Deforming baryons into confining strings

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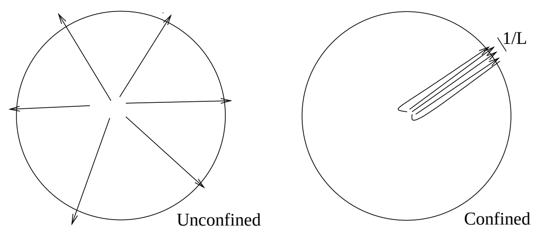
1 Confining strings: expectations

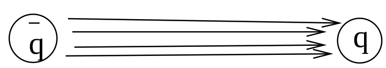
Many Yang-Mills theories are strongly coupled at low energy

$$E \ll \Lambda \quad \Rightarrow \quad g_{YM} \gg 1\,. \tag{1}$$

There are various interesting phenomena associated with the strong coupling regime. One is 'confinement'. This means confinement of chromoelectric flux

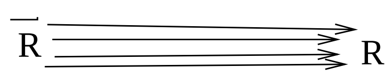


Gauss's law implies that $V \sim 1/r$ for unconfined theories but $V \sim r$ for confined theories. In particular, there is a strong attractive force between sources due to the flux tube or confining string between them:



These are important objects in the low-energy regime. What are their charges? The claim is that confining strings are charged under the centre of the gauge group C_G. For $G = SU(N)$, this implies that the confining strings are charged under $\mathbb{Z}_{\mathbb{N}}$.

Consider the massless gluons in the theory. Gluon number is not conserved because there are interactions in the Lagrangian: A^3 and A^4. The gluons transform in the adjoint of the gauge group. Now consider a confining string between sources in a general representation R of $SU(N)$:



This can be dynamically transformed into another representation by combining the sources with gluons, which may be created at negligible energy cost $R \to R \otimes A$. Dynamically the string will minimise its energy in some representation that can be obtained from R by tensoring with the adjoint representation. Thus one should not associate a given representation with the string at all but rather the equivalence class of representations under adjoint tensoring.

Such an equivalence class is specified by its N-ality. Representations of $SU(N)$ may be specified by tensors that transform as follows under $U^a{}_b \in SU(N)$:

$$R^{a_1\cdots a_n}_{b_1\cdots b_m} \to U^{a_1}{}_{\alpha_1}\cdots U^{a_n}_{\alpha_n} R^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m} U^{\dagger\ \beta_1}_{b_1}\cdots U^{\dagger\ \beta_m}_{b_m}\,. \tag{2}$$

The N-ality of such a representation is $n-m$ mod N. The adjoint representation has one upper and one lower index, so tensoring with the adjoint does not change the N-ality. Therefore N-ality is a well-defined label for confining strings.

The centre of the group are matrices of the form $U^a{}_b = e^{i2\pi k/N}\delta^a{}_b$ for $k = 0\dots N-1$. Under the centre of the group, the tensors transform as

$$R^{a_1\cdots a_n}_{b_1\cdots b_m} \to e^{i2\pi k(n-m)/N} R^{a_1\cdots a_n}_{b_1\cdots b_m}. \tag{3}$$

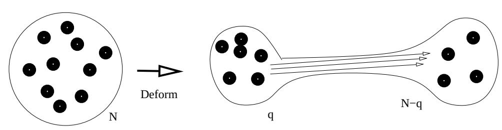
Thus, the charge under the centre of the group is the N-ality. Confining strings are labelled by the N-ality of representations and therefore by their charge under the centre of the gauge group.

A natural question is: what is the tension, T_k, of the k-th confining string when the sources are infinitely separated? It is clear from symmetry that $T_k = T_{N-k}$, but otherwise this is a nontrivial question about the strongly coupled dynamics of the theory. This question can be addressed in various ways via dualities for $\mathcal{N}=1$ SYM

$$\mathcal{L} = \frac{1}{2g}\mathrm{tr}\left[F_{ab}F^{ab} + i\bar{\lambda}\not{D}\lambda\right]\,. \tag{4}$$

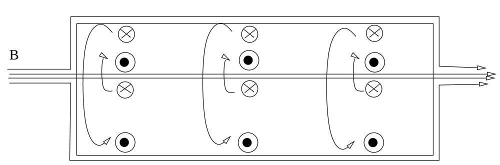
Note that the fermion is also in the adjoint of the gauge group, with N-ality zero, so pair creation of fermions will not be able to break confining strings.

