can be anomalous. More discussion of these points can be found in Davide Gaiotto's lectures.

5. In supergravity one sometimes does not have $su(2)_R$ symmetry.

2.2 Particle representations

The $\mathcal{N} = 2$ supersymmetry algebra acts unitarily on the Hilbert space of our physical theory. Therefore we should understand well the unitary irreps of this algebra.

We will be particularly interested in single-particle representations. We will construct these using the time-honored method of induction from a little superalgebra, going back to Wigner's construction of the unitary irreps of the Poincaré group.

A particle representation is characterized in part by the Casimir $P^2 = M^2$. We will only discuss massive representations with $M > 0$. A massive particle can be brought to rest. It defines a state such that

$$P^m|\psi\rangle = M\delta_0^m|\psi\rangle \tag{2.6}$$

where $M > 0$ is the mass. The little superalgebra is then

$$\mathfrak{s}_\ell^0 \oplus \mathfrak{s}^1 \tag{2.7}$$

with $\mathfrak{s}_\ell^0 = so(3) \oplus su(2)_R \oplus u(1)_R$. (Sometimes the little bosonic algebra will leave off the $u(1)_R$ or even the entire R-symmetry summand.)

The states satisfying (2.6) form a finite dimensional representation ρ of the little superalgebra. The algebra of the odd generators acting on ρ is that of a *Clifford algebra* and therefore we try to represent that.

To make an irreducible representation of the $Q, \bar{Q}$'s we need to diagonalize the quadratic form on the RHS. This can be done as follows (we will find the following a convenient computation in Lecture III on line operators):

Let us assume that $Z \neq 0$. (The case $Z = 0$ can be found in Chapter II of Bagger-Wess [3]. Define a phase α by

$$Z = e^{i\alpha}|Z| \tag{2.8}$$

A particle at rest at the origin $x^i = 0$ of spatial coordinates is invariant under spatial involution. This suggests we consider the *involution* of the the superalgebra given by parity together with $U(1)_R$ symmetry rotation by a phase. That decomposes the supersymmetries into

$$\mathfrak{s}^1 = \mathfrak{s}^{1,+} \oplus \mathfrak{s}^{1,-} \tag{2.9}$$

Define:

$$\mathcal{R}_\alpha{}^A = \xi^{-1}Q_\alpha{}^A + \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}A} \tag{2.10}$$

$$\mathcal{T}_\alpha{}^A = \xi^{-1}Q_\alpha{}^A - \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}A} \tag{2.11}$$

for the supersymmetries transforming as ± 1 under the involution, respectively. Here ξ is a phase: $|\xi| = 1$ and the R-symmetry rotation is by $\zeta = \xi^{-2}$.

These operators satisfy the Hermiticity conditions

$$\begin{aligned} (\mathcal{R}_1{}^1)^\dagger &= -\mathcal{R}_2{}^2 \\ (\mathcal{R}_1{}^2)^\dagger &= \mathcal{R}_2{}^1 \end{aligned} \tag{2.12}$$

Then, on V we compute:

$$\{\mathcal{R}_\alpha{}^A, \mathcal{R}_\beta{}^B\} = 4\left(M + \mathrm{Re}(Z/\zeta)\right)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{2.13}$$

$$\{\mathcal{T}_\alpha{}^A, \mathcal{T}_\beta{}^B\} = 4\left(-M + \mathrm{Re}(Z/\zeta)\right)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{2.14}$$

Together with the Hermiticity conditions we now see that

$$\left(\mathcal{R}_1{}^1 + (\mathcal{R}_1{}^1)^\dagger\right)^2 = \left(\mathcal{R}_1{}^2 + (\mathcal{R}_1{}^2)^\dagger\right)^2 = 4(M + \mathrm{Re}(Z/\zeta)) \tag{2.15}$$

Since the square of an Hermitian operator must be positive semidefinite we obtain the *BPS bound*:

$$M + \mathrm{Re}(Z/\zeta) \geq 0. \tag{2.16}$$

This bound holds for any ζ, and therefore we can get the *strongest* bound by taking $\zeta = -e^{i\alpha}$, in which case

$$M \geq |Z| \tag{2.17}$$

Moreover, when we make the choice $\zeta = -e^{i\alpha}$, a little computation shows that

$$\begin{aligned}
\{\mathcal{R}_\alpha{}^A, \mathcal{R}_\beta{}^B\} &= 4\left(M - |Z|\right)\epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\mathcal{T}_\alpha{}^A, \mathcal{T}_\beta{}^B\} &= -4\left(M + |Z|\right)\epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\mathcal{R}_\alpha^A, \mathcal{T}_\beta^B\} &= 0
\end{aligned} \tag{2.18}$$

2.2.1 Long representations of $\mathcal{N} = 2$

How shall we construct representations of (2.7) ?

Suppose first that $M > |Z|$. Representations in this case are known as "non-BPS" or "long" representations of the superalgebra.

Since $M - |Z| \neq 0$ after making a suitable positive rescaling we can define generators $\hat{\mathcal{R}}$ and $\hat{\mathcal{T}}$ such that

$$\begin{aligned}
\{\hat{\mathcal{R}}_\alpha{}^A, \hat{\mathcal{R}}_\beta{}^B\} &= \epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\hat{\mathcal{T}}_\alpha{}^A, \hat{\mathcal{T}}_\beta{}^B\} &= -\epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\hat{\mathcal{R}}_\alpha^A, \hat{\mathcal{T}}_\beta^B\} &= 0
\end{aligned} \tag{2.19}$$

Then we have two (graded) commuting Clifford algebras. The representation theory of the $\hat{\mathcal{R}}'s$ and $\hat{\mathcal{T}}'s$ can be considered separately, so we search for the most general representation of the superalgebra $\mathfrak{s}_\ell^0 \oplus \mathfrak{s}^{1,+}$ (the representation theory of $\mathfrak{s}_\ell^0 \oplus \mathfrak{s}^{1,-}$ will be identical, and it will then be easy to combine these to obtain the general representation of $\mathfrak{s}_\ell^0 \oplus \mathfrak{s}^1$).

Each of $\mathfrak{s}^{1,\pm}$ is itself a sum of two (graded) commuting Clifford algebras on two generators, e.g. $\mathfrak{s}^{1,+}$ is:

$$(\hat{\mathcal{R}}_1^1)^2 = (\hat{\mathcal{R}}_2^2)^2 = 0 \qquad \& \qquad \{\hat{\mathcal{R}}_1^1, \hat{\mathcal{R}}_2^2\} = -1 \tag{2.20}$$

$$(\hat{\mathcal{R}}_1^2)^2 = (\hat{\mathcal{R}}_2^1)^2 = 0 \qquad \& \qquad \{\hat{\mathcal{R}}_1^2, \hat{\mathcal{R}}_2^1\} = 1 \tag{2.21}$$

The irreducible representation of each of the Clifford algebras (2.20) and (2.21) is two-dimensional, and the irrep of the Clifford algebra of the $\mathcal{R}_\alpha^A$ is then four-dimensional. To construct these we should choose a Clifford vacuum. Clearly it is natural to regard either $\hat{\mathcal{R}}_1^1$ or $\hat{\mathcal{R}}_2^2$ as a creation operator, and then the other serves as an annihilation operator.

Since the algebra of the $\hat{\mathcal{R}}$'s is invariant under $\mathfrak{s}_\ell^0$ this 4-dimensional representation can be promoted to a representation of the superalgebra $\mathfrak{s}_\ell^0 \oplus \mathfrak{s}^{1,+}$. To construct it consider a state $|\Omega\rangle$ of maximal eigenvalue of J_3. Then

$$\mathcal{R}_1{}^A|\Omega\rangle = 0 \tag{2.22}$$

Therefore, the irreducible Clifford representation of (2.20) generated by $|\Omega\rangle$ is the span of $\{|\Omega\rangle, \mathcal{R}_2{}^2|\Omega\rangle\}$ while the irrep of (2.20) and (2.21) is the span of

$$\rho_{hh} = Span\{|\Omega\rangle, \mathcal{R}_2{}^1|\Omega\rangle, \mathcal{R}_2{}^2|\Omega\rangle, \mathcal{R}_2{}^1\mathcal{R}_2{}^2|\Omega\rangle\} \tag{2.23}$$
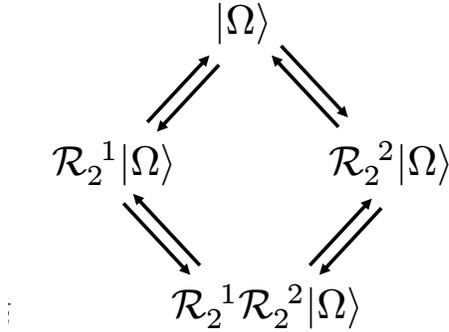
Figure 1: Showing the action of the supersymmetries in the basic half-hypermultiplet representation.

We can get a representation of $so(3) \oplus su(2)_R$ if we take $|\Omega\rangle$ to be the highest weight state in the rep $(\frac{1}{2}; 0)$. Altogether, ρ_{hh} as a representation of $so(3) \oplus su(2)_R$ is

$$\rho_{hh} \cong (0; \frac{1}{2}) \oplus (\frac{1}{2}; 0). \tag{2.24}$$

This important representation is known as the *half-hypermultiplet*. Note that it is $\mathbb{Z}_2$ graded with $\rho_{hh}^0 \cong (0; \frac{1}{2})$ and $\rho_{hh}^1 \cong (\frac{1}{2}; 0)$.

It is shown in [3] that the general representation of $[so(3) \oplus su(2)]_R \oplus \mathfrak{s}^{1,+}$ is of the form

$$\rho_{hh} \otimes \mathfrak{h} \tag{2.25}$$

where $\mathfrak{h}$ is an arbitrary representation of $\mathfrak{s}_\ell^0 \cong so(3) \oplus su(2)_R$.

Now, to get representations of the full superalgebra we apply this construction to the Clifford algebras generated by the $\mathcal{R}_\alpha^A$ and the $\mathcal{T}_\alpha^A$, and the general representation of the little superalgebra is of the form

$$\text{LONG \quad REP}: \qquad \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h} \qquad\qquad (2.26)$$

where $\mathfrak{h}$ is an arbitrary representation of $so(3) \oplus su(2)_R$, and the first and second factors are the half-hypermultiplet representations for $\mathcal{R}$ and $\mathcal{T}$, respectively.

Example: The smallest (long) representation is obtained by taking $\mathfrak{h}$ to be the trivial one-dimensional representation:

$$\rho_{hh} \otimes \rho_{hh} = 2(0;0) \oplus (0;1) \oplus 2(\frac{1}{2};\frac{1}{2}) \oplus (1;0). \qquad\qquad (2.27)$$

2.2.2 Short representations of $\mathcal{N} = 2$

When the bound (2.17) is saturated something special happens: The quadratic form in the Clifford algebra of the $\mathcal{R}_\alpha{}^A$ degenerates and becomes zero. In a unitary representation these operators must therefore be represented as zero. Such representations are called "short" or *BPS representations*.

Definition: We refer to the $\mathcal{R}_\alpha{}^A$ as *preserved supersymmetries* and the $\mathcal{T}_\alpha{}^A$ as *broken supersymmetries*.

The representations are now "shorter" – since we need only represent the clifford algebra $\mathfrak{s}^{1,-}$ generated by the $\mathcal{T}$'s. Thus we have the BPS or short representations:

$$\text{SHORT \quad REP}: \qquad \rho_{hh} \otimes \mathfrak{h} \qquad\qquad (2.28)$$

where $\mathfrak{h}$ is an arbitrary finite-dimensional unitary representation of $so(3) \oplus su(2)_R$.

The two main examples are

1. The simplest representation is the half-hypermultiplet with $\mathfrak{h}$ equals the one-dimensional trivial rep and is just ρ_{hh}. It consists of a pair of scalars in a doublet of $su(2)_R$ and a Dirac fermion.

2. Another important representation is the *vectormultiplet* obtained by taking $\mathfrak{h} = (\frac{1}{2};0)$. As a representation of $so(3) \oplus su(2)_R$ it is

$$\rho_{vm} \cong (0;0) \oplus (\frac{1}{2};\frac{1}{2}) \oplus (1;0) \qquad\qquad (2.29)$$

2.3 Field Multiplets

The particle representations described above have corresponding free field multiplets. The two most important are:

1. Vectormultiplet: Let $\mathfrak{g}$ be a compact Lie algebra. An $\mathcal{N} = 2$ vectormultiplet has a scalar φ, fermions $\psi_{\alpha A}$, in the $(\mathbf{2};\mathbf{1}) \otimes 2$ of $so(1,3) \oplus su(2)_R$, (and their complex conjugates $\bar{\psi}_{\dot{\alpha}A} := (\psi_\alpha{}^A)^\dagger$), an Hermitian gauge field A_m and an auxiliary field $D_{AB} = D_{BA}$ satisfying the reality condition $(D_{AB})^* = -D^{AB}$. After multiplication by i all these fields are valued in the adjoint of $\mathfrak{g}$. Taking the case $\mathfrak{g} = u(1)$ we recognize the particle content of the vm described above.[1]

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[1] A possible source of confusion here: These fields correspond to massless particle representations, which we have not discussed.

2. Hypermultiplet: Consists of 2 complex scalar fields, in the spin $\frac{1}{2}$ representation of $SU(2)_R$ and a pair of Dirac fermions which are singlets under $SU(2)_R$. [2]

3. In $N = 2$ supergravity, there is in addition, a supermultiplet with the graviton and 2 gravitinos. We will not need the explicit form in these lectures.

In these lectures we are mostly focussing on vectormultiplets. The supersymmetry transformations of the vectormultiplet are:

$$
\begin{aligned}
[Q_{\alpha A}, \varphi] &= -2\psi_{\alpha A} \\
[\bar{Q}_{\dot{\alpha}A}, \varphi] &= 0 \\
[Q_{\alpha A}, A_m] &= i\bar{\psi}_{\dot{\beta}A}(\bar{\sigma}_m)^{\dot{\beta}}{}_\alpha \\
[\bar{Q}_{\dot{\alpha}A}, A_m] &= -i(\bar{\sigma}_m)_{\dot{\alpha}}{}^\beta \psi_{\beta A} \\
[Q_{\alpha A}, \psi_{\beta B}] &= \sigma^{mn}_{\beta\alpha} F_{mn} \epsilon_{AB} + iD_{AB}\epsilon_{\beta\alpha} + \frac{i}{2} g \epsilon_{\beta\alpha} \epsilon_{AB}[\varphi^\dagger, \varphi] \\
[\bar{Q}_{\dot{\alpha}A}, \psi_{\beta B}] &= -i\epsilon_{AB}\sigma^m_{\beta\dot{\alpha}} D_m \varphi \\
[Q_{\alpha A}, D_{BC}] &= \left(\epsilon_{AB}\sigma^{m\dot{\beta}}_\alpha D_m \bar{\psi}_{\dot{\beta}C} + B \leftrightarrow C \right) \\
&\quad + g \left(\epsilon_{AB}[\varphi^\dagger, \psi_{\alpha C}] + B \leftrightarrow C \right)
\end{aligned}
\tag{2.30}
$$

2.4 Families of Theories

We are going to be considering *families* of $\mathcal{N} = 2$ theories. One way in which we can make families is by varying coupling constants in the Lagrangian.

Another very important source of families is the fact that the $\mathcal{N} = 2$ theories we will consider have *families of quantum vacua*. These vacua are typically labeled by expectation values of various operators in the theory, and the parameter space of the vacua is known as the moduli space of vacua.

These vacuum moduli spaces form a manifold. We will be restricting our attention to only one kind of vacuum known as the "Coulomb branch." The manifold of vacua in the Coulomb branch will be denoted $\mathcal{B}$ and a generic point will be denoted $u \in \mathcal{B}$. In a weakly-coupled Lagrangian formulation one can think of these parameters as boundary conditions of vm scalars for $\vec{x} \to \infty$.

Example: For example, in pure $SU(N)$ $\mathcal{N} = 2$ gauge theory the classical potential energy is just

$$
\int d^3\vec{x} \mathrm{Tr}(\vec{E}^2 + \vec{B}^2) + \mathrm{Tr}(\vec{D}\varphi)^2 + \mathrm{Tr}([\varphi^\dagger, \varphi])^2
\tag{2.31}
$$

and hence the energy is minimized for flat gauge fields (which are therefore gauge equivalent to zero on $\mathbb{R}^3$ with φ a space-time independent normal matrix. Gauge transformations

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[2]The half-hypermultiplet has $\dim_c \rho^0 = 2$. That corresponds to two *real* scalar fields. The full hypermultiplet field representation has four *real* scalar fields, and hence the corresponding particle representation has $\dim_c \rho^0 = 4$.

act by conjugation by a unitary matrix and hence the vacua are characterized by the (unordered) set of eigenvalues of φ, which may be parameterized by

$$u_i = \lim_{\vec{x} \to \infty} \mathrm{Tr}(\varphi(\vec{x}))^i \tag{2.32}$$

for $i = 2, \ldots, N$. The moduli space thus looks like a copy of $\mathbb{C}^{N-1}$. Now, because of $\mathcal{N} = 2$ supersymmetry it turns out that the classical vacua are not removed by quantum effects, although the low energy physical excitations above those vacua can be very strongly modified by quantum effects.

In fact, it turns out that $\mathcal{B}$ is a *complex* manifold (this is clear in our example above) [3] Moreover, the central charge function Z is a *holomorphic function* on $\mathcal{B}$. In the mid-1990's it was realized by N. Seiberg and others that the constraint of holomorphy, when skillfully employed, is a very powrful constraint on supersymmetric dynamics [24]. As shown by Seiberg and Witten (and sketched below), the central charge function for families of $\mathcal{N} = 2$ theories can be determined exactly.

In a theory with massive particles we can consider the *one-particle Hilbert space*, denoted $\mathcal{H}$. It is the Hilbert space of the theory with a single particle excitation above the groundstate and is a direct sum over the massive particles of a unitary representation of the $\mathcal{N} = 2$ superalgebra $\mathfrak{s}$. [4] When our theories come in families labeled by $u \in \mathcal{B}$ the one-particle Hilbert space, as a representation of $\mathfrak{s}$, will depend on u. In particular, the spectrum of P^2 (i.e. the spectrum of masses of 1-particle states) and the spectrum of central charges Z depends on u. We sometimes write $\mathcal{H}_u$ to emphasize this dependence. In general the spectrum will be a very complicated function of u. However, as we noted above, for the short-representations, or BPS representations $M = |Z(u)|$ and hence, since $Z(u)$ is holomorphic the (possible) BPS spectrum can be determined exactly. This motivates the definition:

Definition: The space of BPS states is the subspace of the one-particle Hilbert space:

$$\mathcal{H}_u^{BPS} = \{\psi : H\psi = |Z|\psi\} \subset \mathcal{H}_u \tag{2.33}$$

An important point here is that knowing $Z(u)$ does not fully determined the BPS spectrum, since a representation might or might not appear in $\mathcal{H}^{BPS}$. When it does appear for some value of u we expect it to remain in the spectrum as a "constant" in some open neighborhood of u. Note that, from the characters it is clear that an irreducible BPS representation cannot become a non-BPS representation. In this sense it is "rigid."

These considerations raise the problem of finding a method of determining the BPS spectrum in $\mathcal{N} = 2$ theories. Despite much progress, in its full generality this is an open problem: There is no algorithm for computing the BPS spectrum of a general $\mathcal{N} = 2$ field theory or supergravity compactification. The only theories for which a complete prescription is known are the so-called A_1 theories in class $\mathcal{S}$. This class of theories is

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[3]In fact, $\mathcal{B}$ has admits a kind of geometry called *special Kähler geometry*. Some aspects of special Kahler geometry are summarized in Sectino 2.6 below.

[4]Although we are definitely going to consider theories with massless particles, we are going to ignore the very subtle issues related to the continuum of states associated with arbitrarily soft massless particles.

discussed in the lectures of Davide Gaiotto. As far as we know, there is no example of a complete BPS spectrum for compactification of type II strings on a compact Calabi-Yau manifold.

2.5 Low energy effective theory: Abelian Gauge Theory

2.5.1 Spontaneous symmetry breaking

At generic points on the moduli space of vacua there is an unbroken ABELIAN gauge symmetry of rank r.

