Special Holonomy Metrics and Hitchin's Functionals

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Stable Forms

Stable forms are defined as follows [1]. Let X be a manifold of real dimension n, and $V = TX$. Then, the form $\rho \in \Lambda^p V^*$ is stable if it lies in an open orbit of the (natural) $GL(V)$ action on $\Lambda^p V^*$. In other words, this means that all forms in the neighborhood of ρ are $GL(V)$-equivalent to ρ. This definition is useful because it allows one to define a volume. For example, a symplectic form ω is stable if and only if $\omega^{n/2} \neq 0$.

Here we give another example, for $n = 7$ and $p = 3$:

$$\begin{aligned} \dim GL(V) = n^2 \qquad &= 49 \\ \dim \Lambda^p V^* = \frac{n!}{p!(n-p)!} \qquad &= 35 \\ &\quad 14 = \dim(G_2) \end{aligned} \tag{1}$$

Volume Functionals

Basic Idea: For a stable p-form ρ on a compact manifold X, construct a volume functional,

$$V(\rho) = \int_X \phi(\rho)$$

such that the critical points of $V(\rho)$ define metrics of reduced holonomy, see Figure 1.

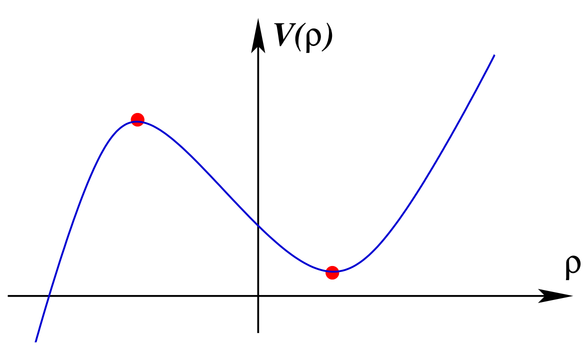

FIG. 1: Critical points of volume functionals define special geometric structures on X.

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<u>$n = 6$ and $p = 4$:</u>

$$V(\sigma) = \int_X |\sigma^3|^{\frac{1}{2}}. \tag{2}$$

<u>$n = 6$ and $p = 3$:</u>

$$V(\rho) = \int_X |\sqrt{-\frac{1}{6} K_a{}^b K_b{}^a}|, \tag{3}$$

where

$$K_a{}^b := \frac{1}{12} \rho_{a_1 a_2 a_3} \rho_{a_4 a_5 a} \epsilon^{a_1 a_2 a_3 a_4 a_5 b}.$$

Hamiltonian Flow

Basic Idea: For a (non-compact) manifold X foliated by a manifold Y, construct a Hamiltonian system, with Hamiltonian $H(x_i, p_i)$, such that the Hamiltonian flow equations

$$\begin{cases} \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \end{cases} \iff \begin{array}{c} \text{Special Holonomy Metric} \\ \text{on } (t_1, t_2) \times Y \end{array} \tag{4}$$

define a reduced holonomy metric on $(t_1, t_2) \times Y$.

Consider, for example, a homogeneous quotient space

$$Y = G/K, \tag{5}$$

where G is some group and $K \subset G$ is a subgroup. Therefore, we can think of X as being foliated by *principal orbits* G/K over a positive real line, $\mathbb{R}_+$, as shown on Figure 2. G/K may collapse into a degenerate orbit:

$$B = G/H \tag{6}$$

where symmetry requires

$$G \supset H \supset K \tag{7}$$

Moreover,

$$H/K = \mathbf{S}^k \quad \Longrightarrow \quad X \ \text{ smooth}$$

also implies

$$X \cong (G/H) \times \mathbb{R}^{k+1} \tag{8}$$

There exists a symplectic structure on the space, $\mathcal{P}$, of G-invariant forms on $Y = G/K$ [2]:

$$\begin{aligned} \mathcal{P} &= \text{ Phase Space} \\ \omega &= \sum dx_i \wedge dp_i \end{aligned} \tag{9}$$

Moreover, there is a canonical construction of a Hamiltonian $H(x_i, p_i)$ for our dynamical system, such that the Hamiltonian flow equations (4) are equivalent to the special holonomy condition [2].