After reviewing the results for infinite strings, we will obtain an expression for the energy of finite confining strings, $E_q(r)$, where the sources are separated by a distance r. This will be achieved by finding solutions representing the deformation of baryon vertices into confining strings by pulling q quarks away from the remaining $N-q$



2 Digression: analogy with superconductors

Type II superconductors can expel magnetic fields by creating Cooper pair currents. These currents reduce the energy of the superconductor by generating an opposing magnetic field to cancel the external field. Small vortices of current, Abrikosov vortices, can confine the magnetic flux into tubes. The dots and crosses in the picture below indicate an azimuthal current.



The vortices carry a topological $\mathbb{Z}$ charge. This is because the Cooper pairs are described by a scalar field ϕ which at infinity transverse to the flux tube, S^1_∞, must take values in a $U(1)$ space of vacua. Therefore the vortex corresponds to a map $S^1 \to U(1)$. Such maps are topologically classified by $\pi_1(U(1)) = \mathbb{Z}$.

There are non-Abelian versions of these magnetic flux strings. In this case there may be non-Abelian scalars Φ which often have potentials constructed from commutators $[\Phi, \Phi]$ which break $SU(N)$ to its centre $\mathbb{Z}_\mathbb{N}$. The topological charge of the vortices is now $\pi_1(SU(N)/\mathbb{Z}_\mathbb{N}) = \mathbb{Z}_\mathbb{N}$. The similarity of these magnetic strings with the confining electric strings suggests that confining strings can be studied using electromagnetic duality.

3 Two ways of engineering $\mathcal{N} = 1$ SYM

Start with a Calabi-Yau spacetime, $\mathcal{M}^{1,3} \times \mathrm{CY}$, preserving $\mathcal{N} = 2$ supersymmetry and halve the supersymmetry by adding a wrapped brane.

- IIA: Wrap N D6-branes on the supersymmetric (calibrated) S^3 of the deformed conifold T^*S^3.
- IIB: Wrap N D5-branes on the supersymmetric S^2 of the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$.

In both these setups, the theory on the four noncompact directions of the brane at low energies is $\mathcal{N} = 1$ SYM. There are no Killing spinors on S^3 and S^2 with the standard spin connection. Therefore in order to preserve supersymmetry, the theory in the compact directions must be twisted, so that the fields are charged under a combination of the $SO(3)$ or $SO(2)$ structure group of the wrapped cycle and the $SO(3)$ or $SO(4)$ group of the transverse directions to the D-brane. The combination is such that the connections cancel, so the Killing spinor equation is

$$(\partial + \omega + A)\eta = \partial\eta = 0\,, \tag{5}$$

which does have solutions.

To decouple[1] the brane from bulk, one takes the near horizon limit. In this limit the geometry undergoes a geometric (conifold) transition. We obtain two backgrounds with RR flux instead of branes.

- IIA: Resolved conifold with $\int_{S^2} G_2^{RR} = N$. This background may be lifted to M theory to give a manifold with G_2 holonomy.
- IIB: Deformed conifold with $\int_{S^3} G_3^{RR} = N$.