We can illustrate this nicely with weakly coupled pure $SU(N)$ $\mathcal{N} = 2$ theory. The Lagrangian has a potential energy term $\mathrm{Tr}(\varphi^\dagger, \varphi])^2$ and minimizing this energy means that φ is a normal matrix - which is unitarily diagonalizable:

$$\langle \varphi \rangle = Diag\{a_1, a_2, \ldots, a_N\} \tag{2.34}$$

with $\sum a_i = 0$. For generic values of the a_i the VEV $\langle \varphi \rangle$ breaks the gauge symmetry to the abelian gauge group semidirect product with the Weyl group. We recognize the u_i as symmetric power functions in a_i, (which are gauge invariant). Moreover, the low energy abelian gauge group has rank $r = N - 1$.

Remark: From (2.34) one might get the impression that when collections of a_i vanish, then nonabelian gauge symmetry is restored. One of the remarkable results of Seiberg-Witten theory is that in fact this does not happen: Strong coupling dynamics drastically alter the weak-coupling picture and in fact there are not massless vector multiplets.

2.5.2 Electric and Magnetic Charges

At low energies we describe the theory by a rank r abelian gauge theory. In this theory there are electric and magnetic charges, and these satisfy the Dirac quantization rule.

Extending the exercise 5.1 below to the case of r abelian gauge fields we learn that Dirac quantization says that the electric and magnetic charges are valued in a Γ and moreover this lattice is endowed with a *symplectic form*, i.e. an antisymmetric integral valued form:

$$\langle \gamma_1, \gamma_2 \rangle = -\langle \gamma_2, \gamma_1 \rangle \in \mathbb{Z} \tag{2.35}$$

which is, moreover, nondegenerate: If $\langle \gamma, \delta \rangle = 0$ for all $\delta \in \Gamma$ then $\gamma = 0$. [5]

The corresponding vector space $V = \Gamma \otimes \mathbb{R}$ is a symplectic vector space.

2.5.3 Self-dual abelian gauge theory

It turns out that in the Seiberg-Witten solution the low energy abelian gauge theory is a *self-dual gauge theory*.

The basic fieldstrength is a two-form on spacetime, just as in Maxwell's theory, but unlike Maxwell's theory the 2-form is valued in a symplectic vector space: $\mathbb{F} \in \Omega^2(M^4) \otimes V$.

In order to define the theory we add the data of a *positive compatible complex structure*. This means

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[5]It is actually better to include flavor charges in the discussion. Then the flavor+gauge charge lattice is only Poisson and not symplectic. For simplicity we restrict attention to the symplectic case.

1. *Complex structure*: There is an $\mathbb{R}$-linear operator $\mathcal{I} : V \to V$ such that $\mathcal{I}^2 = -1$.

2. *Compatible* Moreover $\langle \mathcal{I}(v), \mathcal{I}(v') \rangle = \langle v, v' \rangle$ for all $v, v' \in V$.

3. *Positive* Since $\mathcal{I}$ is compatible with the symplectic product we can introduce the *symmetric* bilinear form

$$g(v, v') := \langle v, \mathcal{I}(v') \rangle \tag{2.36}$$

We will assume that g is positive definite. We will often denote the metric simply by (v, v').

Now we have

$$\mathbb{F} \in \Omega^2(M_4) \otimes V \tag{2.37}$$

Now $*^2 = -1$ on $\Omega^2(M_4)$ for M_4 of Lorentzian signature, so $s := * \otimes \mathcal{I}$ squares to 1 and we can impose the ϵ-self-duality constraint

$$s\mathbb{F} = \epsilon\mathbb{F} \tag{2.38}$$

where $\epsilon = +1$ for a *self-dual field* and $\epsilon = -1$ for an *anti-self-dual field*.

The dynamics of the field is simply the flatness equation:

$$d\mathbb{F} = 0 \tag{2.39}$$

but in order to recognize this as a generalization of Maxwell's theory we need to do some linear algebra.

2.5.4 Lagrangian decomposition of a symplectic vector with compatible complex structure

In our physical considerations we will be choosing "duality frames." This will amount to choosing a Darboux basis for Γ denoted $\{\alpha_I, \beta^I\}$ where $I = 1, \ldots, r$. Our convention is that

$$\begin{aligned}
\langle \alpha_I, \alpha_J \rangle &= 0 \\
\langle \beta^I, \beta^J \rangle &= 0 \\
\langle \alpha_I, \beta^J \rangle &= \delta_I{}^J
\end{aligned} \tag{2.40}$$

The $\mathbb{Z}$-linear span of α_I is a maximal Lagrangian sublattice L_1 while that for β^I is another L_2 and we have a Lagrangian decomposition:

$$\Gamma \cong L_1 \oplus L_2. \tag{2.41}$$

Upon complexification we have $V \otimes_{\mathbb{R}} \mathbb{C} \cong V^{0,1} \oplus V^{1,0}$. The $\mathbb{C}$-linear extension of $\mathbb{C}$ is $-i$ on $V^{0,1}$ and $+i$ on $V^{1,0}$.

Given a Darboux basis we can define a basis for $V^{0,1}$:

$$f_I := \alpha_I + \tau_{IJ}\beta^J \qquad I = 1, \ldots, r \tag{2.42}$$

while $V^{1,0}$ is spanned by

$$\bar{f}_I := \alpha_I + \bar{\tau}_{IJ}\beta^J \qquad I = 1, \ldots, r \qquad (2.43)$$

So we have:

$$\mathcal{I}(f_I) = -if_I \qquad \mathcal{I}(\bar{f}_I) = +i\bar{f}_I \qquad (2.44)$$

Compatibility if $\mathcal{I}$ with the symplectic structure now implies $\langle f_I, f_J \rangle = 0$ and hence $\tau_{IJ} = \tau_{JI}$. It is useful to write τ_{IJ} in its real and imaginary parts

$$\tau_{IJ} = X_{IJ} + iY_{IJ} \qquad (2.45)$$

Positive definiteness of g implies Y_{IJ} is positive definite. It will be convenient to denote the matrix elements of the inverse by Y^{IJ} so

$$Y^{IJ}Y_{JK} = \delta^I{}_K \qquad (2.46)$$

Using the inverse transformations to (2.42) (2.43):

$$\beta^I = -\frac{i}{2}Y^{IJ}(f_J - \bar{f}_J)$$
$$\alpha_I = \frac{i}{2}\bar{\tau}_{IJ}Y^{JK}f_K - \frac{i}{2}\tau_{IJ}Y^{JK}\bar{f}_K \qquad (2.47)$$

We compute the action of $\mathcal{I}$ in the Darboux basis:

$$\mathcal{I}(\alpha_I) = \alpha_K(Y^{-1}X)^K{}_I + \beta^K(Y + XY^{-1}X)_{KI}$$
$$\mathcal{I}(\beta^I) = -\alpha_K Y^{KI} - \beta^K(XY^{-1})_K{}^I \qquad (2.48)$$

2.5.5 Lagrangian formulation

Equations (2.37), (2.38), (2.39) summarize the entire theory in a manifestly invariant way.

Usually physicists choose a Darboux basis or "duality frame" for V. We can then define components of $\mathbb{F}$:

$$\mathbb{F} = \alpha_I F^I - \beta^I G_I \qquad (2.49)$$

If we impose the ϵSD equations (2.38) then, using (2.48) we can then solve for G_I in terms of F^I and the complex structure:

$$G_J = -\epsilon Y_{JK} * F^K - X_{JK}F^K \qquad (2.50)$$

Now the equation (2.39) splits naturally into two

$$dF^I = 0 \qquad (2.51)$$

$$dG_J = -\epsilon d\left(Y_{JK} * F^K + \epsilon X_{JK}F^K\right) = 0 \qquad (2.52)$$

(2.51) is the Bianchi identity and (2.52) is the equation of motion of a generalization of Maxwell theory.

Because of (2.51) we can locally solve $F^I = dA^I$ and thereby write an action principle with action proportional to

$$\int_{M_4} \left(Y_{JK}F^J * F^K + \epsilon X_{JK}F^J F^K\right) \qquad (2.53)$$

This is part of the low energy action in the Seiberg-Witten theory.

1. From this Lagrangian we learn the physical reason for demanding that the bilinear form $g(v, v')$ be positive definite. If Y_{IJ} were diagonal then $Y_{IJ} = \delta_{IJ} \frac{1}{e_I^2}$ would be the matrix of coupling constants, and X_{IJ} would be a matrix of theta-angles. Thus, the data of the complex structure on V summarizes the complexified gauge coupling of the theory.

2. Note that the difference between the self-dual and anti-self-dual case is the relative sign of the parity-odd term, as expected.

3. The self-dual equations (2.37), (2.38), (2.39) do not follow from a relativistically invariant action principle. This is a famous surprising property of self-dual field theories. However, once one chooses a duality frame (which induces a Lagrangian splitting in the space of fields) one can indeed write an action principle, as above. A very similar remark applies to the self-dual theory of (abelian) tensormultiplets in six-dimensional supergravity. Once one chooses a Lagrangian decomposition of fieldspace one can write a covariant action principle for this self-dual theory [4]. This is not unrelated to the above examples: If we compactify (with a partial topological twist described in [15]) the six-dimensional abelian tensormultiplet theory on the Seiberg-Witten curve to get an abelian gauge theory in four dimensions we obtain precisely the Seiberg-Witten effective Lagrangian!

2.5.6 Duality Transformations

There is of course no unique choice of duality frame for V, although in different physical regimes one duality frame can be preferred. The change of description between different duality frames is given by an integral symplectic transformation $(\alpha_I, \beta^I) \to (\tilde{\alpha}_I, \tilde{\beta}^I)$ and leads to standard formulae for strong-weak electromagnetic duality transformations in abelian gauge theories.

In Problem 5.4 we guide the student through a series of formulae expressing how various quantities change under duality transformations.

2.5.7 Grading of the Hilbert space

Finally, we note that since we have unbroken abelian gauge symmetry we have abelian global symmetries (constant gauge transformations) which act as symmetries on the theory, and the Hilbert space must be a representation of these symmetries. The characters of the group of gauge symmetries of the *self-dual* theory is the lattice Γ of electric and magnetic charges and hence, our one-particle Hilbert space will be graded by Γ:

$$\mathcal{H}_u = \oplus_{\gamma \in \Gamma} \mathcal{H}_{u,\gamma} \qquad (2.54)$$

2.6 The constraints of $\mathcal{N} = 2$ supersymmetry

2.6.1 The bosonic part of the action

The action (2.53) is only part of the low-energy effective Lagrangian. At low energies the effective action is still $\mathcal{N} = 2$ supersymmetric, but now it is a theory of *abelian* vectormultiplets. For an $\mathcal{N} = 2$ supersymmetric Lagrangian we must add terms governing the

dynamics of the vectormultiplet scalars and the fermions, and the interactions. We must also explain how the complex structure $\mathcal{I}$, or equivalently, τ_{IJ} is determined.

It turns out that the constraint of $\mathcal{N} = 2$ supersymmetry is very powerful. It says that, once one chooses a duality frame, there is a corresponding collection of vectormultiplet scalars a^I, $I = 1, \ldots, r$ which form a local coordinate system for $\mathcal{B}$ together with a locally-defined holomorphic function $\mathcal{F}(a)$, known as the *prepotential*, such that

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}(a)}{\partial a^I \partial a^J} \tag{2.55}$$

Thus, the complex structure and hence the gauge couplings of the low energy abelian gauge theory are determined by $\mathcal{F}$.

Moreover, the bosonic terms in the low energy effective action are

$$S = -\int \frac{1}{4\pi} \mathrm{Im}\tau_{IJ} \left(da^I * d\bar{a}^J + F^I * F^J \right) + \epsilon \frac{1}{4\pi} \mathrm{Re}\tau_{IJ} F^I F^J \tag{2.56}$$

2.6.2 The central charge function

Now let us return to the electro-magnetic charge grading of the Hilbert space, (2.54).

In the Seiberg-Witten and supergravity examples the central charge operator Z is a *scalar* operator on each component $\mathcal{H}_{u,\gamma}$. This defines the central charge function, $Z(\gamma; u)$, sometimes written as $Z_\gamma(u)$.

Moreover, in all the examples the central charge function Z is *linear* in γ:

$$Z(\gamma_1 + \gamma_2; u) = Z(\gamma_1; u) + Z(\gamma_2; u) \tag{2.57}$$

that is $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$.

Moreover, the central charge function is not an arbitrary linear function on . If our duality frame is α_I, β^I then we will define

$$a^I = Z(\alpha_I; u) \qquad\qquad a_{D,I} := Z(\beta^I; u) \tag{2.58}$$

so that, if $\gamma = p^I \alpha_I - q_I \beta^I$ then

$$Z(\gamma; u) = q_I a^I + p^I a_{D,I}. \tag{2.59}$$

Again, using the constraint of $\mathcal{N} = 2$ supersymmetry it turns out that we necessarily have

$$a_{D,I} = \frac{\partial \mathcal{F}}{\partial a^I} \tag{2.60}$$

Note that $\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a^J}$. Recall that the a^I form a good set of holomorphic local coordinates on $\mathcal{B}$.

We urge to student to work through the problem 5.2

2.7 The Seiberg-Witten IR Lagrangian

We can now summarize the Seiberg-Witten solution in a sentence: The constraints of $\mathcal{N} = 2$ supersymmetry are satisfied by defining a family of (noncompact) Riemann surfaces Σ_u, depending on $u \in \mathcal{B}$ and identifying:

1. $\Gamma_u = H_1(\bar{\Sigma}_u; \mathbb{Z})$. The symplectic form is the intersection form.

2. The complex structure on Σ_u induces one on $V_u = \Gamma_u \otimes \mathbb{R}$ which is positive and compatible with the intersection form.

3. $Z_\gamma(u) = \oint_\gamma \lambda_u$ for a certain special meromorphic one-form λ_u on Σ_u. This differential, called the *Seiberg-Witten differential* is part of the data of the solution and does not simply follow from specifying Σ_u.

♣ Need to answer question: τ_{IJ} and λ are not independent data, so why do we need to specify λ in addition to $\bar{\Sigma}_u$? ♣

2.7.1 A basic example: $SU(2)$ SYM

A fundamental example - historically the first example understood by Seiberg and Witten - is that of $\mathfrak{g} = su(2)$ $\mathcal{N} = 2$ pure super-Yang-Mills theory. In this case $\mathcal{B} = \mathbb{C}$. The parameter $u \in \mathcal{B}$ can be identified with

$$u = \frac{1}{2} \langle \mathrm{Tr} \varphi^2 \rangle \tag{2.61}$$

for large u. The vacuum dynamics is controlled by the family of Riemann surfaces Σ_u

$$\Sigma_u = \{(t, v) : \Lambda^2 (t + \frac{1}{t}) = v^2 - 2u\} \subset \mathbb{C}^* \times \mathbb{C}, \tag{2.62}$$

where Λ is the scale set by the theory. The Seiberg-Witten differential is

$$\lambda_u = v \frac{dt}{t} \tag{2.63}$$

The Riemann surface is a torus with four punctures. Choosing A and B cycles for the torus we define

$$a(u) = \oint_A \lambda_u \qquad a_D(u) = \oint_B \lambda_u \tag{2.64}$$

The BPS spectrum in this example is completely known and will be discussed at the end of Lecture I in Section 2.14.2.

Remark: One very canonical way of viewing the surface is to regarded it as a double-covering of $C = \mathbb{C}^*$ in T^*C given by

$$\lambda^2 = \left(\frac{\Lambda^2}{t^3} + \frac{2u}{t^2} + \frac{\Lambda^2}{t} \right) (dt)^2 \tag{2.65}$$

It is a form of the Seiberg-Witten curve which comes out quite naturally from a viewpoint involving six-dimensional superconformal field theory. This viewpoint is described in the lectures of Davide Gaiotto.
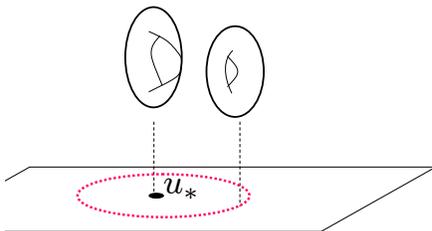
Figure 2: Near the singular point $u = \Lambda^2$ the basis of homology cycles on the SW curve has monodromy.

2.7.2 Singular Points and Monodromy

A very important point is that the family of Riemann surfaces degenerates at special values of u. This is clear from (2.65). There are two branch points from the prefactor at

$$t_\pm = -u \pm \sqrt{u^2 - 1} \tag{2.66}$$

(By a rescaling of u and λ we can set $\Lambda = 1$.) Because of the branch points at $t = 0, \infty$ it is also convenient to define $s^2 = t$. Then there are four branch points at

$$s = \pm\sqrt{-u \pm \sqrt{u^2 - 1}} \tag{2.67}$$

This makes it clear that the Seiberg-Witten curve is an elliptic curve with 4 singular points. (the lifts of $t = 0, \infty$).

When $u = \pm 1$ the branch points coincide and a nontrivial homology cycle of the Riemann surface pinches as shown in Figure 2. Correspondingly there is *monodromy* of the lattice of charges as u circles one of these singular points.

♣♣♣ You can illustrate the monodromy by choosing a natural basis of cycles and looking at the periods. Get a and $a_D \sim \frac{i}{\pi} a \log a$. Give more details here. ♣♣♣

(Historically, the derivation went the other way. Seiberg and Witten realized there had to be monodromy, and from this they deduced that the solution was based on a family of Riemann surfaces.)

2.8 The BPS Index and the Protected Spin Character

Let us now return to our problem of understanding the BPS spectrum in $\mathcal{N} = 2$ theories.

In trying to enumerate the BPS spectrum of a theory one encounters an important difficulty. It can happen that a non-BPS particle representation has a mass $M(u, \dots)$, which depends on u, as well as other parameters, generically satisfies $M(u, \dots) > |Z(u)|$

but for special values of u, (or the other parameters) it satisfies $M(u, \dots) = |Z(u)|$. When this happens, the non-BPS representation becomes a sum of BPS representations: This is clear since in the non-BPS representation $\rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h}$ the $\mathcal{R}$-supersymmetries act as zero on the first factor and hence we obtain a BPS representation $\rho_{hh} \otimes \mathfrak{h}'$ with

$$\mathfrak{h}' = \mathfrak{h} \otimes \left[(\tfrac{1}{2}; 0) \oplus (0; \tfrac{1}{2}) \right] \tag{2.68}$$

as a representation of $\mathfrak{s}_\ell^0$.

The problem is, that while "true" BPS representations which do not mix with non-BPS representations are - to some extent - independent of parameters and constitute a more solvable sector of the theory, the "fake" BPS representations of the above type are more difficult to control. In particular they can appear and disappear as a function of other parameters (such as hypermultiplet moduli).

We need a way to separate the "fake" BPS representations from the "true" BPS representations. One way to do this is to consider an *index*: This is some function which vanishes on the fake representations but is nonzero and counts the true representations. A good way to do this is to define a function which vanishes on all long representations and is continuous as the long-representation is deformed.

When the theory has $SU(2)_R$ symmetry we can introduce a nice index known as the *protected spin character.* [6]

A representation ρ of the massive little superalgebra $\mathfrak{s}_\ell^0 \oplus \mathfrak{s}^1$ has a character, defined by:

$$\text{ch}(\rho) = \text{Tr}_\rho x_1^{2J_3} x_2^{2I_3} \tag{2.69}$$

where J_3 is a generator of $so(3)$ and I_3 is a generator of $su(2)_R$.

Note that

$$\text{ch}(\rho_{hh}) = x_1 + x_1^{-1} + x_2 + x_2^{-1} \tag{2.70}$$

and therefore, for the general long representation (2.26) of $\mathcal{N} = 2$ we get:

$$\text{ch}(LONG) = (x_1 + x_1^{-1} + x_2 + x_2^{-1})^2 \text{ch}(\mathfrak{h}) \tag{2.71}$$

while for the general short representation (2.28) of $\mathcal{N} = 2$ we get:

$$\text{ch}(SHORT) = (x_1 + x_1^{-1} + x_2 + x_2^{-1}) \text{ch}(\mathfrak{h}) \tag{2.72}$$

The difference is by a factor of $(x_1 + x_1^{-1} + x_2 + x_2^{-1})$.

