Black Hole Microstate Counting and Its Macroscopic Counterpart

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Abstract

We survey recent results on the exact dyon spectrum in a class of $\mathcal{N} = 4$ supersymmetric string theories, and discuss how the results can be understood from the macroscopic viewpoint using AdS_2/CFT_1 correspondence. The comparison between the microscopic and the macroscopic results includes power suppressed corrections to the entropy, the sign of the index, logarithmic corrections and also the twisted index measuring the distribution of discrete quantum numbers among the microstates.

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(UNDER CONSTRUCTION)

Contents

1 Introduction

Black holes are classical solutions of the equations of motion of general theory of relativity. Each black hole is surrounded by an event horizon that acts as a one way membrane. Nothing, including light, can escape a black hole horizon. Thus classically the horizon of a black hole behaves as a perfect black body at zero temperature.

This picture undergoes a dramatic modification in quantum theory [1–4]. There a black hole behaves as a thermodynamic system with definite temperature, entropy etc. In particular, the temperature and the Bekenstein-Hawking entropy of a black hole is given by the simple formulæ:

$$T = \frac{\kappa}{2\pi}, \qquad S_{BH} = \frac{A}{4\,G_N}, \tag{1.1}$$

where κ is the surface gravity – acceleration due to gravity at the horizon of the black hole (measured by an observer at infinity), A is the area of the event horizon and G_N is the Newton's gravitational constant. We have set $\hbar = c = k_B = 1$.

Now, for ordinary objects, the entropy of a system has a microscopic interpretation. If we fix the macroscopic parameters (*e.g.* total electric charge, energy etc.) and count the number of quantum states (dubbed microstates), each of which has the same charge, energy etc., then we can define the microscopic (statistical) entropy as:

$$S_{micro} = \ln d_{micro}, \tag{1.2}$$

where d_{micro} is the number of such microstates. This naturally leads to the question whether the entropy of a black hole has a similar statistical interpretation. As pointed out by Hawking, answering this question in the affirmative is essential for any consistent theory of quantum gravity as otherwise it leads to violation of the laws of quantum mechanics.

In order to investigate the statistical origin of black hole entropy, we need a quantum theory of gravity. Since string theory gives a framework for studying classical and quantum properties of black holes, we shall carry out our investigation in string theory. Now, even though there is a unique string (M)-theory, it can exist in many different stable and metastable phases. Without knowing precisely which phase of string theory describes the part of the universe we live in, we cannot directly compare string theory to experiments. However, there are some issues like those involving black hole thermodynamics, which are universal, and hence can be addressed in any phase of string theory. We shall make use of this freedom to study these issues in a special class of phases of string theory with a large amount of unbroken supersymmetry. Since

these phases have Bose-Fermi degenerate spectrum of states, they do not describe the observed world. Nevertheless they contain black hole solutions and hence can be used to study issues involving black hole thermodynamics.

Many aspects of black hole thermodynamics have been studied in string theory, but we shall focus our attention on one particular aspect: entropy of the black hole in the zero temperature limit (i.e., supersymmetric, extremal black holes). The advantage of studying such a black hole is that it is a stable state of the theory. The general strategy is as follows [5, 6]:

1. Identify a supersymmetric black hole carrying a certain set of electric charges $\{Q_i\}$ and magnetic charges $\{P_i\}$, and calculate its entropy $S_{BH}(Q, P)$ using the Bekenstein-Hawking formula.[1]

2. Identify the supersymmetric quantum states in string theory carrying the same set of charges. These can include not only the fundamental strings but also other objects in string theory which are required for consistency of the theory (e.g. D-branes, Kaluza-Klein monopoles). We then calculate the number $d_{micro}(Q, P)$ of these states.

3. Compare $S_{micro} \equiv \ln d_{micro}(Q, P)$ with $S_{BH}(Q, P)$.

For a class of supersymmetric extremal black holes in type IIB string theory on $K3 \times S^1$, Strominger and Vafa [6] computed the Bekenstein-Hawking entropy via (1.1) and found agreement with the statistical entropy defined in (1.2). This agreement is quite remarkable since it relates a geometric quantity in black hole space-time to a counting problem that does not make any direct reference to black holes. At the same time, one should keep in mind that the Bekenstein-Hawking formula is an approximate formula that holds in classical general theory of relativity. While string theory gives a theory of gravity that reduces to Einstein's theory when gravity is weak, there are corrections.[2] Thus the Bekenstein-Hawking formula for the entropy works well only when gravity at the horizon is weak. Typically this requires the charges to be large. Similarly, the computation of d_{micro} in [6] was also carried out in the limit of large charges, so that instead of having to carry out an exact counting of states, one can use some appropriate asymptotic formula to compute it. Thus the agreement between S_{BH} and S_{micro}, seen in [6], can be regarded as an agreement in the limit of large size.

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[1]Since we are considering a generic phase of string theory, it may have more that one Maxwell field and hence multiple charges.

[2]In string theory, even at classical level, we have higher derivative (α') corrections. This is because strings are not point objects. So even at classical level, there will be corrections to the Bekenstein-Hawking formula. Besides this, there will also be quantum corrections.

This leads to the following question: For ordinary systems, thermodynamics provides an approximate description that becomes exact in the limit of large volume. Is the situation with black holes similar, i.e., do they only capture the information about the system in the limit of large charge and mass? Or, could it be that the relation $A/4G_N = \ln d_{micro}$ is an approximation to an exact result? Our goal will be to argue for the second possibility by giving an exact formula to which the above is an approximation.

In order to address this issue, we have to work on two fronts:

1. Count the number of microstates to greater accuracy.

2. Calculate the black hole entropy to greater accuracy.

We can then compare the two to see if they agree beyond the large charge limit. In these lectures we shall describe the progress on both fronts.

Note that on the gravity side we shall not try to identify the individual microstates – this is the goal of the fuzzball program [7]. Our approach will be to find a systematic procedure that allows us to compute the total number of states in the ensemble from the gravity side without having to identify the individual microstates. More generally, we would like to find an algorithm for computing the trace of various observables in this ensemble from the gravity side.

We end this section by giving a summary of the progress, which will be reviewed in detail in the rest of these lecture notes:

1. **Progress in microscopic counting:** In a wide class of phases of string theory with 16 or more unbroken supercharges, one now has a complete understanding of the microscopic 'degeneracies' of supersymmetric black holes [8–53]. Typically, such theories have multiple Maxwell fields and the black hole is characterized by multiple electric and magnetic charges, collectively denoted by (Q, P). It turns out that for a wide class of charge vectors (all charge vectors in some cases), $d_{micro}(Q, P)$ in these theories can be explicitly computed and can be expressed as Fourier expansion coefficients of some functions with remarkable symmetry properties. This provides us with the 'experimental data' to be explained by a 'theory of black holes', giving a powerful tool for checking the internal consistency of string theory. Needless to say, in the large charge limit, these degeneracies agree with the exponential of the Bekenstein-Hawking entropy of black holes carrying the same set of charges. Our goal will be to see how far the agreement can be pushed beyond the large charge limit.

2. **Progress in black hole entropy computation:** On the macroscopic side, we would like to ask whether we can find an exact formula for the black hole entropy that can be compared with $\ln d_{micro}(Q, P)$. This will require us to take into account

   (a) stringy (α') corrections, and

   (b) quantum (g_s) corrections.

   We shall describe an approach to finding such a general formula for black hole entropy using AdS_2/CFT_1 correspondence. We shall then apply this general formalism to the specific case of supersymmetric black holes in $\mathcal{N} = 4$ supersymmetric string theories, and compare the results with the microscopic answer.

2 Microstate counting

In this section we shall survey the known results on the counting of quarter-BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories.

2.1 The role of index

The counting of microstates is always done in a region of the moduli space where gravity is weak and hence the states do not form a black hole. In order to be able to compare it with the black hole entropy, we must focus on quantities which do not change as we change the coupling from small to large value. So we need an appropriate index which is protected by supersymmetry, and at the same time does not vanish identically when evaluated on the microstates of interest. The relevant index in $D = 4$ turns out to be the helicity trace index [54, 55].

Suppose we have a BPS state that breaks $4n$ supersymmetries. Then there will be $4n$ fermion zero modes (goldstinos) on the world-line of the state. Quantization of these zero modes will produce Bose-Fermi degenerate states. Thus the usual Witten index $Tr(-1)^F$, which measures the difference between the number of bosonic and fermionic states, will receive vanishing contribution from these states. To remedy this situation, we define a new index called the helicity trace index:

$$B_{2n} = \frac{1}{(2n)!} Tr\{(-1)^F (2h)^{2n}\} = \frac{1}{(2n)!} Tr\{(-1)^{2h} (2h)^{2n}\}, \qquad (2.3)$$

where h is the third component of the angular momentum in the rest frame. The trace is taken over states carrying a fixed set of charges. For every pair of fermion zero modes, $Tr\{(-1)^F (2h)\}$

gives a non-vanishing result i, leading to a non-zero contribution $(-1)^n$ to B_{2n}. On the other hand, any state that breaks more than $4n$ supersymmetries, will have more then $2n$ pairs of fermion zero modes and will give vanishing contribution to this trace. In particular, non-BPS states will not contribute, and the index will be protected from corrections as we vary the moduli (except at the walls of marginal stability [56–60], which will be discussed in §2.4).