Explicit metrics for these backgrounds are known, but we are only interested in the low energy theory. The far infrared of the field theory is described by the $r \to 0$ region of the backgrounds, where r is the noncompact radial direction of the conifolds. In this limit, the backgrounds become $\mathcal{M}^{1,3} \times S^2$ and $\mathcal{M}^{1,3} \times S^3$ in the IIA and IIB cases respectively, with N units of RR flux through the spheres:

- IIA infrared background

$$\begin{aligned} ds^2_{\mathrm{IIA}} &= e^{2\Phi_0}\left(dx^2_{1,3} + N^2\tfrac{1}{4}\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \\ C_1^{RR} &= N\tfrac{1}{2}\cos\theta d\phi\,. \end{aligned} \tag{6}$$

[1] The decoupling is not complete for these nonconformal theories. The theory is contaminated by both KK modes and by gravitational (IIA) or little string theory (IIB) modes at the same scale as the scale dynamically generated by the theory.

The RR two-form flux is

$$G_2^{RR} = dC_1^{RR} = -N\tfrac{1}{2}\sin\theta\, d\theta \wedge d\phi = -N\tfrac{1}{2}\mathrm{vol}_{S^2}\,. \tag{7}$$

- IIB infrared background

$$\begin{aligned} ds^2_{\mathrm{IIB}} &= e^{\Phi_0}\left[dx^2_{1,3} + N\left(d\psi^2 + \sin^2\psi[d\theta^2 + \sin^2\theta\, d\phi^2]\right)\right] \\ C_2^{RR} &= -N\left(\psi - \frac{1}{2}\sin 2\psi\right)\sin\theta\, d\theta\wedge d\phi\,, \end{aligned} \tag{8}$$

The RR field strength is thus

$$G_3^{RR} = dC_2^{RR} = -2N\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi = -2N\mathrm{vol}_{S^3}\,. \tag{9}$$

The ranges of the angles are $0 \le \psi \le \pi$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$. We are working in the string frame and Φ_0 is the value of the dilaton at the origin.

We will study the deformation of baryons into confining strings in these backgrounds.

4 Baryons (Witten)

Suppose we have an S^p with flux $\int_S^p G_p^{RR} = N$. Can we wrap a Dp-brane on this cycle?

The relevant part of the Dirac-Born-Infeld action for a probe Dp-brane is

$$S_{\mathrm{DBI}} = \int d^{p+1}\xi\mathcal{L} = -T_p\int d^{p+1}\xi\, e^{-\Phi}\sqrt{-\det({}^\star G + \mathcal{F})} + T_p\int \mathcal{F}\wedge^\star C_{p-1}^{RR}\,, \tag{10}$$

where as usual $T_p = 1/[(2\pi)^p]$ and here $\mathcal{F} = 2\pi F$, the field strength of the worldvolume gauge field, A. We use ${}^\star G$ and ${}^\star C_{p-1}^{RR}$ to denote the pullback onto the worldvolume of the metric and the RR potential.

If we try to wrap the D-brane on the cycle, then the Wess-Zumino term in the brane action becomes a source term for the worldvolulme gauge field

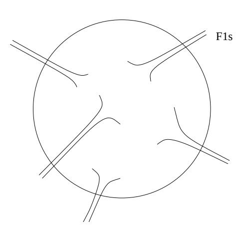
$$T_p\int_{S^p\times\mathbb{R}}\mathcal{F}\wedge^\star C_{p-1}^{RR} = -T_p\int_{S^p\times\mathbb{R}} 2\pi A\wedge^\star G_p^{RR} = N\int_{\mathbb{R}} A\,. \tag{11}$$

However, the compact spatial section cannot carry a total charge. This follows from the equations of motion for F which will be off the form

$$d\star\tilde{F} = \star j = N\mathrm{vol}S^p\,. \tag{12}$$

Although this equation is fine locally, globally it does not make sense because the left hand side is exact but the right hand side is not. We use $\tilde{F}$ to denote some function of F following from the DBI action.

Therefore the source term (11) must be cancelled by fundamental strings ending on the brane. The string contribution to the action is $\int_{\mathbb{R}} A$ and therefore we will have N fundamental strings ending on the Dp-brane.