This suggests that we should consider the specialization:

$$x_1 \frac{\partial}{\partial x_1} \left(\text{Tr} x_1^{2J_3} x_2^{2I_3} \right) |_{x_1 = -x_2 = y} := \text{Tr}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3} \tag{2.73}$$

where $\mathcal{J}_3 = J_3 + I_3$.

---

[6]This was suggested to me by Juan Maldacena.

It is clear that this specialization vanishes on long representations, and therefore it vanishes on "fake" BPS representations. On the other hand from the character of the BPS representation (2.72) we get instead

$$\mathrm{Tr}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3} = (y - y^{-1})\mathrm{ch}(\mathfrak{h})|_{x_1=-x_2=y} \tag{2.74}$$

Now, when we combine the grading of $\mathcal{H}$ by electromagnetic charge (2.54) with the BPS condition we get a grading of the BPS subspace:

$$\mathcal{H}_u^{BPS} = \oplus_{\gamma \in \Gamma} \mathcal{H}_{u,\gamma}^{BPS} \tag{2.75}$$

and on $\mathcal{H}_{u,\gamma}^{BPS}$ the energy is $|Z(\gamma; u)|$.

In all known examples it turns out that $\mathcal{H}_{u,\gamma}^{BPS}$ are finite dimensional, and therefore we can form the traces

$$\mathrm{Tr}_{\mathcal{H}_{u,\gamma}^{BPS}}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3}. \tag{2.76}$$

We define the *Protected Spin Character* by the equation

$$(y - y^{-1})\Omega(\gamma; u; y) := \mathrm{Tr}_{\mathcal{H}_{u,\gamma}^{BPS}}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3} \tag{2.77}$$

that is:

$$\Omega(\gamma; u; y) = \mathrm{ch}(\mathfrak{h}_\gamma)|_{x_1=-x_2=y} = \mathrm{Tr}_{\mathfrak{h}}(-1)^{2J_3}(-y)^{2\mathcal{J}_3} \tag{2.78}$$

Example/Exercise: Show that the contribution of a half-hypermultiplet representation in $\mathfrak{h}_\gamma$ to Ω is just $\Omega(\gamma; u; y) = 1$ and the contribution of a vectormultiplet in $\mathfrak{h}_\gamma$ is $\Omega(\gamma; u; y) = y + y^{-1}$.

Remarks:

1. If we specialize to $y = -1$ then we get the *BPS index* $\Omega(\gamma; u)$.

2. Exercise: Show that we could have defined the BPS index directly via

$$\Omega(\gamma; u) = \frac{1}{2}\mathrm{Tr}_{\mathcal{H}_{\gamma,u}^{BPS}}(2J_3)^2(-1)^{2J_3} = \mathrm{Tr}_{\mathfrak{h}}(-1)^{2J_3} \tag{2.79}$$

   This quantity is known as the *second helicity supertrace*.

3. The BPS index (2.79) can be defined even if the $\mathcal{N} = 2$ superalgebra does not have an unbroken $su(2)_R$ symmetry. This is important since in supergravity we generally do not have that symmetry and should only work with BPS indices, and not protected spin characters.

4. Now, these BPS indices are piecewise constant functions of u, but they can still jump discontinuously. Our next goal is to explain how this can happen.

2.9 Supersymmetric Field Configurations

The analog of BPS particle representations in classical supersymmetric field theory are the "BPS field configurations." A supersymmetry transformation is said to be unbroken in a state $|\Psi\rangle$ if if $Q|\Psi\rangle = 0$. In such a state vev's should satisfy

$$\langle\Psi|[Q, \mathcal{O}]|\Psi\rangle = 0 \tag{2.80}$$

for all local operators $\mathcal{O}$. Now imagine that we have a coherent state in which we can identify vev's with classical fields:

$$\langle\Psi|\varphi(x^\mu)|\Psi\rangle = \varphi^{\text{classical}}(x^\mu) \tag{2.81}$$

Then we can apply the supersymmetry transformations to the classical fields to derive an analog of BPS states in classical field theory.

If we substitute bosonic fields for $\mathcal{O}$ then the identity is trivially satisfied, by Lorentz invariance. However, in the supersymmetry transformation laws

$$[Q, Fermi] = Bose$$

the RHS typically involves nontrivial (first order!) differential operators on the bosonic fields. Thus, a "BPS" or "supersymmetric field configuration" preserving the supersymmetry Q is a solution of the differential equation expressed by $[Q, Fermi] = 0$ for all the fermions in the theory.

Consider the BPS field configuration created by a dyonic BPS particle with central charge $Z = e^{i\alpha}|Z|$. From the representation theory we have seen that the preserved supersymmetries are $\mathcal{R}_\alpha{}^A$ with $\zeta = -e^{i\alpha}$. Therefore we search for a fixed point for the supersymmetries $\mathcal{R}_\alpha{}^A$ acting on the vectormultiplet. This leads to the equations on the bosonic fields: [7]

$$
\begin{aligned}
F_{0\ell} - \frac{i}{2}\epsilon_{jk\ell}F_{jk} - iD_\ell(\varphi/\zeta) &= 0 \\
D_0(\varphi/\zeta) - \frac{g}{2}[\varphi^\dagger, \varphi] &= 0
\end{aligned}
\tag{2.82}
$$

we have written them for a nonabelian vectormultiplet, but of course one can specialize to the abelian case. In that case we choose a duality frame (in order to write out the vectormultiplet and its susy transformations) and we find that the scalar field must be time-independent and moreover

$$F_{0\ell}^{+,I} = i\partial_\ell(\varphi^I/\zeta) \tag{2.83}$$

Now, it is natural to search for dyonic solutions to these field configurations. We therefore take as an ansatz

$$F^I = \frac{1}{2}\left(\omega_s \otimes \rho_M^I + \epsilon\omega_d \otimes \rho_E^I\right) \tag{2.84}$$

---

[7]These are the equations for pure SYM. From the Lagrangian one learns that the auxiliary field $D_{AB} = 0$. If we couple the theory to hypermultiplets then the D_{AB} can be nontrivial leading to Higgs branches of the moduli space. The three equations $D_{AB} = 0$ correspond in mathematics to hyperkahler moment map equations.

where

$$\omega_s := \sin\theta d\theta d\phi$$
$$\omega_d := \frac{drdt}{r^2} \qquad (2.85)$$

and ρ_M^I and ρ_E^I define vectors in $\mathfrak{t}$, a Cartan subalgebra of $\mathfrak{g}$. This would be the typical abelian gauge field created at long distances from a dyonic particle.

Choosing our orientation on $\mathbb{R}^{1,3}$ to be $d^3x \wedge dx^0$ we find $*_4\omega_s = \omega_d$ and therefore the fixed-point equations become

$$2\partial_\ell(\varphi^I/\zeta) = \partial_\ell(\frac{1}{r})(\rho_M^I - i\epsilon\rho_E^I) \qquad (2.86)$$

Now, we can solve this equation, but it will be much more convenient for us to put the solution into a duality-invariant form. Doing so is a little tricky, and we lead the student through the details in Problem 5.5.

The upshot is that the duality invariant fieldstrength is

$$\mathbb{F} = \frac{1}{2}\left(\omega_s \otimes \gamma_s + \epsilon\omega_d \otimes \gamma_d\right) \qquad (2.87)$$

where $\gamma_s, \gamma_d \in V$, and

$$\gamma_d = \mathcal{I}(\gamma_s) \qquad (2.88)$$

Dirac quantization imposes $\gamma_s \in \Gamma$. If we write

$$\gamma_s = p^I\alpha_I - q_I\beta^I \qquad (2.89)$$

with $p^I, q_I \in \mathbb{Z}$ then we find that $\rho_M^I = p^I \in \mathbb{Z}$ is the quantized magnetic charge, but $\rho_E^I = Y^{IJ}(q_J + X_{JK}p^K)$.

As we show in Problem 5.5 the fixed point equations will be solved if the vectormultiplet moduli become r-dependent and satisfy:

$$2\mathrm{Im}\left[\zeta^{-1}Z(\gamma; u(r))\right] = \frac{\langle\gamma, \gamma_c\rangle}{r} + 2\mathrm{Im}\left[\zeta^{-1}Z(\gamma; u)\right] \qquad \forall\gamma \in \Gamma \qquad (2.90)$$

where u on the RHS are the vacuum moduli at $r = \infty$.

Remark: The analog of these equations in $\mathcal{N} = 2$ supergravity are the famous *attractor equations* for constructing dyonic black holes. We will describe them in Lecture II. See (3.10) below.

2.10 BPS boundstates of BPS particles

Let us suppose that we have a very heavy BPS particle of charge γ_c.

Let us consider the dynamics of a second BPS particle of charge γ_h. The γ_h-particle is much lighter than the γ_c-particle:

$$|Z(\gamma_c; u)| \gg |Z(\gamma_h; u)| \qquad (2.91)$$

and therefore we can consider its dynamics in the probe approximation where we view it as moving in a fixed BPS field configuration created by the γ_c-particle. Thus we take $\gamma_s = \gamma_c$ in (3.2) and (2.90).

In this approximation the dynamics of the γ_h-particle is governed by the action

$$\int |Z(\gamma_h; u(r))| ds + \int \langle \gamma_h, \mathbb{A} \rangle \tag{2.92}$$

where we integrate along the worldline of the probe particle and $\mathbb{F} = d\mathbb{A}$ is the fieldstrength created by the heavy γ_c-particle. The energy of such a particle at rest is therefore

$$\begin{aligned} E &= |Z(\gamma_h; u(r))| - \langle \gamma_h, \mathbb{A}_0 \rangle \\ &= |Z(\gamma_h; u(r))| + \frac{(\gamma_h, \gamma_c)}{2r} \end{aligned} \tag{2.93}$$

The second term, $\frac{(\gamma_h, \gamma_c)}{2r}$ is the Coulomb energy and it is expressed in terms of the positive definite symmetric metric on V formed using the complex structure: $(v_1, v_2) := \langle v_1, \mathcal{I} v_2 \rangle$. Note this expression is duality invariant, symmetric in the charges and positive definite, as is physically reasonable. [8]

Now, by taking the real part of the fixed point equations and writing the duality invariant extension we find

$$2\mathrm{Re}\left[\zeta^{-1} Z(\gamma_h; u(r))\right] = \frac{(\gamma_h, \gamma_c)}{r} + 2\mathrm{Re}\left[\zeta^{-1} Z(\gamma_h; u)\right] \tag{2.94}$$

Again, we refer to Problem 5.5 for some hints as to how to show this.

In this way we derive the formula for the energy of a halo particle probing the IR background associated with the core charge γ_c:

$$E_{probe} = |Z_{\gamma_h}(u(r))| \left(1 - \cos(\alpha_h(r) - \alpha_c)\right) - \mathrm{Re}(Z_{\gamma_h}(u)/\zeta). \tag{2.95}$$

Recall that $u \in \mathcal{B}$ stands for the value of the vacuum moduli at infinity. The moduli detected by the probe particle at a distance r from the heavy dyon depends on r, and we denote $Z_{\gamma_h}(u(r)) = |Z_{\gamma_h}(u(r))| e^{i\alpha_h(r)}$. On the other hand $\zeta = -e^{i\alpha_c}$ is independent of r.

The energy is clearly minimized at the value of r for which the cosine is equal to $+1$, that is, for the value of r at which $\alpha_h(r) = \alpha_c$. Note this is the place where the central charge $Z_{\gamma_h}(u(r))$ becomes parallel to $Z_{\gamma_c}(u)$. From the "attractor equation" (2.90) we compute that boundstate radius to be:

$$R_{12} = \frac{1}{2} \langle \gamma_h, \gamma_c \rangle \frac{1}{\mathrm{Im} Z_{\gamma_h} e^{-i\alpha_c}} \tag{2.96}$$

The physics here is that the position-dependent vm scalars give a position dependent mass, leading to attraction or repulsion. Similarly, there is attraction or repulsion due to the electromagnetic field of the dyon. The boundstate radius is the radius at which these two forces balance each other.

Finally, we will not show the details, but by studying the supersymmetric quantum mechanics of the probe BPS particle one finds that this boundstate is a supersymmetric state, (see the analysis in [8]) and therefore

Two BPS particles can form a BPS boundstate.

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[8] A derivation is to use the dyonic field (3.2), with $\epsilon = +1$, to compute $\mathbb{A}_0 = -\frac{\gamma_d}{2r}$.
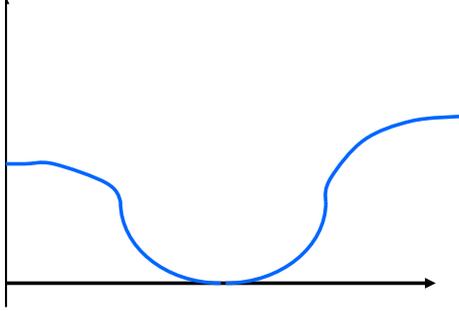
Figure 3: The potential energy of a probe particle in the field of a dyon.

2.11 Denef's boundstate radius formula

In our probe approximation we have treated the probe particle and the source asymmetrically, but it is clear what the symmetric version should be:

If two dyonic BPS particles or black holes of electromagnetic charges γ_1, γ_2 in a vacuum u form a BPS boundstate then that boundstate has total electromagnetic charge $\gamma_1 + \gamma_2$ and boundstate radius :

$$R_{12} = \frac{1}{2}\langle\gamma_1,\gamma_2\rangle\frac{|Z(\gamma_1;u) + Z(\gamma_2;u)|}{\mathrm{Im}Z(\gamma_1;u)\overline{Z(\gamma_2;u)}} \tag{2.97}$$

Remarks:

1. Note that (2.97) can equivalently be written as

$$R_{12} = \frac{1}{2}\langle\gamma_1,\gamma_2\rangle\frac{1}{\mathrm{Im}e^{-i\alpha}Z(\gamma_1;u)} \tag{2.98}$$

   where α is the phase of $Z(\gamma_1;u) + Z(\gamma_2;u)$, so in the limit $|Z(\gamma_2;u)| \gg |Z(\gamma_1;u)|$ (that is, the probe approximation limit) equation (2.98) reduces to (2.96).

2. In the case of supergravity, this formula can be derived very directly by explicit construction of BPS solutions of the supergravity equations representing BPS boundstates of BPS dyonic black holes. We will indicate the relevant solution in the next lecture.

3. Since the boundstate radius must be positive, a crucial corollary of the above result is the *Denef stability condition*: If BPS particles of charges γ_1 and γ_2 form a BPS boundstate in the vacuum u then it must necessarily be that

$$\langle\gamma_1,\gamma_2\rangle\mathrm{Im}Z(\gamma_1;u)\overline{Z(\gamma_2;u)} > 0. \tag{2.99}$$

2.12 Marginal stability

Now that we have found that two BPS particles can make a BPS boundstate we can ask if that boundstate can decay. For example, could there be a tunneling process where the particles fly apart to infinity?

The answer is - in general - NO!

If BPS particles of charges γ_1 and γ_2 form a BPS boundstate of charge $\gamma_1 + \gamma_2$ then we can compute the binding energy:

$$|Z(\gamma_1 + \gamma_2; u)| - |Z(\gamma_1; u)| - |Z(\gamma_2; u)| \qquad (2.100)$$

Because Z is linear in γ we can use the triangle inequality to conclude that this is nonpositive. Moreover, this binding energy is negative, and therefore the particles cannot separate to infinity *unless* $Z(\gamma_1; u)$ and $Z(\gamma_2; u)$ are parallel complex numbers.

This special locus, where the boundstate might become unstable is known as the *wall of marginal stability*:

$$MS(\gamma_1, \gamma_2) := \{u | 0 < Z(\gamma_1; u)/Z(\gamma_2; u) < \infty\} \qquad (2.101)$$

Along such walls boundstates can decay.

2.13 Primitive wall-crossing formula

Now we come to the main result of Lecture I.

Suppose there is a BPS boundstate of BPS particles of charges γ_1, γ_2 and an experimentalist dials the vacuum moduli u, at infinity, so as to approach a wall of marginal stability, $MS(\gamma_1, \gamma_2)$, from the stable side (2.99), crossing at u_{ms}.

It follows from Denef's boundstate radius formula that $R_{12} \to \infty$: We literally see the states leaving the Hilbert space. This confirms our suspicion about the wall.

But now we can be quantitative: How many states do we lose?

The Hilbert space of states of the boundstate is a tensor product of the space of states of the constituents of the particle of charge γ_1, the space of states of the constituents of the particle of charge γ_2 and the states associated to the electromagnetic field of the pair of dyons. Therefore, the PSC changes by:

$$\Delta\Omega(\gamma; u; y) = \pm \mathrm{ch}_{|\langle\gamma_1,\gamma_2\rangle|}(y)\Omega(\gamma_1; u_{ms}; y)\Omega(\gamma_2; u_{ms}; y) \qquad (2.102)$$

where $\mathrm{ch}_n = \mathrm{ch}_{\rho_n}$ is the character of an $SU(2)$ representation of dimension n. (See Problem 5.11.) The rataionale for the first factor is that the electromagnetic field carries a representation of $so(3)$ of dimension $|\langle\gamma_1, \gamma_2\rangle|$, that is, of spin [9]

$$J_{\gamma_1, \gamma_2} := \frac{1}{2}(|\langle\gamma_1, \gamma_2\rangle| - 1) \qquad (2.103)$$

The + sign occurs when we move from a region of instability to stability.

Remarks:

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[9] The classical computation of exercise 5.1 gives $J = \frac{1}{2}|\langle\gamma_1, \gamma_2\rangle|$ but in fact there is a quantum correction and the correct result is $J = \frac{1}{2}(|\langle\gamma_1, \gamma_2\rangle| - 1)$. This quantum correction is best seen by studying the quantum mechanics of the probe particle.

1. Note that the quantity in equation (2.100) is always nonpositive, and is in fact negative even in the Denef-stable region. Thus, negativity of (2.100) is a necessary condition for having a true boundstate, but not a sufficient condition.

2. Similarly, the Denef stability condition is a *necessary* condition, but not a sufficient condition for the existence of a boundstate. Indeed, we can also define an *anti-marginal stability wall* to be a wall where the complex numbers $Z(\gamma_1; u)$ and $Z(\gamma_2; u)$ anti-align (i.e. Z_1 and $-Z_2$ are parallel complex numbers). In this case the anti-marginal wall separates a Denef-stable region from an unstable region. Suppose a boundstate existed in the stable region near a wall of anti-marginal stability. Note that the boundstate radius in Denef's formula also diverges across such a wall, but it is impossible to have a boundstate decay in this case, since that would violate energy conservation! (Show this!). We conclude that such boundstates cannot exist, even in a region of Denef stability. This would appear to pose a paradox if, as does indeed happen, a marginal stability wall can be connected to an anti-marginal stability wall through a region of Denef stability. The resolution of the paradox can be found in [2].

3. Now it is important here that we take γ_1 and γ_2 to be *primitive vectors*, since otherwise there can be more complicated decays and boundstates, as we will see. [10] In particular, a single BPS boundstate of charge $N\gamma_1$ where $N > 1$ is an integer is necessarily a boundstate at threshhold (and therefore very subtle). It can split up adiabatically into N-particle states and these can form more complicated configurations than we have taken into account.

4. Note that $\mathrm{Im} Z_1 \bar{Z}_2 > 0$ means that the complex numbers Z_1 and Z_2 are oriented so that Z_1 is counterclockwise to Z_2 at an angle less than π. As u crosses a wall of marginal stability the vectors Z_1, Z_2 rotate to become parallel and then exchange order.

2.14 Examples of BPS Spectra

As we have mentioned, there is no algorithm for finding the BPS spectrum in a general $\mathcal{N} = 2$ field theory or supergravity theory. In some special $\mathcal{N} = 2$ field theories the BPS spectrum has been determined using special *ad hoc* techniques.

2.14.1 N=3 AD theory

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2.14.2 $SU(2)$, $N_f = 0$ theory

The low energy theory is a $U(1)$ gauge theory. Therefore $\Gamma \cong \mathbb{Z}^2$. At large values of u there is a canonical electric-magnetic splitting of $\Gamma = \mathbb{Z}_e \oplus \mathbb{Z}_m$, but it turns out to be more useful to introduce a basis for Γ consisting of two charges $\gamma_1 = (0, 1)$ and $\gamma_2 = (2, -1)$.