Quarter-BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories preserve four of the sixteen supersymmetries, and hence break twelve supersymmetries. Thus the relevant helicity trace index is B_6. We shall now describe the microscopic results for B_6 in a class of $\mathcal{N} = 4$ supersymmetric string theories. However, we must keep in mind that, since on the microscopic side we compute an index, on the black hole side also we must compute an index. Otherwise we cannot compare the results of microscopic and macroscopic computations. We will show in §3.7 how can we use black hole entropy to compute the index B_6 on the black hole side.

2.2 Microstate counting in heterotic string theory on T^6

The simplest example of an $\mathcal{N} = 4$ supersymmetric string theory is heterotic string theory on T^6 (or equivalently type IIA or IIB string theory on $K3 \times T^2$, as they are related by duality transformations). This theory has 28 U(1) gauge fields arising from the Cartan generators of the $E_8 \times E_8$ (or $SO(32)$) gauge group, and the components of the metric and the 2-form field along the six internal directions. Thus a generic charged state is characterized by 28 dimensional electric charge vector Q and 28 dimensional magnetic charge vector P. Under the $O(6, 22; \mathbb{Z})$ T-duality symmetry of the theory, the charges Q and P transform as vectors. This allows us to define T-duality invariant bilinears in the charges[3]: Q^2, P^2, $Q \cdot P$.

Our goal is to compute the index $B_6(Q, P)$. The computation is done in the dual frame: type IIB on $K3 \times S^1 \times \widetilde{S}^1$, where S^1 and $\widetilde{S}^1$ represent two circles which are not factored metrically.[4] In this frame, we compute B_6 for a rotating D1-D5-p system [61] in Kaluza-Klein (KK) monopole (or equivalently Taub-NUT) background. More specifically, we take a system containing [10]

1. one KK monopole along $\widetilde{S}^1$;

2. one D5-brane wrapped on $K3 \times S^1$;

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[3]Note that these bilinears are not positive definite as $O(6, 22; \mathbb{Z})$-invariant matrices have both positive and negative eigenvalues.

[4]The problem with carrying out this computation in heterotic frame is that there the system will contain $NS5$-branes, and the coupling constant diverges at the core of these branes.

3. $(\widetilde{Q}_1 + 1)$ D1-branes wrapped on S^1;

4. $-n$ units of momentum along S^1;

5. J units of momentum along $\widetilde{S}^1$.

The momentum along $\widetilde{S}^1$ appears as an angular momentum at the center of the Taub-NUT space [62]. Thus, macroscopically, the system describes a rotating BMPV black hole [63] at the center of the Taub-NUT space [10]. In the weak coupling limit, the dynamics is given by that of a system of decoupled harmonic oscillators, and an exact computation of B_6 is possible. The result is then expressed in terms of the T-duality invariant bilinears Q^2, P^2, $Q \cdot P$ in the original heterotic frame, using the fact that the system described above has

$$Q^2 = 2n, \qquad P^2 = 2\widetilde{Q}_1, \qquad Q \cdot P = J. \tag{2.4}$$

If Q^2, P^2 and $Q \cdot P$ were the only T-duality invariants, i.e., if any two dyons with the same Q^2, P^2 and $Q \cdot P$ had been related to each other by a T-duality transformation, then the result for $B_6(Q, P)$ for the specific system described above will give the result for all dyons in the theory. However it turns out that this is not quite correct. Nevertheless, any charge vector satisfying the condition [22]

$$\gcd\{Q_i P_j - Q_j P_i, \quad 1 \le i, j \le 28\} = 1, \tag{2.5}$$

can be related to the above system by a T-duality transformation [31]. Thus the formula we quote below is valid only for this special class of charges. We shall briefly comment on the other charge vectors in §2.5.

Let us denote by $B_6(\widetilde{Q}_1, n, J)$ the sixth helicity trace associated with the system described above. We define the partition function as:

$$Z(\rho, \sigma, v) = \sum_{\widetilde{Q}_1, n, J} (-1)^J B_6(\widetilde{Q}_1, n, J) e^{2\pi i(\widetilde{Q}_1 \rho + n\sigma + Jv)}. \tag{2.6}$$

The computation of Z proceeds as follows. In the weakly coupled type IIB description, the low energy dynamics of the system is described by three weakly interacting pieces:

1. The closed string excitations around the KK monopole.

2. The dynamics of the D1-D5 center of mass coordinate in the KK monopole background.

3. The motion of the D1 branes along $K3$.

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The dyon partition function is obtained as the product of the partition functions of these three subsystems [17].[5] The analysis can be simplified by taking the size of S^1 to be large compared to other dimensions, so that we can regard each subsystem as a 1+1 dimensional CFT. Since BPS condition forces the modes carrying positive momentum along S^1 (right-moving modes) to be frozen into their ground state, only left-moving modes can be excited. We shall now describe the contribution to Z from each subsystem.

First consider the fields describing the dynamics of KK monopole. These include

1. 3 left-moving and 3 right-moving bosons arising from its motion in the 3 transverse directions;

2. 2 left-moving and 2 right-moving bosons arising from the components of 2-form fields along the harmonic 2-form in Taub-NUT space [64, 65];

3. 19 left-moving and 3 right-moving bosons, arising from the components of the 4-form field along the wedge product of the harmonic 2-form on Taub-NUT and a harmonic 2-form on $K3$;

4. 8 right-moving goldstino fermions associated with the eight supersymmetries which are broken by the KK monopole.

Since the right-moving modes are frozen into their ground state, the contribution to the partition function from the KK-monopole dynamics, after separating out the contribution from fermion zero modes which go into the helicity trace, is [17]

$$Z_{KK} = e^{-2\pi i \sigma} \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n \sigma})^{-24} \right\} . \tag{2.7}$$

The overall factor of $e^{-2\pi i \sigma}$ is a reflection of the fact that the ground state of the Kaluza-Klein monopole carries a net momentum of 1 along S^1.

The dynamics of the D1-D5 center of mass motion in the KK monopole background is described by a supersymmetric sigma model with Taub-NUT space as the target space. By taking the size of the Taub-NUT space to be large, we can take the oscillator modes to be those

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[5]A factor of $(-1)^{J+1}$ in (2.6) was missed in [17]. The $(-1)^J$ factor arises because in five dimensions, at the center of the KK monopole, we have $(-1)^F = (-1)^{J+2h}$ instead of $(-1)^{2h}$ [13]. An overall factor of -1, which has been absorbed in the definition of B_6 in (2.6), arises from the partition function of the quantum mechanics describing the D1-D5-brane motion in the KK monopole background [28]. A detailed derivation of many of the results given in this section has been reviewed in [28].

of a free field theory, but the zero mode dynamics is described by a supersymmetric quantum mechanics problem. The contribution is found to be [17]

$$Z_{CM} = e^{-2\pi i v} \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n\sigma})^4 (1 - e^{2\pi i n\sigma + 2\pi i v})^{-2} (1 - e^{2\pi i n\sigma - 2\pi i v})^{-2} \right\} e^{-2\pi i v} (1 - e^{-2\pi i v})^{-2} .$$

(2.8)

The third component comprises D1-brane motion along $K3$. This can be computed as outlined below [66]:

1. First consider a single D1-brane, wrapped k times along S^1 and carrying fixed momenta along S^1 and $\widetilde{S}^1$. The dynamics of this system is described by a supersymmetric sigma model with target space $K3$. The number of states of this system can be counted by the standard method of going to the orbifold limit. After removing a trivial degeneracy factor associated with fermion zero mode quantization, the net number of bosonic minus fermionic states, carrying momentum $-l$ along S^1 and j along $\widetilde{S}^1$, is given by $c(4lk - j^2)$, where $c(n)$ is defined as:

$$F(\tau, z) \equiv 8 \left[\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right] ,$$

(2.9)

$$F(\tau, z) = \sum_{j \in \mathbb{Z}, n} c(4n - j^2) e^{2\pi i n\tau + 2\pi i zj} .$$

(2.10)

Physically, $c(4n - j^2)$ counts the number of BPS states in the supersymmetric sigma model with target space $K3$ with $L_0 = n$ and $\mathcal{J}_3 = j/2$, where $\mathcal{J}_3$ denotes the third component of the $SU(2)$ R-symmetry current.