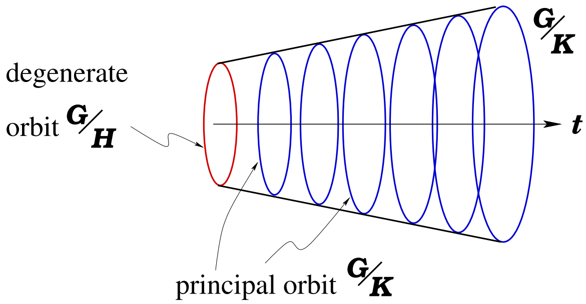

FIG. 2: A non-compact space X can be viewed as a foliation by principal orbits $Y = G/K$. The non-trivial cycle in X correspond to the degenerate orbit G/H, where $G \supset H \supset K$.

An Example

Let us take

$$\begin{array}{ccccc} G & & H & & K \\ \| & & \| & & \| \\ SU(2)^3 & \supset & SU(2)^2 & \supset & SU(2) \end{array} \qquad (10)$$

From (5) it follows

$$Y = SU(2) \times SU(2) \cong \mathbf{S}^3 \times \mathbf{S}^3 \qquad (11)$$

Furthermore, $G/H \cong H/K \cong \mathbf{S}^3$ implies that X is a smooth manifold with topology, *cf.* (8),

$$X \cong \mathbf{S}^3 \times \mathbb{R}^4$$

In order to find a G_2 metric on this manifold, we need to construct the "phase space", $\mathcal{P}$, that is the space of $SU(2)^3$–invariant 3-forms and 4-forms on $Y = G/K$:

$$\mathcal{P} = \Omega^3_G(G/K) \times \Omega^4_G(G/K)$$

It turns out that each of the factors is one-dimensional, generated by a 3-form ρ and by a 4-form σ, respectively,

$$\rho = \sigma_1\sigma_2\sigma_3 - \Sigma_1\Sigma_2\Sigma_3 + x\Big(d(\sigma_1\Sigma_1) + d(\sigma_2\Sigma_2) + d(\sigma_3\Sigma_3)\Big), \qquad (12)$$

$$\sigma = p^{2/5}\Big(\sigma_2\Sigma_2\sigma_3\Sigma_3 + \sigma_3\Sigma_3\sigma_1\Sigma_1 + \sigma_1\Sigma_1\sigma_2\Sigma_2\Big). \qquad (13)$$

where σ_a and Σ_a are left invariant 1-forms:

$$d\sigma_a = -\frac{1}{2}\epsilon_{abc}\sigma_a \wedge \sigma_b, \qquad d\Sigma_a = -\frac{1}{2}\epsilon_{abc}\Sigma_a \wedge \Sigma_b. \qquad (14)$$

Therefore, we have only one "coordinate" x and its conjugate "momentum" p, parametrizing the "phase space" $\mathcal{P} = \Omega^3_{exact}(Y) \times \Omega^4_{exact}(Y)$ of our model. The non-degenerate symplectic structure looks like

$$\omega\left((\rho_1, \sigma_1), (\rho_2, \sigma_2)\right) = \langle \rho_1, \sigma_2 \rangle - \langle \rho_2, \sigma_1 \rangle,$$

where, in general, for $\rho = d\beta \in \Omega^k_{exact}(Y)$ and $\sigma = d\gamma \in \Omega^{n-k}_{exact}(Y)$ one has a nondegenerate pairing

$$\langle \rho, \sigma \rangle = \int_Y d\beta \wedge \gamma = (-1)^k \int_Y \beta \wedge d\gamma. \qquad (15)$$

Once we have the phase space $\mathcal{P}$, it remains to define the Hamiltonian [2]:

$$H = 2V(\sigma) - V(\rho), \qquad (16)$$

where $V(\rho)$ and $V(\sigma)$ are the volume functionals (3) and (2), respectively. Evaluating (16) for the G-invariant forms (12) and (13) we obtain the Hamiltonian flow equations:

$$\begin{cases} \dot{p} &= x(x-1)^2 \\ \dot{x} &= p^2 \end{cases}$$

The solution for $x(t)$ and $p(t)$ determines the evolution of the forms ρ and σ which, in turn, define the associative three-form on the 7-manifold $Y \times (t_1, t_2)$,

$$\Phi = dt \wedge \omega + \rho, \qquad (17)$$

where ω is a 2-form on Y, such that $\sigma = \omega^2/2$. The associative form Φ is automatically closed and co-closed,

$$d\Phi = 0 \qquad d * \Phi = 0 \qquad (18)$$

Therefore, it defines a G_2 holonomy metric on X, *viz.* the G_2 metric on the spin bundle over $\mathbf{S}^3$, originally found in [3, 4]. More examples can be found in [5–7].

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