Geometry of Twistor Spaces

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Simons Workshop Lecture, 7/30/04
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1 Twistor Spaces

Twistor spaces are certain complex 3-manifolds which are associated with special conformal Riemannian geometries on 4-manifolds. This correspondence between complex 3-manifolds and real 4-manifolds is called the *Penrose twistor correspondence.*

1.1 Complex Curves and Conformal Geometry

To motivate the construction, let us begin by looking at the much simpler situation that arises in real dimension 2. First of all, we all know that a complex curve (or Riemann surface) is the same thing as an oriented 2-manifold M^2 equipped with a conformal class $[g]$ of Riemannian (i.e. positive-definite) metrics.

If g is any Riemannian metric on M, and if $u : M \to \mathbb{R}^+$ is any smooth positive function, the new metric $g' = ug$ is said to be *conformally equivalent* to g, and we will convey this relationship here by writing $ug \sim g$. An equivalence class of metrics

$$[g] = \{g' \mid g' \sim g\}$$

is called a *conformal structure* on M. Now if an orientation $\circlearrowleft$ of M is specified, then $(M^2, [g], \circlearrowleft)$ is naturally a complex curve. Why is this true? Well, rotation by $+90°$ defines a certain tensor field $J : TM \to TM$ with $J^2 = -1$. Such a tensor field is called an *almost-complex structure*. If M were of higher dimension, we might *not* be able to find coordinate systems in which the components of J are all constant; see section 2.1 below. However, the relevant obstruction always vanishes in real dimension 2; indeed, Hilbert's theorem on the *existence of isothermal coordinates* asserts that we can always find local coordinates on M^2 in which J takes the standard form

$$\frac{\partial}{\partial y} \otimes dx - \frac{\partial}{\partial x} \otimes dy \ .$$

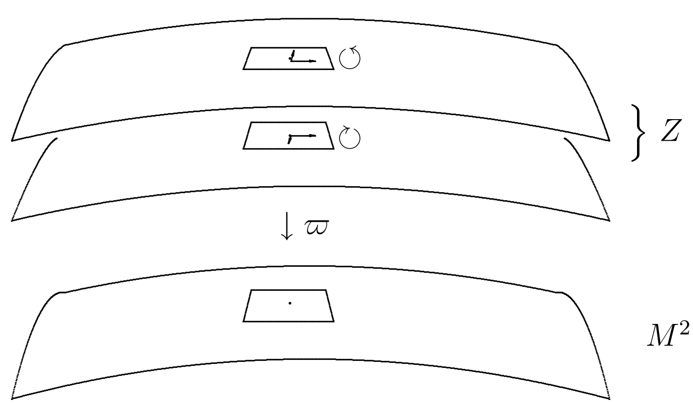
This is of course the same as saying that $z = x + iy$ is a local complex coordinate system with respect to which the given metric g becomes Hermitian.

1.2 Unoriented Surfaces

Now, to motivate the twistor construction, imagine that we are instead given an *unoriented* or even a *non-orientable* surface M^2, together with a conformal structure $[g]$ on M. Can we construct a complex curve from these data in a canonical way? Certainly! The trick is just to consider the bundle $\varpi : Z \to M$ defined by

$$Z = \bigcup_{p \in M} \left\{ j : T_pM \to T_pM \mid j^2 = -1, \quad j^*g = g \right\} .$$

Then Z is a double-covering of M; that is, the inverse image of each point consists of two points j and $j' = -j$. Indeed, the 2-manifold Z can be identified with the set of local orientations of our surface M, and there are two such orientations at each point:



Since the tangent space of Z is naturally identified with the tangent space of M by the derivative of ϖ, there is a natural almost-complex structure J on Z whose value at j is just $\varpi^* j$.

Now (Z, J) becomes a (possibly disconnected) complex curve which is naturally associated with $(M, [g])$. The natural map

$$\sigma : Z \to Z$$

gotten by interchanging the sheets of $\varpi : Z \to M$ satisfies

$$\sigma^* J = -J$$

and

$$\sigma^2 = \text{identity},$$

and so is an *anti-holomorphic involution* of the Riemann surface (Z, J).

The moral of this little parable is that it is a good idea to consider all the bundle of almost-complex structures compatible with a given metric. In the next section we will see where this leads in dimension 4.