2. A generic state contains multiple D1-branes of this type, carrying different amounts of winding along S^1 and different momenta along S^1 and $\widetilde{S}^1$. The total number of states can be determined from the result of step 1 by simple combinatorics.

The net contribution to the partition function from D1-brane motion along $K3$ is [66]:

$$Z_{D1} = e^{-2\pi i \rho} \prod_{\substack{l,j,k \in Z \\ k>0, l\geq0}} \left\{ 1 - e^{2\pi i (l\sigma + k\rho + jv)} \right\}^{-c(4lk - j^2)} ,$$

(2.11)

After taking the product of the component partition functions, we get [17]

$$Z = e^{-2\pi i (\rho + \sigma + v)} \prod_{\substack{l,j,k \in Z \\ k\geq0, l\geq0, j<0 \text{ for } k=l=0}} \left\{ 1 - e^{2\pi i (l\sigma + k\rho + jv)} \right\}^{-c(4lk - j^2)} ,$$

(2.12)

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where we have used the explicit values of $c(u)$ to express the contribution from (2.7) and (2.8) in terms of $c(n)$. Indeed these two factors give the $k = 0$ term in (2.12). Eq.(2.12) can be expressed as

$$Z(\rho, \sigma, v) = 1/\Phi_{10}(\rho, \sigma, v)\,. \tag{2.13}$$

Here Φ_{10} is a well known function, known as the weight 10 Igusa cusp form of $Sp(2,\ \mathbb{Z})$ [67,68].[6] The formula for Z given above was conjectured in [8].

Eq.(2.6) can be inverted to express $B_6(\widetilde{Q}_1, n, J)$ as

$$- B_6(\widetilde{Q}_1, n, J) = (-1)^{J+1} \int d\rho\, d\sigma\, dv\, e^{-2\pi i(\widetilde{Q}_1\rho + n\sigma + Jv)}\, Z(\rho, \sigma, v)\,. \tag{2.14}$$

We shall express this in a more duality invariant notation using (2.4):

$$- B_6(Q, P) = (-1)^{Q\cdot P+1} \int d\rho\, d\sigma\, dv\, e^{-\pi i(P^2\rho + Q^2\sigma + 2Q\cdot Pv)}\, Z(\rho, \sigma, v)\,. \tag{2.15}$$

2.3 Asymptotic expansion

In order to compare (2.15) with the black hole entropy, we need to find its behaviour for large Q^2, P^2, $Q.P$. It turns out that this is controlled by the behaviour of Z at its poles, which in turn are at the zeroes of Φ_{10} [8]. The location of the zeroes of Φ_{10} as well as the behaviour of Φ_{10} around these zeroes can be determined using its modular properties. We perform one of the three integrals using the residue theorem, picking up contributions from various poles. The leading contribution comes from the pole at [8]

$$(\rho\sigma - v^2) + v = 0\,. \tag{2.16}$$

After picking up the residue at this pole, we are left with a two dimensional integral:

$$- B_6(Q, P) \simeq \int \frac{d^2\tau}{\tau_2^2}\, e^{F(Q^2, P^2, Q.P, \tau_1, \tau_2)}\,, \tag{2.17}$$

where (τ_1, τ_2) parametrize the locus of the zeroes of Φ_{10} at (2.16) in the (ρ, σ, v) space and

$$F = \frac{\pi}{2\tau_2}(Q - \tau P)\cdot(Q - \bar{\tau}P) - 24\ln\eta(\tau) - 24\ln\eta(-\bar{\tau}) - 12\ln(2\tau_2) + \ln\left[26 + \frac{\pi}{\tau_2}(Q - \tau P)\cdot(Q - \bar{\tau}P)\right]\,. \tag{2.18}$$

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[6] $Sp(2,\ \mathbb{Z})$ includes the $SL(2,\ \mathbb{Z})$ S-duality group, but it is a much bigger group than the S-duality group of string theory. Thus it is not completely understood why Z has $Sp(2,\ \mathbb{Z})$ symmetry (see [12, 20, 43] for some attempts in this direction). In fact, this property of Z comes out at the very end after combining the results from the individual subsystems. But once we arrive at this final form, these symmetries can be conveniently used to analyse the asymptotic behaviour of Z.

We evaluate this integral by the saddle point method. We expand F around its extremum and carry out the integral using perturbation theory. If we consider a limit in which we scale all the charges by some large parameter Λ, then the perturbation expansion around the saddle point generates a series in inverse power of Λ^2, with the leading semi-classical result being of order Λ^2.

Applying the above procedure, first of all we find that, for large charges, $-B_6(Q, P)$ is positive [28] (i.e., $B_6(Q, P)$ is negative). Furthermore [9, 69]:

$$\ln |B_6(Q, P)| = \pi \sqrt{Q^2 P^2 - (Q.P)^2} - \phi\left(\frac{Q.P}{P^2}, \frac{\sqrt{Q^2 P^2 - (Q.P)^2}}{P^2}\right) + \mathcal{O}\left(\frac{1}{Q^2, P^2, Q.P}\right),$$

(2.19)

where

$$\phi(\tau_1, \tau_2) \equiv 12 \ln \tau_2 + 24 \ln \eta(\tau_1 + i\tau_2) + 24 \ln \eta(-\tau_1 + i\tau_2).$$ (2.20)

The first term, $\pi\sqrt{Q^2 P^2 - (Q \cdot P)^2}$, is indeed the Bekenstein-Hawking entropy of the black hole [70, 71]. The macroscopic origin of the other terms will be discussed in §3.4.

2.4 Walls of marginal stability

Our result for the D1-D5-KK monopole system was derived for weakly coupled type IIB string theory. However, as we move around in the moduli space, we may hit walls of marginal stability, at which the quarter-BPS dyon under consideration becomes unstable against decay into a pair of half-BPS dyons. At these walls, the index jumps, and hence we cannot trust our formula on the other side of the wall. It turns out, however, that with the help of S-duality, we can always bring the moduli to a domain where the type IIB theory is in the weakly coupled domain and we can trust our original formula. The net outcome of this analysis is that, in different domains, the index is given by the formula:

$$- B_6(Q, P) = (-1)^{Q.P+1} \int_C d\rho d\sigma dv \, e^{-\pi i(P^2 \rho + Q^2 \sigma + 2Q.Pv)} / \Phi_{10}(\rho, \sigma, v),$$ (2.21)

where C denotes the choice of 'contour' that picks a 3 real dimensional subspace of integration in the 3 complex dimensional space:

$$Im(\rho) = M_1, \quad Im(\sigma) = M_2, \quad Im(v) = M_3, \quad 0 \le Re(\rho), Re(\sigma), Re(v) \le 1.$$ (2.22)

The three real numbers (M_1, M_2, M_3), which specify the choice of the contour C, depend on the domain in the moduli space where we compute the index [21, 22, 25]. The jumps in the

index, across the walls of marginal stability, are encoded in the residues at the poles in Z that we encounter while deforming the contour corresponding to one domain to the contour corresponding to the other domain. There is a precise correspondence between different walls of marginal stability and different poles of Z. For the decay $(Q, P) \Rightarrow (Q, 0) + (0, P)$, the associated wall is at $v = 0$ [17–19, 21, 22]. This, together with the S-duality invariance of the theory, tells us that for the wall associated with the decay

$$(Q, P) \Rightarrow (\alpha Q + \beta P, \gamma Q + \delta P) + ((1 - \alpha)Q - \beta P, -\gamma Q + (1 - \delta)P), \qquad (2.23)$$

the corresponding pole is at

$$\gamma \rho - \beta \sigma + (\alpha - \delta) v = 0. \qquad (2.24)$$

A precise formula giving (M_1, M_2, M_3) in terms of the moduli and charges can be found in [25]. We should keep in mind, however, that the result is independent of (M_1, M_2, M_3) as long as changing them does not make the contour cross a pole.

On the black hole (macroscopic) side, these jumps correspond to (dis-)appearance of two-centered black holes as we cross walls of marginal stability. There is a precise match between the B_6 index of 2-centered black holes carrying charges given on the right hand side of (2.23), and the change in $B_6(Q, P)$ computed from the residues at the poles (2.24) [24, 25].

In this context, we would like to mention that the changes in the index across the walls of marginal stability are subleading, as these give corrections which grow as exponentials of single power of the charges. This is related to the fact that only decays of a 1/4-BPS dyon into half-BPS dyons contribute to the wall crossing in an $\mathcal{N} = 4$ supersymmetric string theory [26, 38, 48]. However the contribution from the multi-centered solutions can become significant when we study $\mathcal{N} = 2$ dyons [60].