---

[10]A vector γ in a lattice is said to be *primitive* if it is not an nontrivial integral multiple of another lattice vector. That is if $\frac{1}{N}\gamma$ is not in the lattice for any integer $N > 1$.
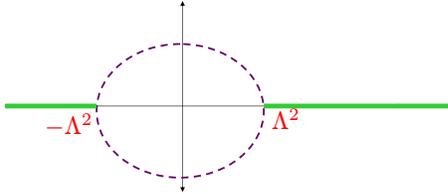
Figure 4: The u-plane for the pure $SU(2)$ gauge theory. The SW curve becomes singular at $u = \pm\Lambda^2$. As a result there is monodromy in the charge lattice Γ around these two points. We have chosen cuts shown in green to trivialize the corresponding local system. The marginal stability curve is shown in dashed purple and separates a strong coupling region near $u = 0$ from a weak coupling region near $u = \infty$.

Γ_u has nontrivial monodromy over the u-plane. If we choose cuts as in Figure 4 then there is a single wall of marginal stability, also shown in 4.

In the strong-coupling region

$$\mathfrak{h}_\gamma^{BPS} = \begin{cases} (0;0) & \gamma = \pm\gamma_1, \pm\gamma_2 \\ 0 & \text{else} \end{cases} \tag{2.104}$$

That is, the strong coupling BPS spectrum consists of two hypermultiplets - traditionally called the monopole and the dyon - and their charge conjugates.

In the weak-coupling region, on the other hand, the spectrum is very different.

$$\mathfrak{h}_\gamma^{BPS}(u) = \begin{cases} (\frac{1}{2};0) & \gamma = \pm(\gamma_1 + \gamma_2) \\ (0;0) & \gamma = \pm[(n+1)\gamma_1 + n\gamma_2], \quad n \geq 0 \\ (0;0) & \gamma = \pm[n\gamma_1 + (n+1)\gamma_2], \quad n \geq 0 \\ 0 & \text{else} \end{cases} \tag{2.105}$$

Evidently, there are plenty of decays/creation of BPS states of charges that involve pairs of *non-primitive* vectors. The primitive wall-crossing formula above is not strong enough to handle these cases, but in the next lecture we will find the proper generalization we need.

3. Lecture II: Halos and the Kontsevich-Soibelman Wall-Crossing Formula

3.1 Introduction

In the previous lecture we derived the *primitive wall-crossing formula*. This is only a small part of the full wall-crossing story.

The problem is that, since Z is linear in γ, the wall of marginal stability $MS(\gamma_1, \gamma_2)$ is *also* a wall of marginal stability for any pair of charges $(N_1\gamma_1, N_2\gamma_2)$ where N_1 and N_2 are nonzero integers of the same sign. As u crosses this wall there can be much more complicated decays and bindings of collections of BPS particles.

In this lecture we will take into account these more complicated sets of decays.

We are going to motivate the main formula - the Kontsevich-Soibelman Wall-Crossing-Formula (KSWCF) by studying solutions of $\mathcal{N} = 2$ supergravity coupled to a collection of vectormultiplets with an abelian gauge group. The KSWCF also applies in $\mathcal{N} = 2$ field theory, by a closely related argument, as we will indicate in Lecture III. The KSWCF was first presented in [19]. For a nice summary see [20].

3.2 Multi-centered solutions of $N = 2$ supergravity

$\mathcal{N} = 2$, $d = 4$ supergravity coupled to abelian vectormultiplets arises naturally from compactifications of type II string theory on Calabi-Yau manifolds. Those theories also have limits in which gravity is decoupled, and in this way results on string compactification can reproduce, say, the Seiberg-Witten solution of $\mathcal{N} = 2$ field theories.

The BPS states in $\mathcal{N} = 2$, $d = 4$ are *boundstates of dyonic BPS black holes*. In the classical approximation these can be written as explicit BPS solutions of the generalized Einstein equations of $\mathcal{N} = 2$ supergravity. These solutions, due to Frederik Denef, and are known as multi-centered solutions, and will be the subject of this section.

The bosonic fields we will work with are the metric $g_{\mu\nu}$, the vectormultiplet scalars u and a self-dual abelian gauge field $\mathbb{F} \in \Omega^2(M^4) \otimes V$ where again $V = \Gamma \otimes \mathbb{R}$ is a symplectic vector space with a compatible complex structure. There is a little complication compared to the field theory case. The $\mathcal{N} = 2$ gravity multiplet has a $U(1)$ gauge field, known as the *graviphoton*. So, if the rank of the gauge group of the vectormultiplets is r then the abelian gauge group of the theory is rank $r + 1$ and hence Γ is of rank $2r + 2$ and V has $\dim_r V = 2r + 2$. Nevertheless, there are r vm scalars and $\dim_c \mathcal{B} = r$.

Example: In compactifications of type II string theory on a Calabi-Yau manifold X, $V = H^{even}(X; \mathbb{R})$ for the type IIA string and $V = H^{odd}(X; \mathbb{R})$ for the type IIB string. Thus, $r = h^2(X)$ for type IIA strings on a Calabi-Yau.

The multicentered solutions have metrics which are asymptotically Minkowski space. The spacetime has coordinates $(\vec{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$ as in Minkowski space but the metric has the following form

$$ds^2 = -e^{2U}(dt + \Theta)^2 + e^{-2U} d\vec{x}^2 \tag{3.1}$$

Here Θ is an $\vec{x}$-dependent one-form on $\mathbb{R}^3$ and U, the warp factor, is a function of $\vec{x}$. Moreover, $U \to 0$ as $\vec{x} \to \infty$.

Then, U, Θ, $u(\vec{x})$ and $\mathbb{F}$ are determined by specifying the following data:

1. A boundary condition $u_\infty \in \mathcal{B}$ for the vectormultiplet scalars at spatial infinity.

2. A choice of centers labeled by charge vectors $\gamma_j \in \Gamma$. These are denoted $(\vec{x}_j, \gamma_j)$.

The dyonic gauge field is very similar to what we had before in (3.2) but now for a single center

$$\mathbb{F} = \frac{1}{2}\left(\omega_s \otimes \gamma_s + \epsilon e^{2U}\omega_d \otimes \gamma_d\right) \tag{3.2}$$

with $\gamma_d = \mathcal{I}(\gamma_s)$, and for many centers ♣ FILL IN ♣.

The complex structure $\mathcal{I}$ is determined by the vectormultiplets, and the vectormultiplets in turn are determined as follows.

First, as a preliminary, since Z is linear it follows that there is a vector $\varpi \in V$, called the *period vector* and denoted ϖ such that

$$Z(\gamma; u) = \langle \gamma, \varpi \rangle \qquad \forall \gamma \in \Gamma \tag{3.3}$$

The vm moduli u determine and conversely from ϖ we can uniquely determine u.

Using the above data we now introduce an harmonic function

$$H : \mathbb{R}^3 \to V \tag{3.4}$$

defined by

$$H(\vec{x}) = \sum_j \frac{\gamma_j}{|\vec{x} - \vec{x}_j|} - 2\text{Im}(e^{-i\alpha_\infty}\varpi_\infty) \tag{3.5}$$

where $e^{i\alpha_\infty}$ is the phase of $Z(\sum \gamma_j; u_\infty)$. Next we consider the equation

$$2e^{-U(\vec{x})}\text{Im}(e^{-i\alpha(\vec{x})}\varpi(\vec{x})) = -H(\vec{x}) \tag{3.6}$$

Taking an inner product with γ this equation reads

$$2e^{-U(\vec{x})}\text{Im}(e^{-i\alpha(\vec{x})}Z_\gamma(u(\vec{x}))) = -\sum_j \frac{\langle \gamma, \gamma_j \rangle}{|\vec{x} - \vec{x}_j|} + 2\text{Im}\left(e^{-i\alpha_\infty}Z_\gamma(u_\infty)\right) \qquad \forall \gamma \in \Gamma \tag{3.7}$$

where $e^{i\alpha(\vec{x})}$ is the phase of $Z(\sum \gamma_i; u(\vec{x}))$.

Equation (3.6), or equivalently, (3.7) determines both $u(\vec{x})$ and $U(\vec{x})$ as a function of $\vec{x}$: In supergravity there are $(r-1)$ complex independent vm moduli, hence $2r-2$ real unknowns together with the unknown $U(\vec{x})$. On the other hand (3.7) gives $2r$ real equations and has one gauge (overall phase) gauge invariance. So in general we expect one unique solution.

To complete the solution we must give a formula for the $\vec{x}-t$ components of the metric. These are determined by the form Θ which is in turn determined from

$$*_3 d\Theta = \langle dH, H \rangle \tag{3.8}$$

Note that this equation can only be solved if the centers $(\vec{x}_j, \gamma_j)$ satisfy the constraint equations:

$$\sum_{j:j\neq i} \frac{\langle \gamma_i, \gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2\text{Im}(e^{-i\alpha_\infty}Z_{\gamma_i}(u_\infty)) \tag{3.9}$$

3.2.1 The single-centered case: Spherically symmetric dyonic black holes

In the case of a single-center of charge γ_c we have $\Theta = 0$ and (3.7) reduces to a simpler, spherically symmetric, equation:

$$2e^{-U(r)}\text{Im}(e^{-i\alpha(r)}Z_\gamma(u(r))) = -\frac{\langle\gamma,\gamma_c\rangle}{r} + 2\text{Im}\left(e^{-i\alpha_\infty}Z_\gamma(u_\infty)\right) \qquad \forall\gamma \in \Gamma \qquad (3.10)$$

This equation is the beautiful and famous attractor equation of Ferrara, Kallosh, and Strominger. It defines a spherically symmetric dyonic black hole of dyonic charge γ_1. As $r \to 0$ one finds that $e^{U(r)} \sim r$, so $r = 0$ is the horizon of a black hole since the g_{tt} part of the metric vanishes, and, as an observer approaches the horizon the local vm moduli approach a fixed point given by

$$2\text{Im}(e^{-i\alpha_*}\varpi_*) = \gamma_c \qquad (3.11)$$

Although it is not our main focus, we cannot resist pointing out a few beautiful properties in Problem 5.3

Remark: There is an important constraint on the charges γ that lead to a valid solution. If we choose a general γ_c it will not always be true that $e^{U(r)}$ is positive definite. This turns out to be ok only for γ_c in a certain noncompact open region of $V = \Gamma \otimes \mathbb{R}$. When γ_c is such that we get a valid solution we say that "γ_c supports a single-centered black hole."

3.2.2 Multi-centered solutions as molecules

In the case of more than one center, the solution is not spherically symmetric and indeed carries angular momentum, as indicated by the $\vec{x} - t$ components of the metric.

As a special case of the above formula, consider the case of just two centers. When $\vec{x}$ is near $\vec{x}_1$ the term in H with the pole dominates, and the solution looks like the single-centered dyonic BPS black hole of charge γ_1. Similarly when $\vec{x}$ is near $\vec{x}_2$ the solution looks like a dyonic BPS black hole of charge γ_2. Therefore, the solution with two centers is a BPS bound state of two dyonic BPS black holes! Now the constraint equation (3.9) on the centers becomes

$$\frac{\langle\gamma_1,\gamma_2\rangle}{|\vec{x}_1 - \vec{x}_2|} = 2\text{Im}(e^{-i\alpha_\infty}Z_{\gamma_1}(u_\infty)) \qquad (3.12)$$

A simple rearrangement of this formula gives Denef's boundstate formula we quoted in Lecture I.

More generally, in equation (3.7) when $\vec{x}$ is near $\vec{x}_j$ one term in the harmonic function dominates, and the solution looks, locally, like a single-centered dyonic black hole of charge γ_j. Thus, the full solution should be regarded as a kind of "molecule" of dyonic black holes. There is an intricate balancing of gravitational, electromagnetic, and scalar forces that binds it together.

Remarks

1. It is in general not possible to make arbitrary choices of charges γ_i and get valid solutions of supergravity. In order to be sure that the solution is legitimate one needs to know that the warp factor $e^{2U(\vec{x})}$ does not become negative in regions of Planck scale or larger and that there are no induced closed timelike curves. These conditions can be difficult to check in general.

2. Comment on α' corrections...

3.3 Halo Fock spaces

The moduli space of solutions to the constraints (3.9) is in general rather complicated, but there is an important class of examples which can be understood very explicitly.

Suppose that we have one center at $\vec{x} = 0$ and charge γ_c (called the "core charge") and all the other centers $\vec{x}_j$, $j = 1, \ldots, N$ have charges parallel to a charge γ_h (called a "halo charge") so $\gamma_j = \lambda_j \gamma_h$ with $\lambda_j > 0$.

In this case (3.9) says that all the centers must lie on a sphere centered at $\vec{x} = 0$ of radius

$$R = \langle \gamma_c, \gamma_h \rangle \frac{1}{2 \mathrm{Im} e^{-i\alpha} Z_{\gamma_h}} \tag{3.13}$$

(where $e^{i\alpha}$ is the phase of $Z(\gamma_c + \sum \lambda_j \gamma_h; u_\infty)$). This is the only constraint - the particle centers can be distributed in any way we like on the sphere of radius R. [11]

Now let us think about the quantum states corresponding to these classical solutions of $\mathcal{N} = 2$ supergravity.

The halo particles are non-interacting BPS particles confined to lie in a sphere. Upon quantization we have a system of noninteracting particles on a sphere which moreover have a "Landau-level degeneracy." We can think of γ_c as a monopole charge and the halo particle of charge $\lambda \gamma_h$ as an electron charge. It is a standard exercise [12] to show that the Hamiltonian for such a system has a degenerate space of groundstates of dimension $|\langle \gamma_c, \lambda \gamma_h \rangle|$ and moreover, this space of groundstates is in an irreducible representation of $Spin(3)$ (the double-cover of spatial rotations around $\vec{x} = 0$) of spin $J_{\gamma_c, \lambda \gamma_h}$ defined in (2.103) of Lecture I.

In addition to the LL degeneracy the $\lambda \gamma_h$ -halo particles themselves might have internal degrees of freedom. Indeed, the space of quantum states of a halo particle at rest is of the form

$$\mathcal{H}^{BPS}_{u, \lambda \gamma_h} = \rho_{hh} \otimes \mathfrak{h}_{\lambda \gamma_h} \tag{3.14}$$

where $\mathfrak{h}_\gamma$ is some representation of the $so(3)$ little group. We next invoke a new interpretation of the meaning of the factor ρ_{hh}. In a first quantization of a halo particle, the half-hypermultiplet degrees of freedom in this expression correspond to the overall center of mass degree of freedom. Indeed, ρ_{hh} is a $\mathbb{Z}_2$-graded vector space

$$\rho_{hh} = \rho^0_{hh} \oplus \rho^1_{hh} \tag{3.15}$$

---

[11] We ignore some subtleties related to singularities in the supergravity solution that occur when centers overlap. The proper way to deal with those is to think about the quantum states, as we do in the following paragraphs.

[12] See Problem 5.6

and the even part $\rho^0_{hh} \cong \mathbb{C}^2 \cong \mathbb{R}^4$ is the center of mass position. This is fixed on the sphere of radius r and the position along the sphere is accounted for by the Landau-level wavefunctions. So, we should certainly drop this factor. In addition by solving the Dirac equation for the halo probe particles (as is done in [8]) one finds that the supersymmetric state has the center of mass multiplet in the spin-1/2 state and the magnetic field forces the electron spin to point inwards. Thus, this degree of freedom is frozen.

The net result of the above discussion is that, quantum-mechanically, each halo particle of charge $\lambda\gamma_h$ gives rise to a vector space of one-particle quantum states which we can identify as

$$(J_{\gamma_c, \lambda\gamma_h}) \otimes \mathfrak{h}_{\lambda\gamma_h} \tag{3.16}$$

where (J) denotes the irrep of $so(3)$ of dimension $2J + 1$. Thus, (3.16) is a representation of $so(3)$.

Now, as we have said, in the groundstate the halo particles are non-interacting. Therefore, *the space of quantum states associated with a halo configuration is a tensor product of the space $\mathfrak{h}_{\mathrm{core}}$ with a subspace of a free particle Fock space made from one-particle states drawn from the vector space* (3.16).

Given this insight it is natural to consider a Hilbert space made from all the halo states together, and from our discussion is this just:

$$\oplus_{N \geq 0} q^N \mathfrak{h}^{\mathrm{Halo}}_{\gamma_c + N\gamma_h} = \mathfrak{h}_{\gamma_c} \bigotimes_{n=1}^{\infty} \mathcal{F}[q^n(J_{\gamma_c, n\gamma_h}) \otimes \mathfrak{h}_{n\gamma_h}] \tag{3.17}$$

where $\mathcal{F}[W]$ denotes a Fock space build from a space of creation operators spanning a vector space W. [13] We have also introduced a formal variable q to account for the $\mathbb{Z}$-grading by the γ_h-charge.

There is an important subtlety we have not yet addressed in writing (3.17). The creation operators in (3.16) include both bosonic and fermionic creation operators. There is a surprising shift of statistics (related to the quantum shift in J_{γ_c, γ_h} from the classical value). As we explained center of mass ρ_{hh} of the BPS particle in the orbit is forced by the magnetic field to be in the spin-1/2 part with the spin pointing radially inwards. Therefore,

The $\mathbb{Z}_2$ grading in the space of creation operators (3.16) *is by* $(-1)^{2J_3 + 1}$ *where J_3 is the generator of spin acting on $\mathfrak{h}_{\lambda\gamma_h}$.*

The net effect is that

1. Hypermultiplets behave like Fock space *fermions*

2. Vectormultiplets behave like Fock space *bosons*

3. The net number of fermionic-bosonic creation operators in (3.16) is $|\langle \lambda\gamma_h, \gamma_c\rangle| \Omega(\lambda\gamma_h; u)$.

---

[13]To be more concrete, suppose $W = W^0 \oplus W^1$ is a $\mathbb{Z}_2$-graded vector space. Then choose a basis α_s for W_0 and b_i for W^1. Then we have bosonic creation operators $[\alpha_s^\dagger, \alpha_t] = \delta_{st}$ and fermionic creation operators $\{b_i, b_j\} = \delta_{ij}$ and we form the corresponding Fock space. This Fock space, which is clearly independent of choice of basis, is $\mathcal{F}[W]$.

Remark: Let us stress that there is no supersymmetry relating the bosonic and fermionic creation operators in the halo Fock space. The $\mathcal{N} = 2$ supersymmetries of our theory are acting on the *overall* center of mass of the halo Fock space states. Thus the BPS representation we are speaking about here is $\rho_{hh} \otimes \mathfrak{h}^{\text{Halo}}_{\gamma_c + N\gamma_h}$ with the $\mathcal{R}^A_\alpha$ supersymmetries acting as 0 and the $\mathcal{T}^A_\alpha$ supersymmetries acting on the overall center of mass factor ρ_{hh}.

3.4 Semi-primitive WCF

When u crosses a wall of marginal stability many bound states can enter or leave the spectrum. Among other things, entire Fock spaces of halo configurations come and go.

In accounting for this it is useful to introduce a generating functional. For each charge γ we introduce a formal variable x_γ with the rule

$$x_{\gamma_1} x_{\gamma_2} = x_{\gamma_1 + \gamma_2} \tag{3.18}$$

(Thus, we could choose a basis γ_i for the lattice and then define $x_i := x_{\gamma_i}$ and if $\gamma = \sum_i n^i \gamma_i$ with $n^i \in \mathbb{Z}$ then x_γ is the Laurent monomial

$$x_\gamma = \prod_i x_i^{n^i} \tag{3.19}$$

in the x_i.)

Now we form a generating function of the contribution of the Fock space (3.17) to the BPS indices

$$G^{\text{Halo}}_{\gamma_c}(u) = \sum_{N=0}^{\infty} \Omega^{\text{Halo}}_{\gamma_c}(N\gamma_h; u) x_{\gamma_c + N\gamma_h} \tag{3.20}$$

This is x_{γ_c} times a polynomial in x_{γ_h}, but we do not write that dependence explicitly in $G^{\text{Halo}}_{\gamma_c}(u)$.