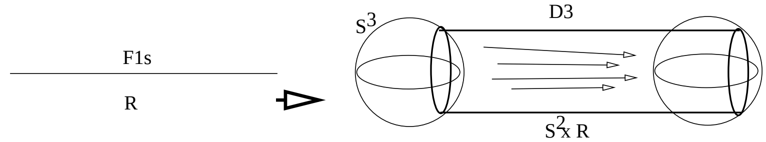


Thus the Dp-brane can only wrap the sphere if it has N fundamental strings ending on it. Such fundamental strings, which go off to infinity, have the gauge theory interpretation of external quark sources. A state of N quarks is known to exist in gauge theory and is called a baryon.

In the IIA background, baryons are D2-branes wrapping the S^2, in the IIB background, they are D3-branes wrapping the S^3.

5 Confining strings in IIB (Klebanov-Herzog)

Consider infinite F1 strings in Minkowski space. These are at a point in the S^3. The Myers effect implies that the strings are blown up into a D3 brane wrapping an S^2 at each point. This is a version of the dielectric effect, where charged matter aligns itself to create a dipole moment that reduces the external electric field.



The S^2 that the branes wrap is at some polar angle ψ_0 in the S^3. The solution for the D3-branes with worldvolume coordinates $(\xi_0, \xi_1, \xi_2, \xi_3)$ is

$$\begin{aligned} t &= \xi_0\,, \quad x = \xi_1\,, \quad \psi = \psi_0\,, \quad (\theta, \phi) = (\xi_2, \xi_3)\,, \\ F &= q d\xi_0 \wedge d\xi_1\,. \end{aligned} \tag{13}$$

The electric flux on the D-brane, F, interacts with the background RR field through the Wess-Zumino term. This interaction supports the D-brane from collapsing due to its tension.

What is the value of ψ_0? We can calculate the energy per unit length of the D-brane configuration

$$T[\psi_0] \propto \left[\sin^4\psi_0 + \left(\psi_0 - \frac{\sin(2\psi_0)}{2} - \frac{\pi q}{N}\right)^{1/2}\right]^{1/2}. \tag{14}$$

The energy is minimised by

$$\psi_0 = \frac{\pi q}{N}\,. \tag{15}$$

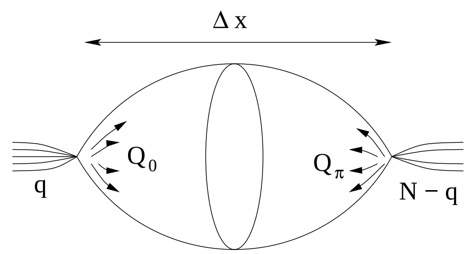
The tension of the resulting minimum-energy D3-brane is

$$T_q = \frac{e^{\Phi_0} N}{2\pi^2} \sin\frac{\pi q}{N}\,. \tag{16}$$

Which is a very clean expression for the confining string tension. It has the required symmetry $q \to N - q$. It is also very similar to the expression for the energy of domain walls of the theory. Unlike the domain walls, however, the confining strings are not BPS objects.

6 Finite length confining strings in IIB

The idea is to find D-brane solutions describing confining strings at finite length. This can be thought of as deformed baryons, in which q of the external quarks are pulled apart in spacetime from the remaining $N - q$.



To find the solutions, generalise the ansatz of the previous section so that the wrapping angle and the flux depend on position in spacetime

$$\psi = \psi(x)\,, \qquad\qquad F = k(x) dt \wedge dx\,. \tag{17}$$

The nontrivial fact is that one can solve the full second order DBI equations of motion for this ansatz. One finds a two parameter family of solutions of the form

$$\begin{aligned} k(\psi) &= -\frac{q\pi}{N} + \psi - \sin\psi\cos\psi\,, \\ \frac{d\psi}{dx} &= F_{q,C}(\psi)\,, \end{aligned} \tag{18}$$

where $F_{q,C}(\psi)$ is an explicitly given function of ψ and two constants q and C. The first of these parameters corresponds to the fraction of quarks separated and the second is related to the distance of separation.