2.5   Other duality orbits

We have already said that the results given above are valid for a subset of dyons satisfying the condition (2.5). These can be related via duality transformation to the D1-D5-p-KK system analyzed here. But we would like to see if we can say something about the dyons which are outside these duality orbits, i.e., which have [22]

$$\gcd\{Q_i P_j - Q_j P_i, \quad 1 \leq i, j \leq 28\} = r, \qquad (2.25)$$

for some integer $r > 1$. These dyons can be related to a system of IIB on $K3 \times S^1 \times \widetilde{S}^1$ with [22, 31, 32]

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1. 1 KK monopole along $\widetilde{S}^1$,

2. r D5-branes wrapped on $K3 \times S^1$,

3. $(\widetilde{Q}_1 + 1)\,r$ D1-branes wrapped on S^1,

4. $-n$ units of momentum along S^1,

5. rJ units of momentum along $\widetilde{S}^1$.

If we can compute the B_6 index for these dyons, we can use this to compute the B_6 index of any other dyon. This has not yet been done from first principles, but a guess has been made by requiring that wall crossing is controlled by the residues at the poles of the partition function as in the $r = 1$ case. In the domain of the moduli space where 2-centered black holes are absent, the proposal for the B_6 index for these dyons is [35]

$$\sum_{s|r} s\, B_6\left(\widetilde{Q}_1\frac{r}{s}, n, J\frac{r}{s}\right), \qquad (2.26)$$

where $B_6(\widetilde{Q}_1, n, J)$ is the function defined in (2.14). An effective string model for arriving at this result has been suggested in [37], but this has not been derived completely from first principles. Note that for large charges, the contribution from the $s > 1$ terms grow as $\exp(\pi\sqrt{Q^2 P^2 - (Q \cdot P)^2}/s)$ and hence are exponentially suppressed compared to the leading $s = 1$ term. Thus the result for the index reduces to that for the $r = 1$ case up to exponentially suppressed corrections.

2.6 Generalization I: Twisted index

Let us take type IIB theory on $K3 \times S^1 \times \widetilde{S}^1$. On special subspaces of the moduli space of $K3$, we encounter enhanced discrete symmetries which preserve the holomorphic (2,0)-form on $K3$ [72, 73]. Thus these symmetries commute with supersymmetry. Let us work on such a subspace of the moduli space with a $\mathbb{Z}_N$ discrete symmetry generated by g. In this subspace, we can define a twisted index:

$$B_6^g = \frac{1}{6!} Tr\{(-1)^F (2h)^6 g\}. \qquad (2.27)$$

This can be calculated using the same method described earlier by keeping track of the g quantum numbers of the various modes contributing to the partition function. The final result

takes the form [51]:

$$B_6^g(Q, P) = (-1)^{Q.P} \int_C d\rho d\sigma dv \, e^{-\pi i (P^2 \rho + Q^2 \sigma + 2Q.Pv)} \, Z^g(\rho, \sigma, v) \,, \qquad (2.28)$$

where the functions Z^g are known explicitly. They also turn out to have nice modular properties and poles in the complex (ρ, σ, v) space.[7] As a result, we can find the behaviour of this index for large charges by the same method described earlier. The important difference is that now there are no poles at (2.24). Instead the poles are at [51]

$$n_2(\rho\sigma - v^2) - m_1\rho + n_1\sigma + m_2 + jv = 0 \,, \qquad m_1 n_1 + m_2 n_2 + \frac{1}{4}j^2 = \frac{1}{4} \,,$$

$$m_1, n_1, m_2 \in \mathbb{Z} \,, \qquad j \in 2\,\mathbb{Z} + 1 \,, \qquad n_2 \in N\,\mathbb{Z} \,. \qquad (2.29)$$

The leading contribution now comes from the poles at (2.29) with $n_2 = N$, and the answer in the large charge limit is [51]:

$$\ln |B_g^6(Q, P)| = \pi\sqrt{Q^2 P^2 - (Q.P)^2}/N + \mathcal{O}(1) \,. \qquad (2.30)$$

A macroscopic explanation of this result will be given in §4.1.

2.7   Generalization II: CHL models

We again start with type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1$ with a $\mathbb{Z}_M$ symmetry generated by $\widetilde{g}$ as described in §2.6, but this time we take an orbifold of this theory by $\widetilde{g}$ accompanied by $2\pi/M$ shift along S^1.[8] This generates a new class of $\mathcal{N} = 4$ supersymmetric string theories known as CHL models [82, 83]. The orbifold operation removes some of the $U(1)$ gauge fields. Thus, in general, CHL models have $(r + 6)$ $U(1)$ gauge fields with $r \leq 22$, and Q and P are $(r + 6)$ dimensional vectors.[9] The precise value of r depends on M, – the order of the orbifold group. The T-duality group is a discrete subgroup of $O(6, r)$ with Q and P transforming as vectors of $O(6, r)$. Thus $O(6, r)$ invariant bilinears Q^2, P^2 and $Q \cdot P$ are T-duality invariants.

In this theory we can take the same D1-D5-KK monopole system as considered earlier since all of these configurations, as well as momenta along S^1 and $\widetilde{S}^1$, are invariant under the orbifold group. The index B_6 in this theory can be calculated in the same way as before, keeping track

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[7]General discussion on such modular forms can be found in [74–81].

[8]The $\mathbb{Z}_M$ symmetries are chosen from the same set as the $\mathbb{Z}_N$ symmetries of the §2.6, but we are using a different label since in the next section we shall combine the analysis of §2.6 and this subsection.

[9]Since 6 of the $U(1)$ gauge fields represent graviphoton fields, they must exist in all $\mathcal{N} = 4$ supersymmetric string theories.

of the $\widetilde{g}$ quantum numbers of the various modes, and the effect of the orbifold projection. The result of this computation is [17]:

$$B_6(Q,P) = (-1)^{Q.P} \int_C d\rho\, d\sigma\, dv\, e^{-\pi i(P^2\rho + Q^2\sigma + 2Q.Pv)}\, \widetilde{Z}^g(\rho,\sigma,v)\,, \qquad (2.31)$$

where $\widetilde{Z}^g(\rho,\sigma,v)$ is yet another new function, also with nice modular properties and poles in the (ρ,σ,v) space. We find that its behaviour for large charges is given by:

$$\ln|B_6(Q,P)| = \pi\sqrt{Q^2 P^2 - (Q.P)^2} - \phi\left(\frac{Q.P}{P^2}, \sqrt{\frac{Q^2 P^2 - (Q.P)^2}{P^2}}\right) + \mathcal{O}\left(\frac{1}{Q^2, P^2, Q.P}\right)\,, \qquad (2.32)$$

where

$$\phi(\tau_1, \tau_2) \equiv (k+2)\ln\tau_2 + \ln\ g(\tau_1 + i\tau_2) + \ln\ g(-\tau_1 + i\tau_2)\,. \qquad (2.33)$$

Here k are known numbers and $g(\tau)$ are known functions, depending on the choice of M. This generalizes (2.19). Furthermore, in each case we have $B_6(Q,P) < 0$. The macroscopic origin of (2.32) will be explained in §3.4, and the macroscopic explanation of the sign of B_6 will be given in §3.7.

2.8   Generalization III: Twisted index in CHL models

Next we consider a special subspace of the moduli space on which type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1$ has a $\mathbb{Z}_M \times \mathbb{Z}_N$ discrete symmetry that commutes with supersymmetry. Let $\widetilde{g}$ and g be the generators of $\mathbb{Z}_M$ and $\mathbb{Z}_N$ respectively. Let us now take an orbifold of this theory by a $\mathbb{Z}_M$ symmetry generated by $\widetilde{g}$ together with $1/M$ unit of shift along S^1. Here g still generates a symmetry of the theory. We now define:

$$B_6^g = \frac{1}{6!}\, Tr\{(-1)^F (2h)^6 g\}\,. \qquad (2.34)$$

The computation of the above index gives the result [52]

$$B_6^g(Q,P) = (-1)^{Q.P} \int_C d\rho\, d\sigma\, dv\, e^{-\pi i(P^2\rho + Q^2\sigma + 2Q.Pv)}\, \widehat{Z}^g(\rho,\sigma,v)\,, \qquad (2.35)$$

where $\widehat{Z}^g$ is yet another set of functions, also with nice modular properties and poles in the complex (ρ,σ,v) space. Its behaviour for large charges is found to be

$$\ln|B_6^g(Q,P)| = \pi\sqrt{Q^2 P^2 - (Q.P)^2}/N + \mathcal{O}(1)\,. \qquad (2.36)$$

A macroscopic explanation of this result will be given in §4.1.