Then, on the side of the wall u_- where halo states are unstable we have

$$G^{\text{Halo}}_{\gamma_c} = x_{\gamma_c} \tag{3.21}$$

If u moves across $MS(\gamma_h, \gamma_c)$ from u_- to u_+ then halo states with core γ_c are created. An entire halo Fock space is created, so we multiply by the partition function of the Fock space of γ_h-particles to obtain

$$G^{\text{Halo}}_{\gamma_c} = (1 - (-1)^{\langle \gamma_h, \gamma_c \rangle} x_{\gamma_h})^{|\langle \gamma_h, \gamma_c \rangle| \Omega(\gamma_h; u_{ms})} x_{\gamma_c} \tag{3.22}$$

Thus, upon crossing the wall, the generating function gains or loses a factor

$$(1 - (-1)^{\langle \gamma_h, \gamma_c \rangle} x_{\gamma_h})^{|\langle \gamma_h, \gamma_c \rangle| \Omega(\gamma; u_{ms})} \tag{3.23}$$

for the Fock space of γ_h particles and therefore gains or loses a factor

$$\prod_{n=1}^{\infty} (1 - (-1)^{\langle n\gamma_h, \gamma_c \rangle} x^n_{\gamma_h})^{|\langle n\gamma_h, \gamma_c \rangle| \Omega(n\gamma; u_{ms})} \tag{3.24}$$

for the Fock space of all the particles with electromagnetic charge parallel to γ_h.

This key observation is known as the *semi-primitive wall-crossing formula*. It generalizes the primitive wall-crossing formula to the case where *one* of the constituent charges is not primitive. This picture of wall-crossing (together with the primitive wcf) was first presented in [9].

Example:[14] A significant example of this is the $D6 - D0$ system in type IIA supergravity on a compact Calabi-Yau manifold. The charge lattice can be taken to be $H^{\text{even}}(CY; \mathbb{Z})$. The core charge is the $D6$-brane which wraps the entire CY and can be identified with $1 \in H^0(CY; \mathbb{Z})$. The D0-brane has charge γ which can be taken to be a generator of $H^6(CY; \mathbb{Z})$ so that $\langle \gamma_c, \gamma \rangle = 1$. For all $n \neq 0$ the Hilbert spaces $\mathfrak{h}_{n\gamma}$ are isomorphic to $H^*(CY; \mathbb{R})$, and the bosonic/fermionic grading is the parity of the differential form. The Lefshetz $sl(2)$ action on $H^*(CY; \mathbb{R})$ defines the spin content. On the unstable side of the wall the generating function is 1 since $\Omega(\gamma_c; u) = 1$ but on the stable side of the wall it jumps to

$$\prod_{n=1}^{\infty} (1 - (-x)^n)^{-n\chi(CY)} \tag{3.25}$$

thus producing the famous McMahon function, familiar from topological string theory.

3.5 BPS Galaxies

In order to solve the full wall-crossing problem we need to come to grips with the possibility of decays involving both constituent charges to be non-primitive.

We are going to present a simple heuristic solution of this problem. It builds on the paper [16] (which we will discuss in Lecture III) and is based on the recent paper [1].

The main problem with the halo picture of semi-primitive wall-crossing is that a halo configuration with core γ_c and halo charge $N\gamma$ is not the most general contribution to the states in $\mathfrak{h}_{\gamma_c + N\gamma}$ and hence

$$\Omega(\gamma_c + N\gamma; u) \neq \Omega(\gamma_c) \Omega_{\gamma_c}^{\text{Halo}}(N\gamma; u) \tag{3.26}$$

First of all, while it is a closed system perturbatively, there can be nonperturbative tunneling to configurations with core charge $\gamma_c + m\gamma$ and halo charge $(N - m)\gamma$. Moreover, there can be entirely different configurations, say with core charge $\gamma_c - \delta$ and orbiting charges $N\gamma + \delta$, for some charge δ.

The idea is to suppress this mixing, and produce a nonperturbatively closed system, by taking a limit with very large core charge.

Thus, we choose a single $U(1)$ from our gauge group with electric and magnetic charges γ_0, γ_0' with $\langle \gamma_0, \gamma_0' \rangle = 1$ and we take our core charge to be

$$\gamma_c = \Lambda^2 \gamma_0 + \Lambda \gamma_0' + \delta \tag{3.27}$$

where

$$\delta \in \Gamma_0^\perp := \{\gamma : \langle \gamma_0, \gamma \rangle = \langle \gamma_0', \gamma \rangle = 0\} \tag{3.28}$$

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[14]This presumes some familiarity with D-branes on Calabi-Yau manifolds.

and we take $\Lambda \to \infty$. [15]

We now consider the ensemble of all BPS multicentered solutions with this large core charge and all other charges in the lattice of charges orthogonal to our distinguished $U(1)$, called $\Gamma_0^\perp$. The total charge of all of other particles will be called $\gamma_{\mathrm{orb}} \in \Gamma_0^\perp$. The charges around the core charge might well be mutually non-local so that these are in general *not* halo configurations. There will be subsystems of particles which should be thought of as bound together, and then the whole subsystem is bound to the core charge. The physical picture one should have in mind is of a *BPS galaxy*: There is a very heavy central object with many complicated boundstates - like solar systems - orbiting around it.

In the limit of large Λ we claim that the mixing described above goes to zero. Establishing this claim is by far the most subtle and important point in [1]. Just to give a taste of the arguments, there are two sources of this suppression:

1. *Entropic suppression*: Black hole fragmentation is exponentially suppressed in the limit of large black hole charge [21]. For example, the amplitude for the fragmentation of a Reisner-Nordstrom black hole of charge $Q = Q_1 + Q_2$ to split into RN black holes of charges Q_1 and Q_2 is suppressed by $\exp[-\frac{1}{2}\Delta S]$ where $\Delta S = \pi Q^2 - \pi Q_1^2 - \pi Q_2^2 = 2\pi Q_1 Q_2$. Taking into account charge quantization we see that as $Q \to \infty$ the fragmentation amplitude is suppressed.

2. *Distance suppression*: If the core and halo charges exchange a charge which is not in $\Gamma_0^\perp$ then the radius for the stable orbit (3.13) goes to infinity for $\Lambda \to \infty$ and hence the tunneling amplitude is infinitely suppressed.

We thus obtain a *closed system* with a fixed central charge γ_c and many BPS particles bound to it in complicated ways governed by the constraints (3.9).

Given this physical picture the Hilbert space of quantum states corresponding to these multicentered configurations has a common factor $\mathfrak{h}_{\gamma_c}$. Mathematically speaking, there is quotient Hilbert space, denoted

$$\mathfrak{h}_{\gamma_c}(\gamma_{\mathrm{orb}}; u) = \mathfrak{h}_{\gamma_c + \gamma_{\mathrm{orb}}}/\mathfrak{h}_{\gamma_c} \tag{3.29}$$

and called the space of "framed BPS states." Here γ_{orb} is the total charge of all the orbiting particles in the BPS galaxy around the core. (We will define an analogous space of "framed BPS states" for $\mathcal{N} = 2$ field theories in Lecture III.) Associated to these we have a "framed BPS index":

$$\overline{\Omega}_{\gamma_c}(\gamma_{\mathrm{orb}}; u) = \lim_{\Lambda \to \infty} \mathrm{Tr}_{\mathfrak{h}_{\gamma_c}(\gamma_{\mathrm{orb}}; u)}(-1)^{2J_3} \tag{3.30}$$

We expect the framed BPS indices to be well-defined and locally constant. There will, however, be wall-crossing of these numbers, as we discuss in the next Section.

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[15]The need to introduce the asymmetric limit in (3.27) is to avoid some technical problems. See [1] for details.

3.6 Wall-crossing for BPS galaxies

Now we come to a key point: While the configurations in the BPS galaxies are in general very complicated, their wall-crossing is very simple. Suppose that as we vary u the central charge of some particle $Z(\gamma; u)$ becomes parallel with the *total* central charge of the galaxy: $Z(\gamma_c + \gamma_{\mathrm{orb}}; u)$. For finite values of Λ the wall of marginal stability depends on γ_{orb} but in the $\Lambda \to \infty$ limit this happens along "BPS walls", defined by:

$$W_\gamma := \{u : Z(\gamma_0; u) \parallel Z(\gamma; u)\} \tag{3.31}$$

The important point here is that since we took the $\Lambda \to \infty$ limit the dependence of the wall on the charge γ_{orb} has dropped out and the wall only depends on γ, the halo particle, and γ_0, the "main" charge in the core.

In order to account for the wall-crossing of the framed BPS indices it is useful to introduce the BPS galaxy generating function:

$$G_{\gamma_c}(u) := \sum_{\gamma_{\mathrm{orb}} \in \Gamma_0^\perp} \overline{\Omega}_{\gamma_c}(\gamma_{\mathrm{orb}}; u) x_{\delta + \gamma_{\mathrm{orb}}} \tag{3.32}$$

Now, when u crosses the BPS wall W_γ, with γ primitive, halo configurations with halo charges proportional to γ will come in from or go out to infinity. For a given configuration with charge $\gamma_t = \gamma_c + \gamma_{\mathrm{orb}}$ the effect of γ-particle halos on the generating function will - by the semiprimitive wcf - be multiplied by

$$(1 - (-1)^{\langle \gamma, \gamma_t \rangle} x_\gamma)^{|\langle \gamma, \gamma_t \rangle| \Omega(\gamma; u)}. \tag{3.33}$$

Now, as we have discussed, we can restrict attention to $\gamma \in \Gamma_0^\perp$. The factor (3.33) then only depends on $\delta + \gamma_{\mathrm{orb}}$. The dependence on γ_{orb} appears to be a nuisance, making the transformation of $G_{\gamma_c}(u)$ complicated - however - and this is a very important point - the transformation can be nicely summarized by introducing an *operator*

$$D_\gamma x_{\gamma'} := \langle \gamma, \gamma' \rangle x_{\gamma'} \tag{3.34}$$

and then the operator

$$\mathcal{K}_\gamma := (1 - (-1)^{D_\gamma} x_\gamma)^{D_\gamma} \tag{3.35}$$

Properties of this interesting transformation, called a Kontsevich-Soibelman transformation are explored in exercise 5.7

Acting with the operator $\mathcal{K}_\gamma^{\Omega(\gamma; u)}$ produces the correct effect on each term in the sum and therefore *if u crosses the wall W_γ then the generating function $G_{\gamma_c}(u)$ transforms by a diffeomorphism $\mathcal{K}_\gamma^{\Omega(\gamma; u)}$ across the point $u \in W_\gamma$.*

To be absolutely correct, we should recall that there can be halo states with charges $k\gamma$ for $k = 1, 2, \ldots$ and hence crossing $W(\gamma)$ at the point u the generating function transforms by the diffeomorphism

$$U_\gamma(u) = \prod_{n=1}^{\infty} \mathcal{K}_{n\gamma}^{\Omega(n\gamma; u)} \tag{3.36}$$

The reader might be concerned by the occurance of the absolute value in (3.33). As the student can explore in exercise 5.8 the operation (3.36) should be applied if u crosses W_γ in the direction of *increasing* $\arg[Z_\gamma e^{-i\alpha_0}]$, where $e^{i\alpha_0}$ is the phase of $Z(\gamma_0; u)$, while the inverse transformation is applied if the wall is traversed in the opposite direction.

3.7 The Kontsevich-Soibelman WCF

So far we have described the wall-crossing for the framed BPS degeneracies. Now we address the problem we originally set out to understand, namely the wall-crossing behavior of the original (or "vanilla") BPS degeneracies $\Omega(\gamma; u)$.

To do that let us make the following simple observation: Consider a (small) closed contractible loop $\mathcal{P}$ in the moduli space $\mathcal{B}$ of vm scalars, and consider the behavior of the BPS galaxy generating function $G_{\gamma_c}(u)$ along this loop. Along this loop $\mathcal{P}$ will intersect a number of walls $W_{\gamma_i}(u_i)$ at various points and therefore going around the loop G_{γ_c} will transform by

$$\prod_i U_{\gamma_i}(u_i) G_{\gamma_c}. \tag{3.37}$$

Note that since the U_{γ_i} do not commute for different i in general, the product must be ordered. Of course the ordering is given path ordering.

On the other hand, since the loop is small and contractible G_{γ_c} is single valued along the loop, so

$$\prod_i U_{\gamma_i}(u_i) G_{\gamma_c} = G_{\gamma_c} \tag{3.38}$$

Given the arbitrary freedom in choosing the charge δ in the core charge γ_c we can "cancel the G_{γ_c}" and therefore we deduce the *operator equation*:

$$\prod_i U_{\gamma_i}(u_i) = 1. \tag{3.39}$$

As we will presently demonstrate, equation (3.39) puts a strong constraint on the *vanilla* BPS degeneracies $\Omega(\gamma_i; u_i)$. Indeed we will now show that it in fact implies the famous Kontsevich-Soibelman wall-crossing formula.

Let us consider a pair of charges γ_1, γ_2 and a generic point on marginal stability wall $u_{ms} \in MS(\gamma_1, \gamma_2)$. In [1] it is shown that we can do a number of technically useful things:

1. It is possible to find a charge γ_0 so that u_{ms} is an attractor point $u_*(\gamma_0)$ supporting single-centered black holes and so that

$$u_{ms} \in W_{\gamma_1} \cap W_{\gamma_2} \tag{3.40}$$

   In particular, $Z_{\gamma_1} \parallel Z_{\gamma_2} \parallel e^{i\alpha_0}$ at u_{ms}. The intersection is real codimension two in moduli space so we will consider a small path $\mathcal{P}$ around u_{ms} in the two transverse dimensions, as in figure 5.

2. We have a "simple root property": The only relevant BPS walls near u_{ms} are of the form $W_{r_1\gamma_1 + r_2\gamma_2}$ for $r_1, r_2 \geq 0$.
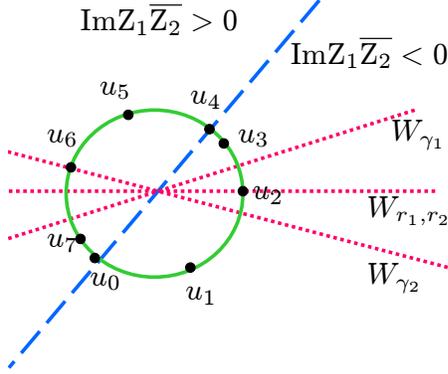
Figure 5: This shows the neighborhood $\mathcal{U}$ in the normal bundle to $W_{\gamma_1} \cap W_{\gamma_2}$. The wall of marginal stability is given by $\text{Im}[Z(\gamma_1; t)\overline{Z(\gamma_2; t)}] = 0$ since $\text{Re}[Z(\gamma_1; t)\overline{Z(\gamma_2; t)}]$ is nonzero throughout $\mathcal{U}$. We choose the ordering of γ_1, γ_2 so that W_{γ_1} is counterclockwise from W_{γ_2} with opening angle smaller than π. Then the BPS walls $W_{r_1,r_2} := W_{r_1\gamma_1 + r_2\gamma_2}$ are ordered so that increasing r_1/r_2 gives walls in the counterclockwise direction. We consider a path $\mathcal{P}$ in $\mathcal{U}$ circling the origin in the counterclockwise direction. The central charges of vectors $r_1\gamma_1 + r_2\gamma_2$ with $r_1, r_2 \geq 0$ at representative points $u_0, \ldots, u_7$ along $\mathcal{P}$ are illustrated in the next figure.
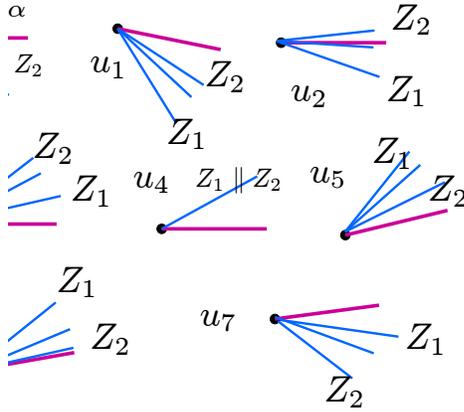

Figure 6: As u moves along the path $\mathcal{P}$ the central charges evolve as in this figure. Note that $\text{Im}(Z_1\overline{Z_2}) > 0$ means that Z_1 is counterclockwise to Z_2 and rotated by a phase less than π. In that case the rays parallel to $r_1Z_2 + r_2Z_2$ for $r_1, r_2 \geq 0$ are contained in the cone bounded by $Z_1\mathbb{R}_+$ and $Z_2\mathbb{R}_+$, and ordered so that increasing r_1/r_2 corresponds to moving counterclockwise. When u crosses the marginal stability wall the cone collapses and the rays *reverse order*. As u moves in the region u_2 the quantity $\arg[Z_\gamma e^{-i\alpha_0}] > 0$ is increasing for all γ_{r_1,r_2} with $r_1, r_2 \geq 0$ while at the point u_6 the argument is decreasing.

Now as u traverses the path $\mathcal{P}$, the values of the central charges evolve as shown in

figure 6. Imposing the condition (3.39) in this case leads to the equation

$$\overset{\leftarrow}{\prod_{\frac{r_1}{r_2}\nearrow}} \mathcal{K}_{r_1,r_2}^{-\Omega_{r_1,r_2}^+} \overset{\leftarrow}{\prod_{\frac{r_1}{r_2}\nearrow}} \mathcal{K}_{r_1,r_2}^{\Omega_{r_1,r_2}^-} = 1 \tag{3.41}$$

where the arrows on the product mean that increasing values of r_1/r_2 are written to the left, and $\Omega_{r_1,r_2}^\pm$ is the BPS index of $r_1\gamma_1 + r_2\gamma_2$ in the region $\mathcal{U}$ with $\mathrm{Im}Z_1\bar{Z}_2 > 0$ and < 0 respectively.

Taking into account the relation between the ordering of r_1/r_2 and the ordering of the phases of the central charges illustrated in figures 5 and 6 we can also write this in the form:

$$\overset{\rightarrow}{\prod_{\mathrm{arg}Z_{r_1,r_2}\nearrow}} \mathcal{K}_{r_1,r_2}^{\Omega_{r_1,r_2}^+} = \overset{\rightarrow}{\prod_{\mathrm{arg}Z_{r_1,r_2}\nearrow}} \mathcal{K}_{r_1,r_2}^{\Omega_{r_1,r_2}^-}. \tag{3.42}$$

This is the famous Kontsevich-Soibelman wall-crossing formula. It tells us how to compute the Ω^+ given a knowledge of the Ω^-. To make that plain we note that the product is a symplectomorphism, and once an ordering is chosen, the factorization in terms of KS transformations is unique. Note that the product has the form:

$$\mathcal{K}_{\gamma_2}^{\Omega^+(\gamma_2)} \cdots \mathcal{K}_{\gamma_1}^{\Omega^+(\gamma_1)} = \mathcal{K}_{\gamma_1}^{\Omega^-(\gamma_2)} \cdots \mathcal{K}_{\gamma_1}^{\Omega^-(\gamma_2)} \tag{3.43}$$

3.7.1 Example 1 of the KSWCF

Suppose $\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ with $\langle\gamma_1,\gamma_2\rangle = 1$. Then one can show the identity on symplecto-morphisms:

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1+\gamma_2}\mathcal{K}_{\gamma_1} \tag{3.44}$$

See Problem 5.10 for some hints on how to establish this identity.

This is the wall-crossing formula for a certain superconformal field theory (the $N = 3$ AD theory). On one side of the wall there are two hypermultiplets, so we know that on -say - the $-$ side of the wall

$$\Omega^-(\gamma) = \begin{cases} 1 & \gamma = \pm\gamma_1, \pm\gamma_2 \\ 0 & \text{else} \end{cases} \tag{3.45}$$

Then we can deduce that on -say- the $+$ side of the wall

$$\Omega^+(\gamma) = \begin{cases} 1 & \gamma = \pm\gamma_1, \pm\gamma_2 \\ 1 & \gamma = \pm(\gamma_1 + \gamma_2) \\ 0 & \text{else} \end{cases} \tag{3.46}$$

3.7.2 Example 2 of the KSWCF

If now $\langle\gamma_1,\gamma_2\rangle = 2$ then the rearrangement lemma for the KS symplectomorphisms becomes considerably more involved, and is:

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1+2\gamma_2}\mathcal{K}_{2\gamma_1+3\gamma_2}\cdots\mathcal{K}_{\gamma_1+\gamma_2}^{-2}\cdots\mathcal{K}_{3\gamma_1+2\gamma_2}\mathcal{K}_{2\gamma_1+\gamma_2}\mathcal{K}_{\gamma_1}. \tag{3.47}$$

On the RHS the slopes of the "populated" central charges $Z_{r_1\gamma_1 + r_2\gamma_2}$ have an accumulation point at $r_1/r_2 = 1$. See Problem 5.10 for some hints on how to establish this identity.