1.3 Dimension Four

Now let us instead consider an oriented 4-manifold M^4, equipped with a conformal class $[g]$. For reasons of technical convenience, let us also temporarily fix some particular metric $g \in [g]$. We then define a bundle $\varpi : Z \to M$ define by setting

$$Z = \bigcup_{p \in M} \left\{ j : T_p Z \to T_p Z \mid j^* g = g,\ j^2 = -1,\ j \text{ orientation compatible} \right\}.$$

Here the notion of *orientation compatibility* is liable to cause some confusion. An almost complex structure on a 4-dimensional oriented vector space is called *orientation compatible* if there is an oriented basis of the form (e_1, je_1, e_3, je_3). Thus the almost-complex structure

$$\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}$$

is compatible with the *standard* orientation of $\mathbb{R}^4$, whereas

$$\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}$$

is instead compatible with the *non-standard* orientation of $\mathbb{R}^4$. Moreover, any metric-compatible almost-complex structure on the vector space $\mathbb{R}^4$ is represented by one of these two matrices relative to a suitable choice of oriented orthonormal basis.

What is the fiber of $\varpi : Z \to M$? To find out, notice that if (M, g) is an oriented Riemannian 4-manifold, the Hodge star operator

$$\star : \Lambda^2 \to \Lambda^2$$

satisfies $\star^2 = 1$, and so yields a decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-,$$

where Λ^+ is the $(+1)$-eigenspace of $\star$, and Λ^- is the (-1)-eigenspace. Both Λ^+ and Λ^- are rank-3 vector bundles over M. Reversing the orientation of M interchanges these two bundles.

Now notice that the matrix

$$\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}$$

corresponds (by index lowering) to the *self-dual* 2-form $dx \wedge dy + dz \wedge dt$ on $\mathbb{R}^4$, whereas the matrix

$$\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}$$

corresponds to the *anti-self-dual* 2-form $dx \wedge dy - dz \wedge dt$. Since $SO(4)$ acts transitively on both the unit sphere in Λ^+ and on the set of orientation-compatible orthogonal complex structures, it follows that the bundle ϖ :

$Z \to M$ can be naturally identified with the bundle $S(\Lambda^+)$ of unit vectors in the rank-3 vector bundle Λ^+. In particular, we see that every fiber of Z is diffeomorphic to S^2. Moreover, every fiber comes with a natural conformal class of metrics and an obvious orientation.

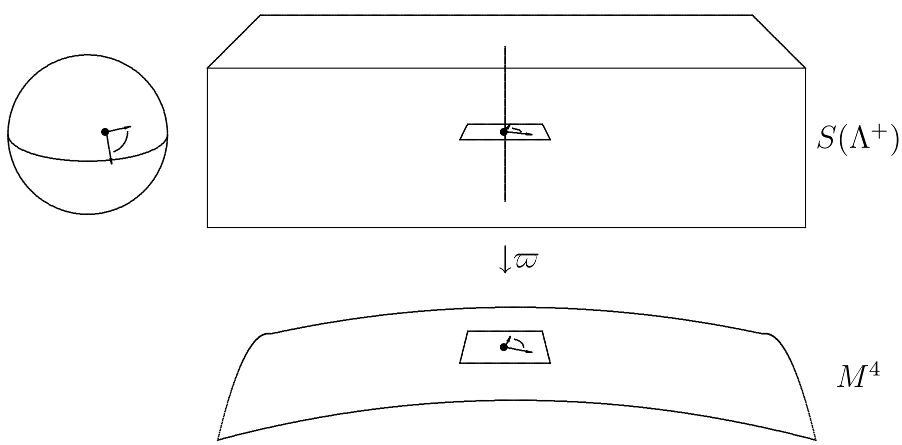
Now, in the spirit of the previous section, we want to make Z into an almost-complex manifold; that is, we want to define a tensor field

$$J : TZ \to TZ, \quad J^2 = -1.$$

Since we have fixed a metric g in our conformal class $[g]$, there seem to be *two* obvious choices of this J. Indeed, since the Levi-Civita connection ∇ of g allows us to parallel transport elements j of Z, there is an associated decomposition $TZ = V \oplus H$ of the tangent space of Z into vertical and horizontal parts. The derivative of ϖ gives us an isomorphism between H and TM, so we may define a bundle endomorphism

$$J_H : H \to H, \quad J_H^2 = -1$$

by letting J_H act at $j \in Z$ by $\varpi^* j$.