2.9 Generalization IV: Twisted index in type II string compactification

The analysis described above has also been generalized to untwisted and twisted indices in type II string compactifications on T^6 and its asymmetric orbifolds. We shall not describe the analysis here; they can be found in [18, 19, 51, 52]. The general feature of all these models is that a $\mathbb{Z}_N$ twisted index B_6^g grows as

$$\ln |B_6^g(Q, P)| = \pi \sqrt{Q^2 P^2 - (Q.P)^2}/N + \mathcal{O}(1) \,. \tag{2.37}$$

This includes the case of $N = 1$, ı.e., the untwisted index, for which $\ln |B_6(Q, P)| \simeq \pi \sqrt{Q^2 P^2 - (Q.P)^2}$. Macroscopic explanation for these results is the same as that for the black holes in heterotic string theories, and hence we shall not discuss these cases separately.

A special mention must be given to the untwisted index in type II string theory on T^6. This theory has $\mathcal{N} = 8$ supersymmetry and the black holes with finite area event horizon are 1/8-BPS. Thus the relevant helicity trace index is B_{14}. For states carrying only NS-NS sector charges (and those related to these by duality symmetry), we can again label the charges by a pair of vectors (Q, P), representing the electric and magnetic charges respectively. In the $r = 1$ sector defined in (2.5), the result for this index is known in terms of Fourier coeffceints of an appropriate combination of Jacobi theta function and Dedekind eta function [11,13,36,84]. For large charges, we get [85]

$$\ln |B_{14}(Q, P)| \simeq \pi \sqrt{Q^2 P^2 - (Q.P)^2} - 2 \ln \left[Q^2 P^2 - (Q.P)^2) \right] + \mathcal{O}(\text{charge}^{-2}) \,. \tag{2.38}$$

Note that there is a logarithmic correction to the entropy, which was absent in the other cases discussed so far. So far this logarithmic term has not been reproduced from direct macroscopic analysis. Some discussion on this can be found in §4.2.

3 Macroscopic analysis

Our next goal is to

- develop tools for computing the entropy of extremal black holes including stringy and quantum corrections,

- relate this entropy to the helicity trace index,

- apply it to black holes carrying the same charges for which we have computed the microscopic index, and

- compare the macroscopic results with the microscopic results.

In this section, we shall mainly address the first and the second issues, i.e., find a general formula for computing the black hole degeneracy and the helicity trace on the macroscopic side. Some aspects of the third and the fourth issues will be discussed in §3.4, but we postpone the major part of this to §4. Since AdS_2 space will play a crucial role in our analysis, we begin by describing some aspects of AdS_2 space.

3.1   What is AdS_2 ?

Take a three dimensional space labelled by coordinates (x, y, z) and metric

$$ds^2 = dx^2 - dy^2 - dz^2 \, . \tag{3.39}$$

AdS_2 may be regarded as a two dimensional Lorentzian space embedded in this 3-dimensional space via the relation:

$$x^2 - y^2 - z^2 = -a^2 \, , \tag{3.40}$$

where a is some constant giving the radius of AdS_2. Clearly, this space has an SO(2,1) isometry.

Introducing the independent coordinates (η, t) such that

$$x = a \sinh \eta \cosh t, \quad y = a \cosh \eta, \quad z = a \sinh \eta \sinh t \, , \tag{3.41}$$

we can write

$$dx^2 - dy^2 - dz^2 = a^2 (d\eta^2 - \sinh^2 \eta \, dt^2) \, . \tag{3.42}$$

Finally, defining

$$r = \cosh \eta \, , \tag{3.43}$$

the metric for AdS_2 can be expressed as:

$$ds^2 = a^2 \left[\frac{dr^2}{r^2 - 1} - (r^2 - 1)dt^2 \right], \quad r \geq 1 \, . \tag{3.44}$$

Using a change of coordinates, one can show that the apparent singularity at $r = 1$ is a coordinate singularity, and one can continue the space-time beyond $r = 1$ to generate what is known as global AdS_2 space-time. This will not play any role in our subsequent discussion.

3.2 Why AdS_2 ?

The reason that AdS_2 plays an important role for extremal black holes is that all known black holes develop an AdS_2 factor in their near horizon geometry in the extremal limit. In particular, the time translation symmetry gets enhanced to the $SO(2,1)$ isometry of AdS_2. We shall illustrate how this happens by considering the example of Reissner-Nordstrom solution in $D = 4$. This is described by the metric

$$ds^2 = -(1 - \rho_+/\rho)(1 - \rho_-/\rho)d\tau^2 + \frac{d\rho^2}{(1 - \rho_+/\rho)(1 - \rho_-/\rho)} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (3.45)$$

Here $\rho_\pm$ are parameters determined in terms of the mass and charges carried by the black hole. In the extremal limit, $\rho_- \to \rho_+$. In order to take this limit, we define:

$$2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}, \qquad (3.46)$$

and take $\lambda \to 0$ limit keeping r, t fixed. In this limit, the metric takes the form [86–88]:

$$ds^2 = \rho_+^2 \left[-(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (3.47)$$

This describes the space $AdS_2 \times S^2$. One can also verify that, in this limit, the near horizon electric and magnetic fields are invariant under the isometries of $AdS_2 \times S^2$.

We will now postulate that *any extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry.* This postulate has been partially proved in [89, 90]. The full near horizon geometry takes the form $AdS_2 \times K$, where K is some compact space. K includes not only the compact space on which string theory is compactified (to get a four dimensional theory), but also the angular coordinates (*e.g.* the S^2 factor for spherically symmetric black holes in four dimensions).

3.3 Higher derivative corrections

In string theory, we expect the Bekenstein-Hawking formula for the black hole entropy to receive

- higher derivative corrections arising in classical string theory, and

- quantum corrections.

Of these, the higher derivative corrections are captured by Wald's general formula for black hole entropy in any general coordinate invariant classical theory of gravity [91–94]. Furthermore, this formula takes a very simple prescription for black holes with an AdS_2 factor in the near horizon geometry [95–97]. We shall illustrate this in the context of spherically symmetric black holes in four dimensional theories. In this case, the near horizon geometry has an $AdS_2 \times S^2$ factor. Consider an arbitrary general coordinate invariant theory of gravity coupled to a set of gauge fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$. The most general form of the near horizon geometry of an extremal black hole, consistent with the symmetry of $AdS_2 \times S^2$, is:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1 \left(-(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) ,$$

$$\phi_s = u_s , \qquad F_{rt}^{(i)} = e_i , \qquad F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi} \sin\theta , \tag{3.48}$$

where v_1, v_2, $\{u_s\}$, $\{e_i\}$ and $\{p_i\}$ are constants. For this background, the components of the Riemann tensor are given by:

$$R_{\alpha\beta\gamma\delta} = -v_1(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}) \quad \text{(where } \alpha, \beta, \gamma, \delta = r, t) ,$$

$$R_{mnpq} = v_2(g_{mp}\, g_{nq} - g_{mq}\, g_{np}) \quad \text{(where } m, n, p, q = \theta, \phi) . \tag{3.49}$$

The covariant derivatives of the Riemann tensor, scalar fields and gauge field strengths vanish.

Let $\sqrt{-\det g}\,\mathcal{L}$ be the Lagrangian density. We define the functions:

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi\, \sqrt{-\det g}\,\mathcal{L} , \qquad \mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\pi(\, e_i\, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})\,) . \tag{3.50}$$

Then for an extremal black hole of electric charge $\vec{q}$ and magnetic charge $\vec{p}$, one finds that [95]

1. the values of $\{u_s\}$, $\{e_i\}$, v_1 and v_2 are obtained by extremizing $\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})$ with respect to these variables:

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0 , \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0 , \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0 , \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0 ; \tag{3.51}$$

2. the Wald entropy of the black hole is given by

$$S_{BH} = \mathcal{E} , \tag{3.52}$$

at the extremum.

Eqs.(3.51) follows from the equations of motion and the definition of the electric charge, while (3.52) follows from Wald's formula for the black hole entropy.

These results provide us with [95–97]

1. an algebraic method for computing the entropy of extremal black holes without solving any differential equation;

2. a proof of the attractor mechanism [98–101], i.e., the black hole entropy is independent of the asymptotic moduli.

However, this approach does not prove the existence of an extremal black hole carrying a given set of charges; it works assuming that the solution exists.

3.4   Quantum corrections: A first look

Next we must address the effect of quantum corrections on the black hole entropy. The first guess would be that we should apply Wald's formula again, but replacing the classical action by the one particle irreducible (1PI) action. This will again give a simple algebraic method for computing the entropy once we compute the 1PI action. However, this prescription is not complete since the 1PI action typically has non-local contribution due to massless states propagating in the loops. In contrast, Wald's formula is valid for theories with local Lagrangian density. This is apparent in (3.50) where the definition of the function f requires explicit knowledge of the local Lagrangian density $\mathcal{L}$.