Comparing with the BPS spectrum in Section 2.14.2 we see that this beautifully encodes the change of the the BPS spectrum of pure $SU(2)$ theory across the wall of marginal stability!. [16]

Here we see how powerful the formula is. Given the knowledge of the finite spectrum of hypermultiplets at strong coupling the formula uniquely determines the weak-coupling BPS indices. The powers of the $\mathcal{K}$-factors also suggest what the exact spectrum should be since $\Omega = +1$ for a half-hypermultiplet and $\Omega = -2$ for a vector-multiplet. This suggests (but, because of potential cancelations, does not prove) the weak coupling spectrum we showed in (2.105) above. In order to deduce the actual spectrum we would need something stronger, like the spin-index, which is the topic of the next subsection.

Remarks:

1. In the A_1 theories of class $\mathcal{S}$ (see [15] for the terminology) it appears that all the wall-crossing identities can be obtained by successive use of (3.44) and (3.47), although I do not know a rigorous argument that this is the case.

2. One might ask for the analogous rearrangement lemmas for $\langle\gamma_1, \gamma_2\rangle > 2$. In this case one finds an infinite product on the RHS corresponding to a *dense* set of slopes for populated central charges in the wedge between Z_{γ_1} and Z_{γ_2}. I am not aware of a physical realization of this wall-crossing formula.

3.8 Including spins

It is straightfoward to take into account the spins in the halo Fock spaces. In this section we will show how doing so leads to the so-called "motivic KSWCF."

Caution! The spin character is not an index; only the protected spin character is an index. But, as we have mentioned, in supergravity there is generically no $SU(2)_R$ symmetry and hence generically no protected spin character. We nevertheless will heedlessly proceed and pretend that the spin character is an index, and see what comes out. Even though this is not justified, it fits in well with the present present discussion, and will be rigorously true when we call upon the same manipulations in the context of framed BPS states associated to line operators in $\mathcal{N} = 2$ *field theory*.

Thus we begin with the (protected) spin character of the halo particle

$$\text{Tr}_{\mathfrak{h}_{\gamma_h}} y^{2J_3} = a_{0,\gamma_h} + a_{1,\gamma_h}(y + y^{-1}) + \ldots \tag{3.48}$$

Here the nonnegative integers a_0 and a_1 denote the number of half-hypermultiplets and vectormultiplets, respectively and in order to be concrete and explicit we will assume these are the only representations which arise. (The generalization to arbitrary halo representations

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can be found in [16].) Our Fock space is generated by a_{m,γ_h} creation operators of $2J_3$ eigenvalue $m + m'$, for each m' of the form $m' = -2J_{\gamma_c,\gamma_h}, -2J_{\gamma_c,\gamma_h} + 2, \ldots, 2J_{\gamma_c,\gamma_h} - 2, 2J_{\gamma_c,\gamma_h}$. The oscillators are fermionic for m even and bosonic for m odd. Note that $\Omega(\gamma_h; u) = a_0 - 2a_1$. If $\langle \gamma_c, \gamma_h \rangle = 0$ then we consider (J_{γ_c,γ_h}) to be the zero vector space, and no halos form. Of course, a_{m,γ_h} is a piecewise continuous integer function of u, but we usually suppress the dependence in the notation.

In view of the above description of oscillators the spin character $\mathrm{Tr} y^{2J_3}$ of the Halo-Fock space (3.17) is

$$\prod_{m'=-2J_{\gamma_c,\gamma_h}}^{2J_{\gamma_c,\gamma_h}} \frac{(1 + y^{m'} x_{\gamma_h})^{a_{0,\gamma_h}}}{[(1 - y^{m'-1} x_{\gamma_h})(1 - y^{m'+1} x_{\gamma_h})]^{a_{1,\gamma_h}}}. \tag{3.49}$$

In order to recover the expression for the BPS index we set $y = -1$ to recover the factor in (3.22).

Now, the appearance of γ_c in the range of the product makes it rather awkward to try to express the spin-character generalization of the generating function $G_{\gamma_c}(u; y)$ in terms of an action of a diffeomorphism.

In order to motivate the general formula, consider a hypermultiplet γ_h-particle with $\langle \gamma_c, \gamma_h \rangle = n$, where n is nonzero. The contribution to the Fock space character is

$$x_{\gamma_c}(1 + y^{n-1} x_{\gamma_h})(1 + y^{n-3} x_{\gamma_h}) \cdots (1 + y^{3-n} x_{\gamma_h})(1 + y^{1-n} x_{\gamma_h}) \tag{3.50}$$

where x_{γ_c} multiplies n factors. If we expand this out and use (3.18) we get

$$x_{\gamma_c}(1 + y^{n-1} x_{\gamma_h})(1 + y^{n-3} x_{\gamma_h}) \cdots (1 + y^{3-n} x_{\gamma_h})(1 + y^{1-n} x_{\gamma_h}) = \sum_{j=0}^{|n|} P_j^{(n)}(y) x_{\gamma_c + j\gamma_h} \tag{3.51}$$

where $P_j^{(n)}(y)$ are symmetric integral Laurent polynomials in y. In fact, as is clear from the Fermionic Fock space interpretation $P_j^{(n)}(y)$ is just the character of the jth antisymmetric product $\Lambda^j \rho_{|n|}$ where ρ_N is the N-dimensional irreducible representation of $SU(2)$.

Now we use a trick. Introduce *non-commuting* variables satisfying a relation generalizing (3.18): [17]

$$X_\gamma X_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}. \tag{3.52}$$

We then claim that the same Laurent polynomials $P_j^{(n)} = ch\Lambda^j \rho_n$ appear when we expand out the product

$$X_{\gamma_c} \Phi_n(X_{\gamma_h}) = \sum_{j=0}^{|n|} P_j^{(n)}(y) X_{\gamma_c + j\gamma_h} \tag{3.53}$$

where

$$\Phi_n(\xi) := \begin{cases} \prod_{s=1}^{n}(1 + y^{-(2s-1)}\xi) & n > 0 \\ 1 & n = 0 \\ \prod_{s=1}^{|n|}(1 + y^{(2s-1)}\xi) & n < 0 \end{cases} \tag{3.54}$$

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[17]Here the noncommutative torus is introduced as a convenient mathematical device. It surely has a compelling physical meaning and motivation, and it would be good to find one.

We prove this by first expanding out $\Phi_n(X_{\gamma_h})$ with coefficients $X_{j\gamma_h}$. For this purpose the X_{γ_h} can be taken as commutative variables satisfying 3.18. It is clear from the "fermionic combinatorics" that the coefficient of $X_{j\gamma_h}$ is (say, for $n > 0$) a polynomial in y^{-1} which is the character $\mathrm{ch}\Lambda^j \rho_n$ up to an overall multiplication by a power of y^{-1}. By comparing the lowest power of y^{-1} with that for the character we see that that power is precisely canceled by

$$X_{\gamma_c} X_{j\gamma_h} = y^{jn} X_{\gamma_c + j\gamma_h} \tag{3.55}$$

Now, let us introduce a noncommutative generalization of the generating function (3.32):

$$\mathbb{G}_{\gamma_c}(u) := \sum_{\gamma_{\mathrm{orb}} \in \Gamma_0^{\perp}} \overline{\underline{\Omega}}_{\gamma_c}(\gamma_{\mathrm{orb}}; u; y) X_{\delta + \gamma_{\mathrm{orb}}} \tag{3.56}$$

The creation of halo Fock spaces across W_{γ_h} are accounted for by the transformation of the noncommutative variables:

$$X_\gamma \to X_\gamma \frac{\Phi_n(X_{\gamma_h})^{a_{0,\gamma_h}}}{[\Phi_n(-y^{-1} X_{\gamma_h}) \Phi_n(-y X_{\gamma_h})]^{a_{1,\gamma_h}}} \tag{3.57}$$

where $n = \langle \gamma, \gamma_h \rangle$. For annihilation of halos we multiply the inverse factor on the RHS of (3.57).

As before, we would like to eliminate the γ dependence (the γ-dependence enters through $n = \langle \gamma, \gamma_h \rangle$) and express this transformation in terms of a single operator, which only depends on γ_h. This is nicely done by introducing the quantum dilogarithm

$$\Phi(X) := \prod_{k=1}^{\infty} (1 + y^{2k-1} X)^{-1} \tag{3.58}$$

and observing that conjugation by the product (notice this is Φ, not Φ_n)

$$S_{\gamma_h} = \frac{\Phi(X_{\gamma_h})^{a_{0,\gamma_h}}}{[\Phi(-y^{-1} X_{\gamma_h}) \Phi(-y X_{\gamma_h})]^{a_{1,\gamma_h}}} \tag{3.59}$$

produces (3.57):

$$S_{\gamma_h} X_\gamma S_{\gamma_h}^{-1} = X_\gamma \frac{\Phi_n(X_{\gamma_h})^{a_{0,\gamma_h}}}{[\Phi_n(-y^{-1} X_{\gamma_h}) \Phi_n(-y X_{\gamma_h})]^{a_{1,\gamma_h}}} \tag{3.60}$$

Again, in a way analogous to the product on n in (3.36), in order to take into account all the halos of charges parallel to γ_h we define (for γ_h primitive),

$$\mathbb{S}_{\gamma_h} := \prod_{n=1}^{\infty} S_{n\gamma_h}. \tag{3.61}$$

The final rule, then, is that we should conjugate

$$\mathbb{G}_{\gamma_c}(u) \to \mathbb{S}_{\gamma_h} \mathbb{G}_{\gamma_c}(u) \mathbb{S}_{\gamma_h}^{-1} \tag{3.62}$$

In particular, for a single hypermultiplet we just conjugate by $\Phi(X_{\gamma_h})$.

Remark: By the same sorts of arguments as before we can arrive at the "motivic KSWCF." This will assert that a suitably ordered product of $\mathbb{S}_\gamma$'s is invariant as parameters such as u are varied across walls of marginal stability. In the formulation of Kontsevich and Soibelman of the motivic WCF they introduce the "quantum torus algebra" (3.52) (with deformation parameter written as q instead of y). The parameter q is associated with objects derived from number theory known as "motives." (In [19, 20] q is the "motive of the affine line $\mathbb{A}^1$".) The theory of motives is a very deep subject in algebraic geometry going back to Grothendieck and is usually mentioned by mathematicians with a certain amount of reverence. The identification of the motivic parameter q with the parameter y measuring the spin of BPS states was first proposed in [11, 12]. It seems a little disappointing that from the physical point of view it involves a relatively trivial generalization of the BPS index to a spin character. Also, in view of the absence of a protected spin character in compactification of type II string theory on compact CY manifolds the validity of the KSWCF in these cases seems very surprising from a physical point of view. Indeed, there are known examples in string theory where variations of hypermultiplet moduli will change the BPS spin character. We find the present situation slightly odd, and in need of clarification.

4. Lecture III: From line operators to the TBA

In this lecture we are going to touch on a few aspects of a series of three papers [13, 15, 16] relating wall-crossing to some other interesting topics in mathematical physics. In this lecture we concern ourselves exclusively with $\mathcal{N} = 2$ field theory.

We have several goals in the present lecture:

1. We will begin by discussing a class of nonlocal observables known as "line operators."

2. Thinking about boundstates of BPS states and line operators defines an analog of the "framed BPS states" discussed in Lecture II but now in the case of field theory. In a very similar way to Lecture II we can then derive the KSWCF for $\mathcal{N} = 2$ field theories.

3. We then turn to correlation functions of line operators. In the simplest case these are functions on a hyperkähler manifold $\mathcal{M}$.

4. We show how analyzing the correlation functions of line operators in terms of framed BPS degeneracies leads to a distinguished set of functions on $\mathcal{M} \times \mathbb{C}^*$ called "Darboux coordinates" and denoted $\mathcal{Y}_\gamma$, in terms of which one can construct the hyperkähler metric on $\mathcal{M}$.

5. It turns out that the construction of the $\mathcal{Y}_\gamma$ involves an integral equation, formally equivalent to the Thermodynamic Bethe Ansatz equation of Zamolodchikov. Moreover, the techniques used for constructing these functions have been applied in considerations of scattering theory in $\mathcal{N} = 4$ SYM as described in the lectures of Maldacena and Veiera.

4.1 Line Operators

A "line operator" is a one-dimensional defect in a quantum field theory. The word "operator" is misleading. A line operator is really a way of modifying the path integral or Hilbert space of the theory.

One approach to defining line operators [Cite:Kapustin, Kapustin-Witten] involves cutting out a small tubular neighborhood around that operator and specifying boundary conditions. The neighborhood of a straightline in four Minkowski space dimensions has a metric

$$-dt^2 + dr^2 + r^2 ds_{S^2}^2 = r^2 \left(ds_{AdS_2}^2 + ds_{S^2}^2 \right) \tag{4.1}$$

In the ultraviolet the quantum field theory flows to some conformal field theory $\mathcal{S}$ and hence one approach to a definition of line operators is to declare that they are boundary conditions of a (super)conformal fixed point theory on $AdS_2 \times S^2$.

We will consider line operators in $\mathbb{R}^{1,3}$ located at a point in space and stretching along the time direction. We think of these as pointlike defects in space, say, at $\vec{x} = 0$.

Line operators clearly break some symmetry. For example, they break Poincaré symmetry down to $so(3)$ rotations around $\vec{x} = 0$ and time translations. We will choose our

line operators to preserve some supersymmetry. Given the discussion in Lecture I around equation (2.10) it is natural to require that the sub-superalgebra that is preserved be that generated by $\mathcal{R}_\alpha{}^A$.

Note that here we have made a *choice* of the phase ζ. This is part of the specification of the line operator, and when we wish to stress that a line operator preserves this subalgebra we write L_ζ.

4.1.1 Examples: Wilson and 't Hooft operators

There are two basic examples of line operators in Lagrangian $\mathcal{N} = 2$ theories: Wilson operators and 't Hooft operators.

To construct supersymmetric Wilson lines along a path p along the time direction, at some fixed $\vec{x}$, in a representation $\rho_\mathcal{R} : G \to \mathrm{End}(V)$ of the gauge group we take

$$L_\zeta(\vec{x}; \mathcal{R}) = \rho_\mathcal{R} P \exp \oint_p \left(\frac{\varphi}{2\zeta} - iA - \frac{\zeta\bar{\varphi}}{2} \right). \tag{4.2}$$

Exercise: Check that this is indeed annihilated by $\mathcal{R}_\alpha{}^A$.

To construct 't Hooft operators we put boundary conditions on a small linking S^2 around $\vec{x} = 0$ corresponding to a magnetic monopole configuration. We will not need the detailed construction in the remainder of this lecture.

4.2 Hilbert space in the presence of line operators

In the presence of a line operator the Hilbert space of states $\mathcal{H}$ is modified (in the sense of representation theory) to a new space $\mathcal{H}_L$.

Since the $\mathcal{R}$-supersymmetries are preserved the argument in Section 2.2 of Lecture I still applies and the Hamiltonian on this Hilbert space satisfies the BPS bound

$$E \geq -\mathrm{Re}(Z/\zeta) \tag{4.3}$$

Once again the Hilbert space can be decomposed into charge sectors: [18]

$$\mathcal{H}_L = \oplus_{\gamma \in \Gamma} \mathcal{H}_{L,\gamma} \tag{4.4}$$

To get some intuition for why there should be a decomposition like this consider the expectation value of (4.2) where it wraps the S^1 in $\mathbb{R}^3 \times S^1$. This should be a trace over $\mathcal{H}_L$.

In the weak coupling region, if we specify a vacuum u then the leading approximation is the trace:

$$\mathrm{Tr}_\mathcal{R} P \exp \oint_p \left(\frac{\varphi}{2\zeta} - iA - \frac{\zeta\bar{\varphi}}{2} \right) \tag{4.5}$$

where the fields are evaluated by their vev's. This trace is a sum over the weight states, which may be interpreted as a sum over electric charges

---

[18] Here we are skipping over an important subtlety. Actually the charge sector is a *torsor* for Γ. See [16] for the full story.

4.3 Framed BPS States

Given the BPS bound we can define framed BPS states as states in $\mathcal{H}_L$ saturating the bound and we can enumerate them with protected spin characters, following the discussion of Lecture I.

A good IR model for the spectrum of states in $\mathcal{H}_{L,\gamma}$ is the following. We think of the framed BPS states as states due to an infinitely heavy BPS dyon of charge γ. A good heuristic here is the following:

Extend the lattice Γ by one extra flavor charge γ_f, and consider a very heavy particle, of charge $\gamma_f - \gamma$ and central charge $Z = \zeta M - Z_\gamma$, where $M > 0$. The renormalized BPS bound in the limit $M \to +\infty$ is just

$$E \geq \lim_{M \to +\infty} (|\zeta M - Z_\gamma| - M) = -\mathrm{Re}(Z_\gamma/\zeta). \tag{4.6}$$
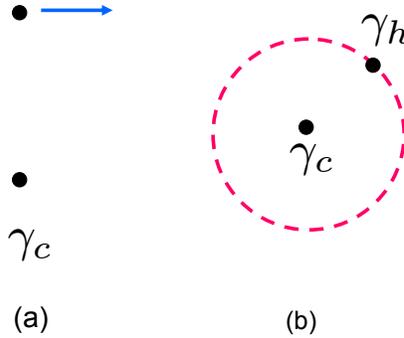

Figure 7: a.) A state in the continuum starting at energy $-\mathrm{Re}(Z_{\gamma_c}/\zeta) + |Z_{\gamma_h}|$ b.) A halo boundstate saturating the BPS bound $-Re(Z_\gamma/\zeta)$

Now, consider a heavy BPS dyon of charge γ_c and another BPS particle of charge γ_h so that $\gamma = \gamma_c + \gamma_h$. There will be states where the γ_h-particle is far away and travels on some unbound orbit. The energy is

$$E = -\mathrm{Re}(Z_{\gamma_c}/\zeta) + |Z_{\gamma_h}| + \frac{1}{2}|Z_{\gamma_h}|v^2 + \cdots \tag{4.7}$$

However, exactly as in our probe computation of Lecture I, there also will be bound orbits where the total energy is

$$E = -\mathrm{Re}(Z_{\gamma_c}/\zeta) - \mathrm{Re}(Z_{\gamma_h}/\zeta) = -\mathrm{Re}(Z_\gamma/\zeta) \tag{4.8}$$

Exactly as in (2.98) of Lecture I the boundstate radius is

$$r_{\mathrm{halo}} = \frac{\langle \gamma_h, \gamma_c \rangle}{2\mathrm{Im}(Z_{\gamma_h}(u)/\zeta)} \tag{4.9}$$

The radius diverges, and the gap to the continuum vanishes across *BPS walls*

$$W_\gamma := \{(u, \zeta) : Z_\gamma(u)/\zeta < 0\} \subset \mathcal{B} \times \mathbb{C}^* \tag{4.10}$$

Thus, across BPS walls W_γ the framed BPS indices and protected spin characters can have wall-crossing.

4.4 Wall-crossing of framed BPS states

We can describe the wall-crossing just as we did in Lecture II.

Associated to a line operator L we introduce a generating function G_L analogous to the generating function we introduced for BPS galaxies in Lecture II:

$$G_L(u, \zeta) := \sum_{\gamma \in \Gamma} \overline{\Omega}(\gamma; \zeta, u) X_\gamma \tag{4.11}$$

The BPS walls divide the space $\mathcal{B} \times \mathbb{C}^*$ into chambers, which we will denote as c. The framed PSC's $\overline{\Omega}(\gamma; \zeta, u)$ are constant in each chamber so we can just write $G_L(c)$.