On the other hand, V is just the tangent space to the fibers, and we have already indicated that there is a natural way of thinking of every fiber as a Riemann surface once we choose one of the two possible ways of *orienting* it. Thus, we have two natural choices for the fiberwise almost-complex structure, say

$$J_V : V \to V, \quad J_V^2 = -1,$$

and $-J_V$. This gives us two natural-looking choices of almost-complex structure on Z, namely $J_H \oplus J_V$ and $J_H \oplus (-J_V)$. However, exactly one of these is conformally invariant! That is, if we replace our metric $g \in [g]$ with a conformally related metric $\hat{g} = ug$, H will be changed in a manner involving the first derivative of u, and it is hardly obvious that one of the discussed choices of almost-complex structure is conformally invariant. In fact, the conformally invariant choice is

$$J = J_H \oplus J_V$$

where J_V has been chosen to correspond to the "in-pointing" orientation of the fibers of $Z = S(\Lambda^+)$.

This may seem interesting, but it may not suffice to convince you that this constitutes the "right" choice of J. However, if you just consider the example of $M =$ flat Euclidean $\mathbb{R}^4$, so that $Z = \mathbb{R}^4 \times S^2$, you will find that the above (conformally invariant) choice of J turns out to be *integrable*, in the sense that it makes (Z, J) into a complex manifold, whereas the other choice is definitely *not*. To make this precise, a few words on the integrability of complex manifolds are now in order.

2 Integrability of Almost-Complex Structures

2.1 Complex and Almost-Complex Manifolds

Suppose we have a manifold X^{2n} with an almost-complex structure

$$J : TX \to TX, \quad J^2 = -1.$$

Then

$$C \otimes TX = T^{0,1}X \oplus T^{1,0}X,$$

where $T^{1,0}$ is the $-i$ eigenspace of J, and $T^{0,1}$ is the $+i$ eigenspace. Notice that

$$\overline{T^{1,0}} = T^{0,1},$$

so that these two eigenspaces are interchanged by complex conjugation. Moreover,

$$T^{0,1} \cap \overline{T^{0,1}} = 0.$$

One says that (X, J) is *integrable* if there exist coordinates $(x^1, y^1, \ldots, x^n, y^n)$ near any point of X in which J becomes the standard, constant-coefficient almost-complex structure

$$J_0 = \sum_{j=1}^{n} \left(\frac{\partial}{\partial y^j} \otimes dx^j - \frac{\partial}{\partial x^j} \otimes dy^j \right)$$

on $\mathbb{R}^{2n} = \mathbb{C}^n$; when this happens, the various choices of such coordinates systems are interrelated by biholomorphic transformations of $\mathbb{C}^n$, thereby endowing X with a complex-manifold structure. Since J_0 has

$$T^{1,0} = \text{span } \{\frac{\partial}{\partial z^j}\} \quad \text{and} \quad T^{0,1} = \text{span } \{\frac{\partial}{\partial \bar{z}^j}\},$$

a necessary condition for (X, J) to be integrable is that $[T^{0,1}, T^{0,1}] \subset T^{0,1}$, meaning that the Lie bracket of two differentiable sections of $T^{0,1}$ should again be a section of $T^{0,1}$. Remarkably enough, this condition is not only necessary, but also *sufficient*:

Theorem 2.1 (Newlander-Nirenberg) *The almost-complex manifold (X^{2n}, J) is integrable, in the sense that there exist local complex coordinates $(z^1, \ldots, z^n)$ on X in which*

$$T^{0,1} = span \; \{\frac{\partial}{\partial \bar{z}^j}\} \; ,$$

if and only if J satisfies the integrability condition

$$[T^{0,1}, T^{0,1}] \in T^{0,1}.$$

Newlander and Nirenberg's proof actually only works if J is highly differentiable, but a recent result of Hill and Taylor shows that the above theorem holds even if we just assume that J is C^1.

Notice that the statement of the theorem is exactly parallel to that of the Frobenius theorem. Indeed, the Frobenius theorem was originally used to prove the Newlander-Nirenberg theorem in the *real-analytic* case.

It is also worth remarking that the integrability condition can be restated without explicit mention of $T^{1,0}$. Indeed, it turns out to be equivalent to the vanishing of the *Nijenhuis tensor*, which can be expressed, for example, in terms of an arbitrary symmetric connection ∇ on X as

$$N_{ab}^c = (\nabla_d J_{[a}{}^c) J_{b]}{}^d + J_d^c \nabla_{[a} J_{b]}{}^d.$$

2.2 Application to Twistor Spaces

In dimension 4, the Riemann curvature tensor of an oriented manifold splits up into four invariantly defined pieces:

$$\mathcal{R} = s \oplus \overset{\circ}{r} \oplus W^+ \oplus W^-.$$

These pieces are the scalar curvature s, the trace-free Ricci curvature $\overset{\circ}{r}$, the self-dual Weyl tensor W^+, and the anti-self dual W^-. To see, this, remember that the 2-forms on such a manifold decompose as