Nevertheless, this procedure has been used to compute corrections to black hole entropy from local terms in the 1PI action with significant success [102–111]. If we consider the CHL models obtained by $\mathbb{Z}_N$ orbifold of type IIB on $K3 \times S^1 \times \widetilde{S}^1$, then at tree level there are no corrections to the black hole entropy from the four derivative terms in the effective action. But at one loop, these theories get corrections proportional to the Gauss-Bonnet term in the 1PI action [55, 112]:

$$\sqrt{-\det g}\, \Delta\mathcal{L} = -\frac{1}{64\pi^2}\, \phi(\tau, \bar{\tau})\, \sqrt{-\det g}\, \left\{ R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}\, R^{\mu\nu} + R^2 \right\},\qquad (3.53)$$

where τ is the modulus of the torus ($S^1 \times \widetilde{S}^1$) and ϕ is the same function that appeared in (2.33). Adding this correction to the supergravity action, we find that the Wald entropy of a black hole in the CHL model is given by [87]

$$S_{BH} = \pi\sqrt{Q^2 P^2 - (Q.P)^2} - \phi\left(\frac{Q.P}{P^2}, \sqrt{\frac{Q^2 P^2 - (Q.P)^2}{P^2}}\right) + \mathcal{O}\left(\frac{1}{Q^2, P^2, Q.P}\right),\qquad (3.54)$$

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in exact agreement with the result (2.32) for $\ln |B_6(Q, P)|$ for large charges.[10]

3.5   Quantum corrections to horizon degeneracy

Let us denote by d_{hor} the degeneracy associated with the horizon of an extremal black hole. We shall now turn to the full quantum computation of d_{hor} from the macroscopic side, and describe a proposal for computing quantum corrected entropy in terms of a path integral of string theory in this near horizon geometry [116, 117]. The steps for computing d_{hor} are as follows:

1. Go to the Euclidean formalism by the replacement $t \to -i\theta$ and represent the AdS_2 factor by the metric:

$$ds^2 = v \left((r^2 - 1)\, d\theta^2 + \frac{dr^2}{r^2 - 1} \right), \quad 1 \le r < \infty, \quad \theta \equiv \theta + 2\pi. \tag{3.55}$$

   With the change of variable $r = \cosh\eta = (1 + \rho^2)/(1 - \rho^2)$, we get the metric on a unit disk:

$$ds^2 = v \left(\sinh^2\eta\, d\theta^2 + d\eta^2 \right) = \frac{4v}{(1 - \rho^2)^2} (d\rho^2 + \rho^2 d\theta^2), \quad 0 \le \eta < \infty, \quad 0 \le \rho < 1. \tag{3.56}$$

2. Regularize the infinite volume of AdS_2 by putting a cut-off $r \le r_0 f(\theta)$, for some smooth periodic function $f(\theta)$. This makes the AdS_2 boundary have a finite length L.

3. Define:

$$Z_{AdS_2} \equiv \left\langle \exp[-iq_k \oint d\theta\, A_\theta^{(k)}] \right\rangle, \tag{3.57}$$

   where the symbol $\langle\ \rangle$ denotes the unnormalized path integral over string fields in the Euclidean near horizon background geometry weighted by $\exp[-Action]$. Here $\{q_k\}$ stands for the electric charges carried by the black hole, representing the electric fluxes of the U(1) gauge fields $A^{(k)}$'s through AdS_2. The integral $\oint$ runs over the boundary of the infrared regulated AdS_2.

---

[10]In fact the original computation involved a more refined version of the 1PI action, where the complete supersymmetric completion of the curvature squared terms in the 1PI action was included in the computation [103–111, 113–115]. Surprisingly, the result is the same as in (3.54). Nevertheless, there can be additional four derivative corrections to the action which could give additional contribution to the entropy to this order. One expects that a suitable non-renormalization theorem will make these additional contributions vanish, but this has not been proven so far.

Note that near the boundary of AdS_2, the θ-independent solution to the Maxwell's equations has the form:

$$A_r = 0, \quad A_\theta = C_1 + C_2 r, \tag{3.58}$$

where C_1 (chemical potential) represents a normalizable mode and C_2 (electric charge) represents a non-normalizable mode. Hence the path integral (3.57) must be carried out keeping C_2 (charge) fixed and integrating over C_1 (chemical potential).[11] Another way to motivate this is the following: in AdS_2, if we try to add charge/mass, it will destroy the asymptotic boundary conditions as it is a two dimensional spacetime. With this new rule, the first order variation of the action will contain a boundary term besides the terms proportional to the equations of motion. This boundary term must be cancelled by some other term in order to have a well-defined path integral. The boundary term $\exp[-iq_k \oint d\theta\, A_\theta^{(k)}]$ precisely serves this purpose.

4. Now, by AdS_2/CFT_1 correspondence, string theory on $AdS_2 \times K$ must be dual to a one dimensional conformal field theory, which we shall call CFT_1, living on the boundary of AdS_2. Furthermore, we must have

$$Z_{AdS_2} = Z_{CFT_1} = Tr(e^{-LH}), \tag{3.59}$$

where H is the Hamltonian of CFT_1 and L is the length of the boundary circle of the infrared regulated AdS_2. The standard rule of AdS/CFT correspondence also gives us some insight into how to identify the CFT_1, – it must be given by the infrared limit of the quantum mechanics that describes the black hole microstates.[12] Now in all known examples, including the ones discussed in §2, the quantum mechanics describing the dynamics of the microscopic system has a finite gap that separates the ground state from the first excited state.[13] Thus in the infrared limit $(L \to \infty)$, only the ground state of this quantum mechanics (in a fixed charge sector) survive, and CFT_1 will consist of a finite number d_0 of degenerate ground states of some energy E_0. This gives, from (3.59),

$$Z_{AdS_2} = d_0\, e^{-L E_0}. \tag{3.60}$$

---

[11]This is different from the standard rules in higher dimensional space-time where the asymptotic value of the gauge field is held fixed.

[12]We emphasize that here, since the boundary theory is on a circle, its partition function can be given a Hilbert space interpretation. This is not possible in higher dimensional AdS_{d+1} spaces where the boundary theory lives on S^d.

[13]Even though the dynamics was described by a two dimensional CFT, the CFT was compactified on a circle of finite size, and hence had a gap in its spectrum.

This suggests that we define d_{hor} to be the finite part of Z_{AdS_2}, defined by expressing Z_{AdS_2} as

$$Z_{AdS_2} = e^{CL + O(L^{-1})} \times d_{hor}, \quad \text{as } L \to \infty. \tag{3.61}$$

Here C is a constant. The above definition of d_{hor} will be called the *quantum entropy function*.

5. Finally we note that, since the AdS_2 path integral is evaluated by keeping fixed the asymptotic value of the electric field (and hence the electric charge for a given action), the AdS_2 path integral computes the entropy in the microcanonical ensemble where all the charges are fixed.

One of the consistency tests this proposal must satisfy is that, in the classical limit, it should reproduce the exponential of the Wald entropy. This can be seen as follows: In the classical limit,

$$
\begin{aligned}
Z_{AdS_2} &= \exp[-Action - iq_k \oint d\theta \, A_\theta^{(k)}]\Big|_{classical} \\
&= \exp\left[\int dr d\theta \left\{ \sqrt{\det g_{AdS_2}} \mathcal{L}_{AdS_2} - iq_k F_{r\theta}^{(k)} \right\} \right],
\end{aligned} \tag{3.62}
$$

where g_{AdS_2} is the metric on AdS_2, and $\mathcal{L}_{AdS_2}$ is the two dimensional Lagrangian density obtained after dimensional reduction on K and is evaluated on the near horizon geometry.[14] Taking the infrared cut-off to be $\eta \le \eta_0$ for simplicity, using the Euclidean version of the near horizon background given in (3.48), and evaluating the r, θ integral, we get,

$$
\begin{aligned}
Z_{AdS_2} &= \exp\left[-2\pi \left(q_i e_i - \sqrt{\det g_{AdS_2}} \mathcal{L}_{AdS_2} \right) (\cosh \eta_0 - 1) \right] \\
&= \exp\left[2\pi \left(q_i e_i - \sqrt{\det g_{AdS_2}} \mathcal{L}_{AdS_2} \right) + CL + \mathcal{O}(L^{-1}) \right] \\
&= \exp\left[S_{wald} + CL + \mathcal{O}(L^{-1}) \right],
\end{aligned} \tag{3.63}
$$

where

$$L = \sqrt{v} \sinh \eta_0 \Rightarrow \cosh \eta_0 = L/\sqrt{v} + \mathcal{O}(L^{-1}). \tag{3.64}$$

The constant C can receive additional corrections from boundary terms in the action which we have ignored. The important point is that these boundary terms do not affect the value of the finite part, and hence d_{hor} is well defined.