Wall-crossing story is essentially identical to that for the generating function of BPS galaxies in Lecture II. The main difference is that the presence of the $SU(2)_R$ symmetry makes the derivation for the protected spin characters rigorously true.

Therefore, across W_γ get a diffeomorphism:

$$G(L_\zeta, c^+) = S_\gamma G(L_\zeta, c^-) S_\gamma^{-1} \tag{4.12}$$

and hence we derive the KSWCF for field theories, where c^+ is the side of the wall with $\mathrm{Im}(Z_\gamma(u)/\zeta) > 0$.

One new element which we will need later today is the specialization of this formula to $y = -1$. In this case the Heisenberg relations give a *commutative* algebra, the twisted group algebra:

$$\hat{x}_{\gamma_1} \hat{x}_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \hat{x}_{\gamma_1 + \gamma_2} \tag{4.13}$$

In this case (4.12) becomes a diffeomorphism of

$$\hat{G}(L_\zeta) = \sum \overline{\Omega}(L_\zeta, \gamma) \hat{x}_\gamma \tag{4.14}$$

given by :

$$\hat{x}_\gamma \to (1 - \hat{x}_\gamma)^{-\langle \gamma, \gamma_h \rangle \Omega(\gamma_h)} \hat{x}_\gamma = \mathcal{K}_{\gamma_h}^{-\Omega(\gamma_h)}(\hat{x}_\gamma) \tag{4.15}$$

4.5 Expectation values of loop operators

We now turn to the study of correlation functions of line operators.

The most natural thing to do is to consider Euclidean space $\mathbb{R}^3 \times S^1$ and consider the line operator wrapped on the Euclidean circle. Then the path integral with the line operator wrapped around the circle can be considered to be a trace.

We claim it is

$$\langle L_\zeta \rangle = \mathrm{Tr}_{\mathcal{H}_{u, L_\zeta}} (-1)^{2J_3} e^{-2\pi R H} e^{i\theta \cdot \mathcal{Q}} \sigma(\mathcal{Q}). \tag{4.16}$$

where

1. R is the radius of the circle S^1

2. H is the Hamiltonian.

3. $\mathcal{Q}$ is the charge operator, valued in Γ.

4. $\theta \in Hom(\Gamma, \mathbb{R}/\mathbb{Z})$ is described below.

5. $\sigma(\mathcal{Q})$ is a tricky sign, which we will not attempt to explain here. See [16] for a discussion.

The main point we need to explain here is θ. In order to define the path integral on $\mathbb{R}^3 \times S^1$ we must specify boundary conditions for the fields at $\vec{x} \to \infty$. The boundary conditions of the vm scalars are given by u. We must also specify some for the electromagnetic fields. Since we want finite action configurations the fields should approach flat fields at infinity, but now that we have introduced S^1 there can be nontrivial holonomy of the electromagnetic field. Naively one might think that it is only necessary to specify the "electric Wilson lines"

$$\theta_e^I := \lim_{\vec{x} \to \infty} \oint_{S^1_{\vec{x}}} A^I \tag{4.17}$$

where $S^1_{\vec{x}}$ is the fiber of $\mathbb{R}^3 \times S^1 \to \mathbb{R}^3$ above the point $\vec{x}$. Note that in order to write this we needed to choose a duality frame. It is better to work in a frame independent formalism. Then $\mathbb{F} = d\mathbb{A}$ should be flat at infinity and we specify

$$\theta := \lim_{\vec{x} \to \infty} \oint_{S^1_{\vec{x}}} \mathbb{A} \tag{4.18}$$

This should be viewed as a linear operator from charges to elements of $\mathbb{R}/\mathbb{Z}$.

If the gauge group is $U(1)$ and not $\mathbb{R}$ then we can consider gauge transformations on $S^1_{\vec{x}}$ which have nontrivial winding number $S^1_{\vec{x}} \to U(1) \cong S^1$. These shift they scalar fields θ and thus make them periodic scalar fields.

We must also specify boundary conditions for the fermions. We take them to be *periodic* around the circle. This is the boundary condition which preserves supersymmetry (because only with this boundary condition is there a covariantly constant spinor). That is the reason that $(-1)^{2J_3}$ is inserted in the trace.

Now, the correlation function $\langle L_\zeta \rangle$ is an interesting function of u, θ and ζ which we would like to understand. But we must first make a digression into several geometrical subjects. We return to its analysis in Section 4.9 below.

4.6 A three-dimensional interpretation

The trace in (4.16) only depends on the vacuum structure of the theory. It therefore only depends on physics in the far IR and physics at distance scales much larger than R should have a three-dimensional interpretation.

We therefore suspect that we can interpret (4.16) as a one point function of a local operator in an effective three-dimensional theory.

Since our boundary conditions preserve all supersymmetries the effective three-dimensional theory should have 8 real supersymmetries. (The pointlike operator at $\vec{x} = 0$ corresponding to L_ζ will then preserve four of those supersymmetries.)

General theorems of supersymmetric field theory tell us that the three dimensional theory must be a sigma model with a hyperkähler target space $\mathcal{M}$. We will explain what this means below.

4.6.1 The torus fibration

The low energy effective field theory of the $d = 4$ $\mathcal{N} = 2$ theory on $\mathbb{R}^3 \times S^1$ is a three-dimensional sigma model of maps $\mathbb{R}^3 \to \mathcal{M}$. The low energy bosonic fields in three dimensions are the vectormultiplet scalars $u(x^\mu)$ and the gauge field. Let us choose a duality frame so that the independent scalars are $a^I(\vec{x})$ and the the gauge field is a one-form $A^I(x^\mu)$ with $I = 1, \ldots, r$. Upon reduction to three dimensions we have:

a.) $a^I(x^\mu) \to a^I(\vec{x})$.

b.) The Wilson lines $\theta_{el}^I(\vec{x}) = \oint_{S^1_{\vec{x}}} A^I$

c.) The 3-dimensional gauge field $A_i^I(\vec{x})dx^i$. Now, in three-dimensions a gauge field with compact gauge group is dual to a compact scalar field. In this way we can define scalar fields $\theta_{m,I}(\vec{x})$, $I = 1, \ldots, r$ satisfying:

$$*_3 d\theta_{m,I} = dA^I \tag{4.19}$$

The pair $(\theta_{el}^I, \theta_{m,I})$ are the scalar fields defined in (4.18) in a particular duality frame.

Mathematically, $\mathcal{M}$ is a torus bundle over $\mathcal{B}$ as shown in 8. More properly, there are singularities at points where BPS particles become massless. $\mathcal{M}$ is a torus bundle over the complement $\mathcal{B}^*$ but can be continued, albeit with singularities, to a fibration over all of $\mathcal{B}$.
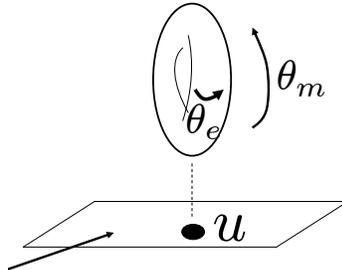

Figure 8: The moduli space $\mathcal{M}$ is a fibration over $\mathcal{B}$ whose generic fiber is the torus of electric and magnetic Wilson lines.

4.6.2 The Semi-Flat Sigma Model

If we dimensionally reduce the Seiberg-Witten Lagrangian then we get a three-dimensional sigma model with action

$$\int_{\mathbb{R}^3} -\frac{R}{2}\mathrm{Im}\tau_{IJ}da^I * d\bar{a}^J - \frac{1}{2R}(\mathrm{Im}\tau)^{-1,IJ}dz_I * d\bar{z}_J \tag{4.20}$$

where

$$dz_I = d\theta_{m,I} - \tau_{IJ}d\theta_e^J \tag{4.21}$$

The details of this computation are spelled out in Problem 5.12. As explained there, the scalar field z_I has periods $n_I - \tau_{IJ}m^J$ where $n_I, m^I \in \mathbb{Z}$.

This is a sigma model with target space $\mathcal{M}$ where $\mathcal{M}$ carries the *semi-flat metric*

$$g^{sf} = R(\mathrm{Im}\tau)|da|^2 + R^{-1}(\mathrm{Im}\tau)^{-1}|dz|^2 \tag{4.22}$$

An important remark is that (4.22) is *not* the exact metric of the effective three-dimensional model at finite R. It only becomes exponentially close to the exact metric as $R \to \infty$. The reason for this is that there are instanton corrections to the metric. These instantons can be thought of as due to worldlines of BPS particles wrapped around the compactification circle S^1. The amplitude for such a worldline behaves roughly like

$$\exp[-const.R|Z_\gamma|] \tag{4.23}$$

for a BPS particle of charge γ. At the end of the lecture we will see how to construct the exact metric and we will obtain the exact instanton expansion for that metric as a series of corrections to the semiflat metric.

4.7 Complex structures

The target space $\mathcal{M}$ has a complex structure with complex coordinates a^I, and z_I. Recall the relation of supersymmetry to complex structure. Already for $\mathcal{N} = 1$ chiral multiplets φ^i, the action of the supersymmetries on a general function $F(\varphi, \bar{\varphi})$ defined on the target space takes the form:

$$[Q_\alpha, F] = \psi_\alpha^i \frac{\partial}{\partial \varphi^i}F$$
$$[\bar{Q}_{\dot{\alpha}}, F] = \bar{\psi}_{\dot{\alpha}}^i \frac{\partial}{\partial \bar{\varphi}^i}F \tag{4.24}$$

This shows that Q_α acts by a vector field corresponding to a holomorphic differential operator while $\bar{Q}_{\dot{\alpha}}$ acts by an anti-holomorphic differential operator.

In our case with $\mathcal{N} = 2$ supersymmetry, Q_α^A acts as a holomorphic differential operator. However, when we reduce to three dimensions there is no difference between the α and $\dot{\alpha}$ index. It turns out that we can form a doublet of supersymmetries

$$Q_{a\alpha A} = (Q_{\alpha A}, \sigma_{\alpha\dot{\beta}}^0 \bar{Q}_A^{\dot{\beta}}) \tag{4.25}$$

where $a = 1, 2$ and that linear combinations of $Q_{1\alpha A}$ and $Q_{2\alpha A}$ also act as holomorphic differential operators but in a *different* complex structure. Indeed, our supersymmetries $\mathcal{R}_\alpha^A$ are of this type. This gives a *family* of $\bar{\partial}$ operators.

The general result, which we do not derive here, is that supersymmetry implies that $\mathcal{M}$ is a *hyperkahler manifold*. In the next section we will summarize some aspects of hyperkahler geometry.

4.8 HyperKahler manifolds

Let us summarize a few basic definitions and facts about hyperkähler manifolds. A nice review of this beautiful subject can be found in the review of Hitchin [17].

Definition: A *hyperkähler* manifold is a Riemannian manifold M with three orthogonal transformations on the tangent bundle $J_a \in \text{End}(TM)$, $a = 1, 2, 3$, such that

1. J_a satisfy the algebra of the quaternions:

$$J_a J_b = -\delta_{ab} + \epsilon_{abc} J_c \qquad (4.26)$$

2. $\nabla J_a = 0$

Since the tangent space has a quaternionic structure the real dimension of M must be a multiple of four. Let us say that $\dim M = 4r$. Then, near any point $p \in M$ we can choose an orthonormal basis for the quaternionic vector space $T_p^* M$ so that, in complex structure, say, J_3 a basis of the cotangent space can be written as dz^I, dw_I, $I = 1, \ldots, r$ and in this basis the complex structures act as:

$$
\begin{aligned}
J_3 &: (dz^I, dw_I) \rightarrow (i dz^I, i dw_I) \\
J_2 &: (dz^I, dw_I) \rightarrow (-\overline{dw_I}, \overline{dz^I}) \\
J_1 &: (dz^I, dw_I) \rightarrow (i\overline{dw_I}, -i\overline{dz^I})
\end{aligned}
\qquad (4.27)
$$

For a hyperkähler manifold M one can show that it is Kähler with respect to each complex structure J_a and hence there are three Kähler forms ω_a, $a = 1, 2, 3$. In the local coordinates given above we can write:

$$\omega_3 = \frac{i}{2} dz^I \overline{dz^I} + \frac{i}{2} dw_I \overline{dw_I} \qquad (4.28)$$

while

$$\omega_1 + i\omega_2 := \omega_+ = dz^I dw_I \qquad (4.29)$$

is of type $(2, 0)$.

In fact, M has a whole sphere's worth of complex structures! If n^a is a real unit three-vector, $n^a n^a = 1$ then note that

$$(n^a J_a)^2 = -1 \qquad (4.30)$$

This sphere of complex structures is known as the *twistor sphere*. The space $Z = M \times S^2$ is known as *twistor space*. It can be given the structure of a complex manifold by taking

$S^2 = \mathbb{C}P^1$ and then, letting $\zeta \in \mathbb{C}$ be the inhomogeneous coordinate on $\mathbb{C}P^1$, the fiber above $\zeta \in \mathbb{C}P^1$ is M considered as a complex manifold in complex structure ζ, denoted M^ζ.

Let us choose the north pole to correspond to the complex structure J_3 and consider stereographic projection of $S^2 = \mathbb{C}P^1 \to \mathbb{C}$. the holomorphic symplectic form is

$$\omega_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_- \tag{4.31}$$

4.8.1 The twistor theorem

Hitchin's twistor theorem says - roughly speaking - that the holomorphic family of of holomorphic symplectic manifolds $\mathcal{M}^\zeta$ equipped with ω_ζ uniquely characterizes the hyperkähler metric. Indeed, some technical points in the statement of the theorem imply that ω_ζ has a three-term Laurent expansion in ζ together with an antiholomorphic symmetry under

$$\zeta \to -1/\bar{\zeta} \tag{4.32}$$

and from this one has the expansion (4.31), from which, in turn, one can extract the hyperkähler metric from the Kähler form ω_3.

4.9 The Darboux expansion

Now let us return to our correlation function $\langle L_\zeta \rangle$.

As we have just explained, these are functions on the hyperkähler manifold $\mathcal{M}$, and moreover, they are *holomorphic* functions on $\mathcal{M}^\zeta$.

We expect $\langle L_\zeta \rangle$ to be a sum over sectors $\gamma \in \Gamma$. Moreover, since L_ζ preserves supersymmetry the quantity should be "BPS saturated." Since the BPS states in $\mathcal{L}_{L,\gamma}$ are the framed BPS states, we expect the correlation function to have the form:

$$\langle L_\zeta \rangle = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma)\mathcal{Y}_\gamma \tag{4.33}$$

where there is a set of "universal" functions $\mathcal{Y}_\gamma$, independent of the line operators L_ζ.

Equation (4.33) should be contrasted with the formal generating function (4.14). They are formally similar, but (4.33) is expanded in terms of honest functions $\mathcal{Y}_\gamma$ on $\mathcal{M}$.

What do we know about these functions?

1. $\mathcal{Y}_\gamma$ is a function of u, θ, ζ

2. For fixed ζ it is holomorphic on $\mathcal{M}^\zeta$.

3. From the $R \to \infty$ limit we know that we project onto framed BPS states and so

$$\mathcal{Y}_\gamma \to \mathcal{Y}_\gamma^{\text{semiflat}} := e^{\pi R Z_\gamma/\zeta + i\hat{\theta}\cdot\gamma + \pi R \zeta \bar{Z}_\gamma} \tag{4.34}$$

4. Similar, but more subtle arguments imply that we also have the asymptotics

$$\mathcal{Y}_\gamma \to \mathcal{Y}_\gamma^{\text{semiflat}} \tag{4.35}$$

for $\zeta \to 0, \infty$

5. With (4.5) as motivation we see that $\mathcal{Y}_\gamma$ should satisfy the reality condition

$$\overline{\mathcal{Y}_\gamma(\zeta)} = \mathcal{Y}_{-\gamma}(-1/\bar{\zeta}) \tag{4.36}$$

6. Now argue that as u, ζ vary $\langle L_\zeta \rangle$ must be continuous. On the other hand, $\underline{\overline{\Omega}}(L_\zeta, \gamma; u)$ has wall-crossing as we saw above. Thus the transformations of (4.14) summarized by (4.15) above must be compensated by compensating discontinuities in the functions $\mathcal{Y}_\gamma$, and hence

$$\mathcal{Y}_\gamma \to \mathcal{K}_{\gamma_h}^{\Omega(\gamma_h; u)} \mathcal{Y}_\gamma \tag{4.37}$$

across the walls $W(\gamma_h)$.

4.10 The TBA

The above 6 properties uniquely characterize the $\mathcal{Y}_\gamma$.

Now it turns out that we can write an integral equation whose solutions satisfy precisely these properties. This equation states that

$$\mathcal{Y}_\gamma(\zeta) = \mathcal{Y}_\gamma^{\mathrm{sf}}(\zeta) \exp\left[-\sum_{\gamma'} \frac{\langle \gamma, \gamma' \rangle \Omega(\gamma'; u)}{4\pi i} \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{Y}_{\gamma'}(\zeta')) \right] \tag{4.38}$$

Here the integrals are along the rays:

$$\ell_\gamma := \{\zeta : Z_\gamma(u)/\zeta < 0\} \tag{4.39}$$

that is, it is the projection of the BPS wall at fixed u into the ζ-plane.

Note that, if we change variables on ℓ_γ to $\zeta = -e^{\eta + i\alpha}$ then the ray is described by $-\infty < \eta < \infty$. Along this ray

$$\mathcal{Y}_\gamma^{\mathrm{sf}} = \sigma(\gamma) e^{-2\pi R |Z_\gamma| \cosh \eta + i\theta_\gamma} \tag{4.40}$$

and hence, in an iterated expansion the integrals along ℓ_γ converge very well. Indeed, taking a logarithm and solving the integral equation by iteration we obtain an expansion valid at large R in quantities of the form (4.23).

One can check that the functions $\mathcal{Y}_\gamma$ defined by the solutions to (4.38) indeed solve the properties of Section 4.9. The most important point to check is discontinuities across W_γ. Note that if we compare the solution to the equation with the analytic continuation across the wall the difference is due to the residue of the kernel in the integral over ℓ_γ. The kernel is just such as to give the discontinuity by a KS transformation.

Remark: By taking the logarithm and making a few changes of variables we can identify (4.38) precisely with a version of Zamolodchikov's Thermodynamic Bethe Ansatz. See [13] for details.

4.11 The construction of hyperkähler metrics

Finally, let us indicate how the hyperkähler metric on $\mathcal{M}$ can be constructed.

The $\mathcal{Y}_\gamma$ satisfy the twisted group algebra

$$\mathcal{Y}_\gamma \mathcal{Y}_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \mathcal{Y}_{\gamma+\gamma'} \tag{4.41}$$

Ignoring the twisting, they therefore generate the algebra of holomorphic functions on the algebraic torus $T_c = \Gamma \otimes \mathbb{C}^*$. The latter is a holomorphic symplectic manifold with holomorphic symplectic form

$$\omega_{T_c} = C^{ij} \frac{dY_i}{Y_i} \wedge \frac{dY_j}{Y_j} \tag{4.42}$$

where $C_{ij} = \langle \gamma_i, \gamma_j \rangle$ and $Y_i = Y_{\gamma_i}$ are computed relative to a basis γ_i for Γ.

It is therefore natural to consider the holomorphic symplectic form ω_ζ given by the "pullback" under $\mathcal{Y}_\gamma$ of (4.43):

$$\omega_\zeta = C^{ij} \frac{d\mathcal{Y}_i}{\mathcal{Y}_i} \wedge \frac{d\mathcal{Y}_j}{\mathcal{Y}_j} \tag{4.43}$$

This is properly holomorphic symplectic, and moreover, it is *continuous* across walls of marginal stability.

In [13] it is argued that this is indeed the proper physical metric which corrects the singular semi-flat metric at finite values of R to a smooth(er) hyperkähler metric on $\mathcal{M}$. As we have noted, the $\mathcal{Y}_\gamma$ when expanded around $\mathcal{Y}_\gamma^{\rm sf}$ have an expansion in quantities of the form (4.23). As promised, we can now interpret that expansion as an exact instanton expansion for the quantum-corrected metric on $\mathcal{M}$.