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-.$$

On the other hand, raising an index of the Riemann curvature tensor to obtain $\mathcal{R}^{ij}{}_{kl}$, we may think of it as a linear map $\mathcal{R} : \Lambda^2 \to \Lambda^2$, and this linear map may be considered as consisting of various blocks:

$$\left(\begin{array}{c|c} \Lambda^+ \to \Lambda^+ & \Lambda^- \to \Lambda^+ \\ \hline \Lambda^+ \to \Lambda^- & \Lambda^- \to \Lambda^- \end{array}\right)$$

This allows us to decompose the curvature tensor as

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W_- + \frac{s}{12} \end{array}\right).$$

Here $W_\pm$ are the trace-free pieces of the appropriate blocks, and the scalar curvature s is understood to act by scalar multiplication; the trace-free Ricci curvature $\overset{\circ}{r} = r - \frac{s}{4}g$ acts on 2-forms by

$$\varphi_{ab} \mapsto \overset{\circ}{r}_{ac}\, \varphi^c{}_b - \overset{\circ}{r}_{bc}\, \varphi^c{}_a.$$

The above decomposition of the curvature tensor is perhaps simplest to understand in terms of spinors. Indeed, in terms of 2-spinor indices, the decomposition of the curvature tensor becomes

$$\begin{aligned} R_{abcd} &= W^+_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + W^-_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ &\quad -\frac{1}{2}\mathring{r}_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} - \frac{1}{2}\mathring{r}_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ &\quad +\frac{s}{12}(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}) \end{aligned}$$

where $W^+_{ABCD} = W^+_{(ABCD)}$, $W^-_{A'B'C'D'} = W^-_{(A'B'C'D')}$, and $\mathring{r}_{ABC'D'} = \mathring{r}_{(AB)(C'D')}$.

A metric is called *anti-self-dual* (or, more briefly, ASD) if W^+ vanishes. This is a conformally invariant condition: if g is ASD, so is $g' = ug$, for any positive function u.

Theorem 2.2 (Penrose, Atiyah-Hitchin-Singer) *Let $(M, [g])$ be an oriented 4-manifold with conformal structure. Then the previously-discussed almost-complex structure J on the 6-manifold $Z = S(\Lambda^+)$ is integrable if and only if $(M, [g])$ has $W^+ = 0$.*

We thus obtain a complex 3-manifold (Z, J) associated to each anti-self-dual 4-manifold $(M, [g])$. This complex manifold is called the *twistor space* of (M, g). It turns out that complex structure J of Z completely encodes the conformal metric of $[g]$, and that one can completely characterize the complex 3-manifolds which arise as twistor spaces.

Conversely, a complex 3-manifold arises by this construction iff it admits a fixed-point-free anti-holomorphic involution $\sigma : Z \to Z$ and a foliation by σ-invariant rational curves $\mathbb{CP}_1$, each of which has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Moreover, the complex manifold (Z, J) and the real structure σ suffice to determine the metric g on M up to conformal rescaling. This aspect of the correspondence is particularly important in practice, as it reduces the problem of constructing ASD metrics to that of constructing certain complex 3-folds.

We now present two simple examples:

Example. Let $M = S^4$, and let $[g]$ be the conformal class of the usual "round" metric. This metric is conformally flat, and hence ASD. The corresponding complex 3-manifold is just $\mathbb{CP}_3$. This may be is a bundle over S^4

via the so-called Hopf map, which arise by thinking of S^4 as the quaternionic projective line $\mathbb{HP}_1$.

$$\begin{array}{rcl} & \mathbb{CP}_3 & = (\mathbb{C}^4 - 0)/\mathbb{C}^\times \\ & \downarrow & \\ S^4 = & \mathbb{HP}_1 & = (\mathbb{H}^2 - 0)/\mathbb{H}^\times \end{array}$$

The Hopf map is natural map of quotient spaces induced by the inclusion $\mathbb{C}^\times \hookrightarrow \mathbb{H}^\times$ of the multiplicative group of non-zero complex numbers into the multiplicative group of non-zero quaternions. $\diamondsuit$

Example. Recall that the complex projective plane $\mathbb{CP}_2$ has a standard orientation. Let us write $\overline{\mathbb{CP}}_2$ to denote $\mathbb{CP}_2$ equipped with the *opposite* orientation. Set $M = \overline{\mathbb{CP}}_2$, and let us equip this manifold with the *Fubini-Study metric* g. Then $[g]$ turns out to be ASD.