---

[14]Note that the Euclidean action is given by $-\int dr d\theta \sqrt{g_{AdS_2}} \mathcal{L}$, where $\mathcal{L}$ is the analytic continuation of the Lagrangian density for Lorentzian signature.

This establishes that $d_{hor} = \exp[S_{wald}]$ in the classical limit.

We conclude this section with two comments:

- By choosing the boundary terms appropriately, we could cancel the constant C and reinterprete the full partition function Z_{AdS_2} as d_{hor}. In the dual CFT_1, this corresponds to shifting the ground state energy by adding appropriate counterterms.

- Our interpretation of the AdS_2 partition function as the degeneracy associated with the horizon is based on representing Euclidean AdS_2 as a disk with a single boundary. If instead we represent it as a strip with two boundaries, with the help of the standard conformal transformation $\tan \frac{w}{2} = \frac{z-1}{z+1}$ mapping the unit disk in the $z = \rho\, e^{i\theta}$ plane to a strip in the w plane, then we have two copies of CFT_1 living on the two boundaries of the strip, each with degeneracy d_{hor}. Standard argument [118] shows that the Hartle-Hawking state of this system will represent the maximally entangled state between these two copies of the CFT_1, and as a result, d_{hor} can be reinterpreted as the entanglement entropy between the two boundaries in this state. This has been verified explicitly in [119] in the classical limit.

3.6   Hair contribution

In general, the macroscopic degeneracy, denoted by d_{macro}, can have two kinds of contributions [120, 121]:

1. From the the degrees of freedom living on the horizon.

2. From the degrees of freedom living outside the horizon (hair).[15]

Denoting the degeneracy associated with the horizon degrees of freedom by d_{hor} and those associated with the hair degrees of freedom by d_{hair}, we can write down a general formula for d_{macro}:

$$d_{macro}(Q) = \sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} d_{hair}(\vec{Q}_{hair}; \{\vec{Q}_i\}). \qquad (3.65)$$

---

[15]For example, the fermion zero modes associated with the broken supersymmetry generators are always part of the hair modes, since the effect of supersymmetry-breaking by the classical black hole solution can be felt outside the horizon of the black hole.

The nth term in the sum represents the contribution from an n-centered black hole, $\vec{Q}_i$'s denote the charges carried by the i-th center[16] and $\vec{Q}_{hair}$ denotes the charges carried by the hair modes. In principle, d_{hair} can be calculated by explicitly identifying and quantizing the hair modes. On the other hand, $d_{hor}(Q_i)$ for each center can be computed using the quantum entropy function formalism described in §3.5.

3.7 Degeneracy to index

As discussed before, on the microscopic side we usually compute an index. On the other hand, d_{hor} computes degeneracy. More generally, eq.(3.65) gives us a general formula for computing the degeneracy on the macroscopic side. Thus this cannot be directly compared with the B_6 index computed on the microscopic side.

We shall now describe a strategy for using d_{hor} to compute the index on the macroscopic side [117,122]. We shall illustrate this for the helicity trace B_n for four dimensional black holes, but it can be generalized to five dimensional black holes as well [123]. For a black hole that breaks $2k$ supercharges, we had defined

$$B_k = \frac{1}{k!} Tr\{(-1)^{2h}(2h)^k\}, \tag{3.66}$$

where h is the third component of angular momentum in the rest frame. Since the total contribution to h can be regarded as a sum of the contributions from the horizon and the hair degrees of freedom, we can express B_k as

$$B_k = \frac{1}{k!} Tr\{(-1)^{2h_{hor}+2h_{hair}}(2h_{hor} + 2h_{hair})^k\}, \tag{3.67}$$

where h_{hor} and h_{hair} denote the contribution to h from the horizon and the hair degrees of freedom respectively.

Now, typically all the fermion zero modes associated with the broken supersymmetries are hair degrees of freedom, since the effect of supersymmetry-breaking by the black hole solution can be felt outside the horizon. Thus the hair modes contain $2k$ fermion zero modes, and in order that the trace over these zero modes do not make the whole trace vanish, we need an insertion of $(2h_{hair})^k$ into the trace. In other words, if we expand the $(2h_{hor} + 2h_{hair})^k$ factor in a binomial expansion, then only the $(2h_{hair})^k$ term will contribute. This gives

$$B_k = \frac{1}{k!} Tr\{(-1)^{2h_{hor}+2h_{hair}}(2h_{hair})^k\} = \sum B_{0;hor} \, B_{k;hair} . \tag{3.68}$$

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[16]In this section we shall use $\vec{Q}$ to denote all the electric and all the magnetic charges.

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This can be expanded in the spirit of (3.65) as

$$B_{k;macro}(Q) = \sum_n \sum_{\substack{\{\vec{Q}_i\}, \vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n B_{0;hor}(\vec{Q}_i) \right\} B_{k;hair}(\vec{Q}_{hair}; \{\vec{Q}_i\}). \tag{3.69}$$

A further simplification follows from the fact that in four dimensions, only the $h_{hor} = 0$ black holes are supersymmetric. This is of course known to be true for a classical black hole, but more generally it follows from the fact that unbroken supersymmetries, together with the $SL(2, R)$ isometry of the near horizon geometry, generate the full $SU(1, 1|2)$ supergroup which includes $SU(2)$ as a symmetry group. This implies a spherically symmetric horizon, and hence zero angular momentum since the partition function on AdS_2 computes the entropy in a fixed angular momentum sector (microcanonical ensemble). Thus $B_{0;hor} = Tr_{hor}(1) = d_{hor}$, and we can express (3.69) as

$$B_{k;macro}(Q) = \sum_n \sum_{\substack{\{\vec{Q}_i\}, \vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} B_{k;hair}(\vec{Q}_{hair}; \{\vec{Q}_i\}). \tag{3.70}$$

Most of our analysis involves 1/4-BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories in $D = 4$ which preserves 4 out of 16 supersymmetries, i.e., such a black hole configuration breaks 12 supersymmetries. Thus the relevant helicity trace index is B_6. In these theories, the contribution from multi-centered black holes is known to be exponentially suppressed [26,38,48]. Furthermore, for single-centered black holes, often the only hair modes are the fermion zero modes. In this case, $Q_{hair} = 0$. Furthermore, since for each pair of fermion zero modes $Tr\{(-1)^F(2h)\} = i$, we have $B_{6;hair} = i^6 = -1$. Thus

$$B_6(Q) = -d_{hor}(Q), \tag{3.71}$$

up to exponentially suppressed contribution from multi-centered black holes. This explains how we can compare the helicity trace index computed in the microscopic theory with d_{hor} computed in the macroscopic theory. Note that since $d_{hor}(Q) > 0$, we get $B_6 < 0$. This agrees with the explicit microscopic results stated above (2.19) and below (2.33).

4 Applications of quantum entropy function

Eq.(3.54) shows how Wald's formula applied to 1PI action can be used to calculate some of the subleading corrections to the black hole entropy, and reproduce the results known from

microscopic computation. Since quantum entropy function reduces to the exponential of Wald entropy in the classical limit, we expect that as long as the quantum corrections generate a local contribution to the 1PI action, Wald's formula applied to 1PI action and quantum entropy function will give the same results. In this section, we shall describe how quantum entropy function can be used to compute some other corrections to the entropy which could not be calculated by direct use of Wald's formula.

4.1 Computation of twisted index

Suppose we have a $\mathbb{Z}_N$ symmetry generated by g that commutes with all the supersymmetries of an $\mathcal{N} = 4$ supersymmetric string theory. We can then define a twisted index:

$$B_6^g = \frac{1}{6!} Tr\{(-1)^{2h}(2h)^6 g\}. \tag{4.72}$$

In §2.6 and §2.8, we described the results for such indices in a wide variety of $\mathcal{N} = 4$ supersymmetric string theories. We shall now describe how to compute them from the macroscopic side.

We proceed as in §3.7. After separating out the contribution from the hair degrees of freedom, we see that the relevant quantity associated with the horizon is

$$Tr_{hor}\{(-1)^{2h_{hor}}g\} = Tr_{hor}(g), \tag{4.73}$$

since $h_{hor} = 0$ for a supersymmetric black hole. By following the logic of AdS/CFT correspondence, we find that d_{hor} is now given by the finite part of a twisted partition function

$$Z_g = \left\langle \exp[-iq_k \oint d\theta\, A_\theta^{(k)}] \right\rangle_g, \tag{4.74}$$

where the subscript g denotes that in carrying out the path integral, we are instructed to integrate over field configurations with a g-twisted boundary condition on the fields under $\theta \to \theta + 2\pi$. Other than this, the asymptotic boundary conditions must be identical to that of the attractor geometry since the charges have not changed.