4.12 Omissions

The above lecture has only touched briefly on the project [13, 15, 16]. In addition to many important details we have actually had to ignore some rather important aspects of the subject. These include

1. 6d viewpoint.
2. Hitchin systems.
3. A_1 theories: Fock-Goncharov coordinates, WKB triangulations, and the algorithm for computing the BPS spectrum.
4. How to compute framed BPS degeneracies in A_1 theories: Laminations.
5. Cluster algebras and cluster varieties.
6. Quantization of Hitchin moduli spaces.

5. Problems

5.1 Angular momentum of a pair of dyons

Consider two dyons of (magnetic, electric) charge (p_i, q_i), $i = 1, 2$.

    a.) By computing the Poynting vector of the electromagnetic field show that the two-dyon system carries classical angular momentum (in cgs units) around the midpoint between the two dyons given by

$$\vec{J} = \frac{1}{c}(p_1 q_2 - p_2 q_1)\hat{r} \tag{5.1}$$

where $\hat{r}$ is the unit vector pointing from dyon 2 to dyon 1.

    b.) Using quantum mechanical quantization of angular momentum conclude that

$$(p_1 q_2 - p_2 q_1) = \frac{\hbar c}{2} n \tag{5.2}$$

where n is an integer.

    c.) Show that the antisymmetric bilinear form

$$\langle (p_1, q_1), (p_2, q_2) \rangle := p_1 q_2 - p_2 q_1 \tag{5.3}$$

defines a *symplectic form* on $\mathbb{R}^2$.

5.2 The period vector

Since the central charge Z is linear on Γ it can be extended to a linear functional $Z \in \mathrm{Hom}(V, \mathbb{C})$.

    a.) Show that there is necessarily a vector $\varpi \in V$ such that

$$Z(\gamma) = \langle \gamma, \varpi \rangle \tag{5.4}$$

This vector is known as the *period vector*.

    b.) Show that, in the notation of Section 2.7 that the period vector is

$$\varpi = a^I \alpha_I + a_{D,I} \beta^I \tag{5.5}$$

    c.) Show that the vector space $\mathrm{Hom}(V, \mathbb{C})$ is symplectic and, if we choose a duality frame the symplectic form can be written as

$$da^I \wedge da_{D,I} \tag{5.6}$$

(Here a^I and $a_{D,I}$ are considered to be independent coordinates on $\mathrm{Hom}(V, \mathbb{C})$.)

    d.) Show that the central charge functions that arise in physical theories define a Lagrangian subspace of $\mathrm{Hom}(V, \mathbb{C})$ and that the generating function for this Lagrangian subspace is the prepotential $\mathcal{F}$.

5.3 Attractor Geometry

Show that for the single-centered attractor equation the near horizon geometry $r \to 0$ is $AdS_2 \times S^2$.

Note that the fixed point values of the moduli are *independent* of the boundary conditions u_∞ at $r = \infty$.

5.4 Duality Transformations

We regard the symplectic form and $\mathcal{I}$ as fixed. We define our basic symplectic transformation by

$$\left(\tilde{\alpha} \ \tilde{\beta} \right) = \left(\alpha \ \beta \right) M \tag{5.7}$$

or, written out with indices:

$$\left(\tilde{\alpha}_I \ \tilde{\beta}^I \right) = \left(\alpha_J \ \beta^J \right) \begin{pmatrix} A^J_{\ I} & B^{JI} \\ C_{JI} & D_J^{\ I} \end{pmatrix} \tag{5.8}$$

Preserving the Darboux basis means

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \langle \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix}, \left(\tilde{\alpha} \ \tilde{\beta} \right) \rangle = M^{tr} J M \tag{5.9}$$

So M is a symplectic matrix. Since we are interested in integral bases for Γ we have $M \in Sp(2r, \mathbb{Z})$.

a.) Show that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \in Sp(2r) \tag{5.10}$$

is equivalent to

$$
\begin{aligned}
A^{tr} D - C^{tr} B &= 1 \\
A^{tr} C - C^{tr} A &= 0 \\
B^{tr} D - D^{tr} B &= 0
\end{aligned}
\tag{5.11}
$$

b.) Show that

$$M^{-1} = -J M^{tr} J = \begin{pmatrix} D^{tr} & -B^{tr} \\ -C^{tr} & A^{tr} \end{pmatrix} \tag{5.12}$$

c.) Show that we have an automorphism of the symplectic group

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \tag{5.13}$$

and an anti-automorphism $M \to M^{tr}$. The latter does not interact very well with the raised/lowered index structure of the sub-blocks.

d.) Compute the transformation of the period matrix as follows: The complex structure $\mathcal{I}$ is held fixed and we are just changing Darboux basis. Therefore the $\mathcal{I} = -i$ space is unchanged. It will be spanned by f_I and by

$$\tilde{f}_I = \tilde{\alpha}_I + \tilde{\beta}^J \tilde{\tau}_{JI} \tag{5.14}$$

Show that

$$f_I = \tilde{f}_J (D_I{}^J - B^{KJ} \tau_{KI}) \tag{5.15}$$

and hence

$$\tilde{\tau} = (A^{tr} \tau - C^{tr})(D^{tr} - B^{tr} \tau)^{-1} \tag{5.16}$$

e.) Show that we can change the ugly equation (5.16) to the more standard

$$\tilde{\tau} = (A\tau + B)(C\tau + D)^{-1} \tag{5.17}$$

if we apply the anti-automorphism of Sp:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A^{tr} & -C^{tr} \\ -B^{tr} & D^{tr} \end{pmatrix} \tag{5.18}$$

f.) Define $F^{I,+} := F^I + i *_4 F^I$. Show that

$$*_4 F^{\pm} = \mp i F^{\pm} \tag{5.19}$$

g.) Show that when $\epsilon = -1$,

$$\tilde{F}^+ = (D^{tr} - B^{tr}\tau)F^+ \tag{5.20}$$

and when $\epsilon = +1$ we replace $F^+ \rightarrow F^-$ or $\tau \rightarrow \bar{\tau}$.

5.5 Duality invariant version of the fixed point equations

a.) Show that charge quantization and the fixed point equation (2.86) implies

$$2\mathrm{Re}(\varphi^I/\zeta) = p^I/r + 2\mathrm{Re}(\varphi^I_\infty/\zeta) \tag{5.21}$$

b.) Warning: The fixed point equations are solved by taking ρ_M^I and ρ_E^I constant but this is not a solution of the full equations of motion! (This is a very unusual situation.) The r-dependence of the vm scalars means that τ_{IJ} is r-dependent. Substitute (2.84) back into the Maxwell equations to show that they are only satisfied if

$$X_{JK}\rho_M^K - Y_{JK}\rho_E^K \tag{5.22}$$

is constant. Since $\rho_M^I = p^I$ is quantized, this means that ρ_E^I cannot be constant if the vm scalars are not constant.

c.) Show that under duality transformations

$$\begin{pmatrix} \tilde{q}_I \\ \tilde{p}^I \end{pmatrix} = M^{tr} \begin{pmatrix} q_I \\ p^I \end{pmatrix} \tag{5.23}$$

d.) By demanding duality invariance of the fixed point equations, and using the duality transformation laws for $F^{+,I}$ show that the vm scalars φ^I and their duals $\varphi_{D,I}$ must transform under duality like

$$
\begin{pmatrix} \tilde{\varphi}_{D,I} \\ \tilde{\varphi}^I \end{pmatrix} = M^{tr} \begin{pmatrix} \varphi_{D,I} \\ \varphi^I \end{pmatrix} \tag{5.24}
$$

e.) Define the period vector to be

$$
\varpi = i(\varphi^I \alpha_I + \varphi_{D,I} \beta^I) \tag{5.25}
$$

and $Z(\gamma; u) = \langle \gamma, \varpi \rangle$ and obtain equation (2.90):

$$
2\text{Im}\left[\zeta^{-1} Z(\gamma; u(r))\right] = \frac{\langle \gamma, \gamma_c \rangle}{r} + 2\text{Im}\left[\zeta^{-1} Z(\gamma; u)\right] \qquad \forall \gamma \in \Gamma \tag{5.26}
$$

f.) The fixed point equations also imply $F_{0\ell}^I = \text{Re}(i\partial_\ell(\varphi^I/\zeta))$. Now whos that under duality transformations

$$
\begin{pmatrix} \tilde{G}_I \\ \tilde{F}^I \end{pmatrix} = M^{tr} \begin{pmatrix} G_I \\ F^I \end{pmatrix} \tag{5.27}
$$

and hence the duality invariant form of $F_{0\ell}^I = \text{Re}(i\partial_\ell(\varphi^I/\zeta))$ becomes

$$
\mathbb{F}_{0\ell} = \text{Re}\partial_\ell(\varpi/\zeta) \tag{5.28}
$$

g.) Writing $\mathbb{F}_{0\ell} = -\partial_\ell \mathbb{A}_0$, derive (2.94).

5.6 Landau Levels on the Sphere

Consider a dyon of charge γ_1 confined to a sphere of radius r surrounding a dyon of charge γ_2. Suppose that $\langle \gamma_1, \gamma_2 \rangle \neq 0$.

a.) By rotating to an appropriate duality frame show that the action for the particle can be written as

$$
\int \frac{1}{2}\mu r^2 (\sin^2\theta \dot{\phi}^2 + \dot{\theta}^2) - \int \kappa \cos\theta \dot{\phi} \tag{5.29}
$$

where $\kappa = \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle$.

b.) Show that the action is not well-defined when $\theta = 0, \pi$ but can be made well-defined by adding a suitable multiple of $\int \dot{\phi}$ to the Lagrangian.

c.) Show that e^{iS} is well-defined provided $\kappa \in \frac{1}{2}\mathbb{Z}$. This gives another derivation of the Dirac quantization condition. (Essentially, it is Dirac's original derivation.)

d.) Show that the Hamiltonian for this particle is (setting $\hbar = 1$):

$$
H = \frac{1}{2\mu r^2}\left(p_\theta^2 + p_\phi^2\right) = \frac{1}{2\mu r^2}\bar{H} \tag{5.30}
$$

which, upon quantization with proper operator ordering (coming from becomes

$$
\bar{H} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(-i\frac{\partial}{\partial\phi} + \kappa\cos\theta\right)^2 \tag{5.31}
$$

e.) Show that the groundstates have energy $E = \frac{|\kappa|}{2\mu r^2}$ and the space of groundstates is spanned by the wavefunctions:

$$\Psi_m = e^{im\phi}(1 + \cos\theta)^{\frac{1}{2}(|\kappa|-m)}(1 - \cos\theta)^{\frac{1}{2}(|\kappa|+m)} \tag{5.32}$$

where $m \in \mathbb{Z}$ and normalizability requires

$$-(|\kappa| + 1) < m < (|\kappa| + 1) \tag{5.33}$$

f.) If κ is an integer then m must be an integer, but if κ is a half-integer then m must be a half integer. With this in mind show that the number of solutions of (5.33) is $|\langle\gamma_1, \gamma_2\rangle|$

5.7 Kontsevich-Soibelman transformations

a.) Show that the operators $\tau_\gamma := (-1)^{D_\gamma} x_\gamma$ satisfy $\tau_\gamma \tau_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle}\tau_{\gamma+\gamma'}$,

b.) Introduce the symplectic form

$$C^{ij}\frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j} \tag{5.34}$$

where $C_{ij} = \langle\gamma_i, \gamma_j\rangle$ is the matrix of the symplectic form on Γ in basis γ_i. Show that $\{x_{\gamma_1}, x_{\gamma_2}\} = \langle\gamma_1, \gamma_2\rangle x_{\gamma_1} x_{\gamma_2}$.

c.) Show that the KS transformation is a symplectic transformation.

d.) For the case $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ with $\langle\gamma_1, \gamma_2\rangle = 1$ define $x = x_{\gamma_1}$, $y = x_{\gamma_2}$, and $K_{a,b} = K_{a\gamma_1+b\gamma_2}$. Show that
♣ CONTINUE ♣

5.8 Creation vs. Annihilation of Halos

a.) Show that the stable side of the wall W_γ is

$$\langle\gamma, \gamma_c + \gamma_{\text{orb}}\rangle\text{Im}(e^{-i\alpha_0}Z(\gamma; u)) > 0 \tag{5.35}$$

where α_0 is the phase of $Z(\gamma_0; u)$.

b.) Whether a halo Fock space is created or destroyed when u crosses W_γ depends on whether u crosses from the unstable to the stable side, or the other way around. Verify the rule that if u crosses in the direction of increasing $\arg[Z_\gamma e^{-i\alpha_0}]$ then the transformation (3.36) should be applied.

5.9 Deriving the Primitive and Semiprimitive WCF from the KSWCF

One of the original presentations of Kontsevich and Soibelman proceeded along the following lines:

a.) Define a Lie algebra with one generator e_γ for each $\gamma \in \Gamma$ by the relations:

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle}\langle\gamma_1, \gamma_2\rangle e_{\gamma_1+\gamma_2} \tag{5.36}$$

b.) Note that there is a natural subalgebra generated by $e_{r_1\gamma_1+r_2\gamma_2}$ with $r_1, r_2 \geq 0$, and that this subalgebra has a filtration by subalgebras with generators $r_1, r_2 \geq N$.

c.) Consider the group element $U_\gamma = \exp\left(\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2}\right)$. Then for any convex wedge $\mathcal{V}$ in the complex plane, the ordered product

$$\prod_{Z_\gamma(u)\in\mathcal{V}} U_\gamma^{\Omega(\gamma;u)} \tag{5.37}$$

ordered by the slope of $Z_\gamma(u)$ is constant across walls of marginal stability.

d.) Note that the filtration on the Lie algebra mentioned in 1 can be used to truncate the group elements. Show that if we quotient by the subalgebra with $N = 2$ we get a copy of the Heisenberg Lie algebra and the Heisenberg group.

e.) Show that the truncation of the KSWCF to this Heisenberg group (together with the Baker-Campbell-Hausdorff formula) is equivalent to the primitive wall-crossing formula.

f.) Using a similar truncation, derive the semi-primitive wall-crossing formula.

5.10 Verifying Some Wall-Crossing Identities

In this problem you will verify (3.44) and (3.47) as mathematical identities on products of KS-transformations.

♣ TO BE COMPLETED. See appendix of [13] ♣

5.11 Characters of $SU(2)$ representations

a.) Let ρ_n denote the n-dimensional representation of $SU(2)$. What is the maximal spin?

b.) The character of a representation V of $SU(2)$ is defined to be $\chi_V(y) = \text{Tr} y^{2J_3}$ where J_3 is any generator. Show that

$$\chi_{\rho_n}(y) = \frac{y^n - y^{-n}}{y - y^{-1}} \tag{5.38}$$

Evaluate the limits $y \to \pm 1$ using L'Hopital's rule.

c.) An arbitrary finite dimensional representation of $SU(2)$ is completely reducible and hence isomorphic to $\sum_{n\geq 1} a_n \rho_n$ for some integers $a_n \in \mathbb{Z}_+$. Show that the character of a representation V of $SU(2)$ determines V uniquely up to isomorphism.

d.) A *virtual* representation is a formal sum $\sum a_n \rho_n$ where a_n are integers. Show that the virtual representations form a ring. Show that the character of a virtual representation does not determine it uniquely.

e.) Show that the character of the jth antisymmetric product of ρ_n is the q-binomial:

$$P_j^{(n)}(y) = \frac{[n]_y!}{[j]_y![n-j]_y!} := \begin{bmatrix} n \\ j \end{bmatrix} \tag{5.39}$$

where

$$[n]_y := \frac{y^n - y^{-n}}{y - y^{-1}}. \tag{5.40}$$

f.) ♣ Something about the q-binomial theorem ♣

5.12 Reduction of a $U(1)$ gauge field to three dimensions and dualization

Consider a $U(1)^r$ gauge field on $\mathbb{R}^{1,2} \times S^1$ with metric $ds^2 = dx^\mu dx_\mu + R^2(dx^3)^2$ and $x^3 \sim x^3 + 2\pi$. Here $\mu = 0, 1, 2$ and we denote x^μ by $\bar{x}$. The action is

$$\int -\frac{1}{4\pi}\mathrm{Im}\tau_{IJ}F^I * F^J + \frac{1}{4\pi}\mathrm{Re}\tau_{IJ}F^I F^J \tag{5.41}$$

where $I, J = 1, \ldots r$, F^I is the 2-form fieldstrength and τ_{IJ} is a symmetric complex matrix with positive definite imaginary part. It may be spacetime-dependent.

Show that the low energy effective action in three dimensions is a sigma model with a torus as target space and action

$$\int -\frac{1}{2R}(\mathrm{Im}\tau)^{-1,IJ}dz_I * d\bar{z}_J \tag{5.42}$$

where $dz_I = d\theta_{m,I} - \tau_{IJ}d\theta_e^J$ where θ_e^I and $\theta_{m,J}$ are real scalar fields with period 1.

Hints:

a.) Consider the dimensional reduction to $\mathbb{R}^3$. Write

$$\begin{aligned} A^I(\bar{x}, x^3) &\to \theta_e(\bar{x})dx^3 + \bar{A}^I(\bar{x}) \\ F^I(\bar{x}, x^3) &\to d\theta_e^I \wedge dx^3 + \bar{F}^I \end{aligned} \tag{5.43}$$

where θ_e^I is a scalar in $\mathbb{R}^{1,2}$.

b.) Show that due to certain gauge transformations the scalar field θ_e^I has with period 1.

c.) Now substitute into the action and do the integral over x^3 to get

$$\int -\frac{1}{2R}\mathrm{Im}\tau_{IJ}d\theta_e^I *_3 d\theta_e^J - \frac{R}{2}\mathrm{Im}\tau_{IJ}\bar{F}^I *_3 \bar{F}^J - \mathrm{Re}\tau_{IJ}\bar{F}^I d\theta_e^J \tag{5.44}$$

d.) Now dualize the 3d vector field with fieldstrength $\bar{F}^I$ by introducing

$$\exp i \int \bar{F}^I d\theta_{m,I} \tag{5.45}$$

into the path integral and integrating out $\bar{F}^I$ through a Gaussian integral. Show that $\theta_{m,I}$ is also a periodic scalar of period 1. To do the Gaussian integral recall that the stationary action is half the value of the linear term at the stationary point which is easily seen to be

$$\bar{F}^I = -\frac{1}{R}(\mathrm{Im}\tau)^{-1,IJ}\left(d\theta_{m,J} - \mathrm{Re}\tau_{JK}d\theta_e^K\right) \tag{5.46}$$

For more help see [30] and [29].

5.13 Dual torus

Let Γ be a symplectic lattice of rank r.

a.) Show that $T := \Gamma^* \otimes \mathbb{R}/\mathbb{Z}$ is an algebraic torus of dimension r, i.e. it is isomorphic to $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (with r factors).

b.) Show that for any vector $\gamma \in \Gamma$ there is a canonical $\mathbb{C}^*$-valued function X_γ on T.

c.) Show that T has a holomorphic symplectic form, and express it in terms of functions X_γ.

6. Some Sources for the Lectures

The course will cover material primarily from papers by Denef and Moore and by Gaiotto, Moore, and Neitzke.

A previous knowledge of some aspects of N=2 susy and of the attractor mechanism and the split attractor flows would be helpful.

For general background on N=2 supergravity, special geometry, the attractor mechanism, and black hole entropy see [22].

The viewpoint on the attractor mechanism we will use is reviewed in Section 2 of [23]. For a nice introductory discussion of split attractor flows see [7].

In lecture one we will begin with wall-crossing formulae from the viewpoint of supergravity. For a brief qualitative overview see [10]. More details are in [9].

For essential background for the paper [13] see [17] for a nice review of hyperkahler geometry.

A key role will be also played by reduction of N=2 theory from four to three dimensions. We recommend [26].

For the paper [15] an important role will be played by a hypothetical six-dimensional superconformal theory. For some background on this theory see [27, 32].

The essential geometrical construction of some N=2 d=4 theories from M5 branes was introduced by Witten in [31]. See [15], section 3 and [14] for further explanation and development.

The geometrical picture of BPS states in this context was first discussed in [18] and some nice aspects of wall-crossing in a special class of theories was discussed in [28]. This geometrical picture for BPS states is used extensively in [15].

Acknowledgements

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