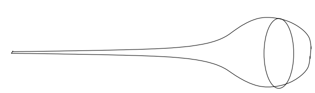
From the solution, one can derive the following expression for the energy as a function of q and C

$$E_q(C) = \frac{N^{3/2}e^{\Phi_0}}{2\pi^2}\int_0^\pi \frac{\sin^2\psi - (\psi - \pi q/N)(2\sin\psi\cos\psi - \psi + \pi q/N)}{\sqrt{-e^{2\Phi_0}/C^4 + \sin^4\psi + (\sin\psi\cos\psi - \psi + \pi q/N)^2}} d\psi. \tag{19}$$

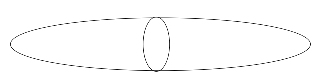
The separation between the quarks is given by

$$\Delta x = \int_0^\pi \frac{d\psi}{F_{q,C}(\psi)} = \int_0^\pi \frac{N^{1/2}e^{\Phi_0} d\psi}{\sqrt{-e^{2\Phi_0} + C^4\sin^4\psi + C^4(\sin\psi\cos\psi - \psi + \pi q/N)^2}}\,. \tag{20}$$

An infinite length limit in (q, C) arises whenever $F_{q,C}(\psi)$ has a zero at some point $\psi_0 \in [0, \pi]$. This can be at one end of the string:



or in the middle



In the former case the energy becomes

$$E_q(\Delta x) = \frac{e^{\Phi_0}}{2\pi} q\Delta x + \cdots \qquad \text{as} \qquad \Delta x \to \infty\,, \tag{21}$$

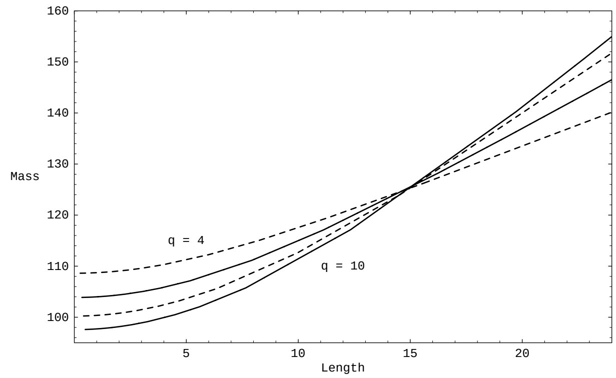
which is just the energy of q F1s, as we should expect.

In the latter case the energy tends to

$$E_q(\Delta x) = \frac{e^{\Phi_0} N}{2\pi^2} \sin \frac{\pi q}{N} \Delta x + \cdots \qquad \text{as} \qquad \Delta x \to \infty\,, \tag{22}$$

recovering the previous result for infinite confining strings.

More interestingly, we can plot the energy as a function of length for different values of q



Mass against length for confining strings with $q = 4, 6, 8, 10$. The background has $N = 30$ and $\Phi_0 = 1$.

As the length goes to infinity, the tension becomes the linear result for infinite confining strings. It is curious that the curves almost intersect at a single point, call it (L_0, M_0), although they do not seem to go through precisely the same point. It seems plausible that the intersection is exact to subleading order at large Δx. Mathematically, this fact requires that the subleading correction to the energy takes the following form

$$E_q(\Delta x) \sim \frac{e^{\Phi_0} N}{2\pi^2 \alpha'} \sin \frac{\pi q}{N} \; [\Delta x - L_0] + M_0 + \cdots, \qquad \text{as} \qquad \Delta x \to \infty\,. \tag{23}$$

This is an interesting result that completely determines the q dependence of the subleading term. It seems difficult to derive (23) directly from the integral for the mass.

The other limit, $\Delta x \to 0$, should not be trusted within the the DBI approximation because ∂F becomes large.

The main result is the expression for the energy (19) and in particular the subleading term in the expansion at large separation (23). It would be nice to see whether the subleading dependence on q/N is generic in other dualities or not, it is curious that it is precisely such that the expressions for the energy intersect at some separation.

In the IIA/G_2 geometry, the only long length limits are of the type when one end collapses rather than there being a zero of $F_{q,C}(\psi)$ in the interior of the string.