From the Euclidean AdS_2 metric given in (3.55), we find that the circle at infinity, parametrized by θ, is contractible at the origin $r = 1$. Thus a g-twist under $\theta \to \theta + 2\pi$ is not admissible. Hence we conclude that the $AdS_2 \times S^2$ geometry is not a valid saddle point of the path integral. This however is not the end of the story, since according to the rules of quantum gravity we must sum over all geometries and field configurations keeping fixed the asymptotic boundary

conditions. Thus we should investigate if there are other saddle points which could contribute to the path integral. To find out possible candidates, we must keep in mind the following constraints:

1. It must have the same asymptotic geometry as the $AdS_2 \times S^2$ geometry.

2. It must have a g-twist under $\theta \to \theta + 2\pi$.

3. It must preserve sufficient amount of supersymmetries so that integration over the fermion zero modes do not make the integral vanish [124, 125].

There are indeed such saddle points in the path integral, constructed as follows [51]:

1. Take the original near horizon geometry of the black hole.

2. Take a $\mathbb{Z}_N$ orbifold of this background with $\mathbb{Z}_N$ generated by the simultaneous action of

   (a) $2\pi/N$ rotation in AdS_2,

   (b) $2\pi/N$ rotation in S^2 (needed for preserving SUSY), and

   (c) g.

To see that this has the same asymptotic geometry as the attractor geometry, we make a rescaling

$$\theta \to \theta/N, \qquad r \to N\,r\,. \tag{4.75}$$

After this rescaling, the metric takes the form:

$$ds^2 = v\left((r^2 - N^{-2})d\theta^2 + \frac{dr^2}{r^2 - N^{-2}} \right)\,, \tag{4.76}$$

with the orbifold action given by:

$$\theta \to \theta + 2\pi\,, \qquad \phi \to \phi + 2\pi/N\,, \qquad g\,. \tag{4.77}$$

For large r, the metric approaches the AdS_2 metric.[17] The g transformation provides us with the correct boundary condition under $\theta \to \theta + 2\pi$. The shift along the ϕ-direction can be

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[17]In contrast, we note that for two dimensional flat spacetime, orbifolding not only introduces a conical singularity but also changes the asymptotic spacetime.

regarded as a Wilson line, and hence is an allowed fluctuation in AdS_2. In other words, by a coordinate change $\phi \rightarrow \phi + \theta/N$, we can remove the shift in ϕ, but this will generate a constant $g_{\theta\phi}$ at the boundary, which describes a normalizable mode and hence is an allowed fluctuation.

The classical action associated with this orbifold can be obtained by dividing the action associated with the parent geometry by N. Thus the classical action associated with this saddle point, after removing the divergent part proportional to the length of the boundary, is S_{wald}/N. As a result, the contribution to the twisted partition function B_g from this saddle point is

$$Z_g^{finite} \sim \exp\left[S_{wald}/N\right] . \tag{4.78}$$

This is exactly what we have found in the microscopic analysis of the twisted index in §2.6 and §2.8.

Note that $\exp\left[S_{wald}/N\right] << \exp\left[S_{wald}\right]$. Thus the $\mathbb{Z}_N$ quantum numbers must be delicately distributed among the microstates of the black hole so that a charge of order unity, averaged over $\exp\left[S_{wald}\right]$ number of states, gives a contribution of order $\exp\left[S_{wald}/N\right]$. In other words, there is a large cancellation going on among terms of order unity to give this result. Nevertheless we see that black holes are able to capture information about this highly sensitive data.

4.2   Logarithmic corrections to the black hole entropy

As already discussed before, the effect of integrating out the massive mode contribution to Z_{AdS_2} can be regarded as a modification of the effective Lagrangian density, and can be accommodated using Wald's formula. However, for calculating the one loop contribution due to the massless modes, we need to compute directly the determinant of the kinetic operator in the $AdS_2 \times S^2$ background.

Let us consider an example where we have a massless scalar field with the standard kinetic term in the near horizon $AdS_2 \times S^2$ background for a spherically symmetric extremal black hole in $D = 4$. All the eigenvalues and eigenfunctions of $\square$ on $AdS_2 \times S^2$ can be found explicitly, which can then be used to compute $det \, \square$, and hence the one loop contribution to Z_{AdS_2}. The result for the contribution to $\ln d_{hor}$ from this massless scalar is of the form[18] [126]:

$$-\frac{1}{180}\ln A . \tag{4.79}$$

---

[18]A different approach to computing logarithmic corrections to extremal black hole entropy can be found in [87].

For black holes in supergravity/superstring theory, the kinetic operator for fluctuations around the near horizon geometry mixes scalars, vectors and tensors. Thus one needs to diagonalize the kinetic operator, find the determinant, and then compute its contribution to Z_{AdS_2} and hence d_{hor}. For the quarter-BPS black holes in $\mathcal{N} = 4$, so far the contribution from the matter multiplet fields has been calculated. The outcome of such calculation is that all contributions cancel [126]. While this does not lead to a complete computation of logarithmic corrections to the extremal black hole entropy, it already gives us some non-trivial information, namely, that the logarithmic correction to the entropy of quarter-BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories cannot depend on the number of matter multiplets in the theory. This agrees with the microscopic result given in (2.32), which has no logarithimic correction for any value of r, – the number of matter multiplets in the theory.

One might tend to conclude that the reason for this cancellation is supersymmetry. However, the above system turns out to be a special case, since the microscopic result (2.38) for 1/8-BPS dyons in $\mathcal{N} = 8$ supersymmetric string theories, which have the same amount of supersymmetry in the near horizon geometry, is known to have logarithmic correction to its entropy [85]. Thus we expect that the cancellation of logarithmic corrections is not merely a consequence of supersymmetry.

5   Other applications

Quantum entropy function has also been used to explain several other features of the microscopic formula. For example, we see from the microscopic formula (2.26) that for charge vectors (Q, P) with $r(Q, P) > 1$, there are additional contributions to the B_6 index whose leading term takes the form $\exp\left(\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}/s\right)$, where s is a factor of r. It turns out that precisely for $r(Q, P) > 1$, the functional integral for Z_{AdS_2} receives extra contribution from saddle points obtained by taking a freely acting $\mathbb{Z}_s$ quotient – for $s|r$ – of the original near horizon geometry. The leading semi-classical contribution from such a saddle point is given by $\exp(S_{wald}/s) = \exp\left(\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}/s\right)$, precisely in agreement with the microscopic results [85, 117].

For $r = 1$, the result for B_6 for large charges takes the form of a sum of the contributions from different poles. The leading asymptotic expansion comes from a specific pole and is given by (2.17). It turns out that the contributions from the other poles have the leading term of the form $\exp\left(\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}/N\right)$, for $N \in \mathbb{Z}$, $N > 1$. On the other hand the contribution to

Z_{AdS_2} receives contribution from, besides the original near horizon geometry, its $\mathbb{Z}_N$ orbifolds which do not change the boundary condition at infinity. The leading semiclassical contribution from these saddle points is given by $\exp\left(\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}/N\right)$, precisely in correspondence with the leading contribution from the subleading poles in the microscopic formula [46, 127].

6 Discussion

All these results provide us with the 'experimental verification' of the theory of extremal black holes, based on Wald's formula and AdS_2/CFT_1 correspondence. The results described here show that quantum gravity in the near horizon geometry contains detailed information about not only the total number of microstates but also finer details (*e.g.* the $\mathbb{Z}_N$ quantum numbers carried by the microstates). Thus, at least for extremal black holes, there seems to be an exact duality between

$$Gravity\ \ description\ \ \Leftrightarrow\ \ Microscopic\ \ description\,. \tag{6.80}$$

The gravity description contains as much information as the microscopic description, but in a quite different way.

It is clear from our discussions that whereas the α'-corrections are well-understood through Wald's formalism, we need to understand the g_s corrections better. The quantum entropy function formalism provides us with a tool for investigations in that direction. Eventually we hope to reproduce the complete asymptotic expansion of the microscopic result for $\ln|B_6|$ from the string theory path integral over AdS_2. Our basic tool is the localization of the path integral to a finite dimensional subspace using supersymmetry. This has been pursued to some extent in [125]. In this process, we hope to learn not only about black holes but also about string theory, *e.g.* the rules for carrying out path integral over string fields.

Another useful direction of study is the generalization of these results to $\mathcal{N} = 2$ supersymmetric string theories. Some attempts at generalizing the microscopic results of §2 in special $\mathcal{N} = 2$ supersymmetric string theories can be found in [128–130].

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