Why? Up to scale, the Fubini-Study metric is completely characterized by the fact that it is invariant under $SU(3)$. Now the stabilizer of a point of $\mathbb{CP}_2$ is $U(2)$:

$$\mathbb{CP}_2 = SU(3)/U(2).$$

Thus the curvature tensor $\mathcal{R}$ must be $U(2)$-invariant at each point of $\mathbb{CP}_2$, and hence $SU(2)$-invariant. But at a point of $\mathbb{CP}_2$, $SU(2)$ acts on Λ^- via the standard representation of $SO(3) = SU(2)/\mathbb{Z}_2$ on $\mathbb{R}^3$. This is enough to imply that $(\mathbb{CP}_2, [g])$ has $W^- = 0$. Reversing orientation, we conclude that $(\overline{\mathbb{CP}}_2, [g])$ has $W^+ = 0$, as claimed.

The twistor space of $(\overline{\mathbb{CP}}_2, [g])$ is quite interesting. Let $\mathbb{V} \cong \mathbb{C}^3$ be a 3-dimensional complex vector space, let $\mathbb{V}^*$ denote its dual vector space, and let $\mathbb{P}(\mathbb{V})$ and $\mathbb{P}(\mathbb{V}^*)$ be the two complex projective planes $\mathbb{CP}_2$ one gets from these two vector spaces. Then the relevant twistor space is the hypersurface in $\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V}^*) \cong \mathbb{CP}_2 \times \mathbb{CP}_2$ defined by the incidence relation $v \cdot w = 0$, where $v \in \mathbb{V}$, $w \in \mathbb{V}^*$.

You might guess that the twistor projection $\varpi : Z \to M$ might be a factor projection $([v], [w]) \to [v]$ or $([v], [w]) \to [w]$. Wrong! In fact, it is given by $([v], [w]) \to [\overline{v} \times w]$, where $v \to \overline{v}$ is the anti-linear "index lowering" map $\mathbb{V} \to \mathbb{V}^*$ induced by some chosen Hermitian inner product, and where $\times$ is the bilinear cross-product $\mathbb{V}^* \times \mathbb{V}^* \to \mathbb{V}$ induced by contraction with a non-zero element of $\Lambda^3\mathbb{V}$. $\diamondsuit$

3 Aside on Super-Manifolds

3.1 Definition of Super-manifolds

A complex super-manifold is consists of an ordinary complex manifold X, called the bosonic manifold, and an enriched class $\mathcal{S}$ of "functions" on X, where $\mathcal{S}$ is a sheaf of graded-commutative algebras which is locally isomorphic to holomorphic functions with values in a Grassmann algebra $\Lambda^\bullet \mathbb{C}^k$. In the following discussion, we will in fact only consider examples which are globally of the simple form

$$\mathcal{S} = \mathcal{O}(\Lambda^\bullet E^*) = \mathcal{O}(\oplus_{j=0}^k \Lambda^j E^*),$$

where E is a holomorphic vector bundle of rank k, called the fermionic tangent bundle, and E^* is its dual vector bundle. The Berezinian line-bundle of such a complex super-manifold is then defined to be $B = K \otimes \Lambda^k E$, and plays a rôle similar to the canonical line bundle $K = \Lambda^{n,0}$ of a complex manifold. (Notice that the formula for B involves E, rather than E^*; thus, fermionic directions make a contribution with a "sign" opposite to the contribution of the bosonic directions, in accordance with the the fact that Berezin's super-integration involves derivatives in fermionic directions.) We will say that a super-manifold is formally *super-Calabi-Yau* if its Berezinian line bundle is holomorphically trivial. This idea is important for Witten's twistor string theory.

An important example of a complex super-manifold is the super-projective space $\mathbb{CP}_{(n|m)}$, gotten by taking $X = \mathbb{CP}_n$, and setting $E = \underbrace{\mathcal{O}(1) \oplus \cdots \mathcal{O}(1)}_{m}$. The Berezinian line bundle of $\mathbb{CP}_{(n|m)}$ is $B = \mathcal{O}(m-n-1)$. Thus $\mathbb{CP}_{(n|n+1)}$ is formally super-Calabi-Yau for any n. Witten has drawn our attention to the special case of $\mathbb{CP}_{(3|4)}$, which is to be viewed as a super-symmetric version of the twistor space of S^4.

3.2 Twistor Spaces are Spin

I would now like to point out a recipe for producing an analog of this construction for *any* ASD 4-manifold. We begin with the observation, due to Hitchin, that *any twistor space* Z of a 4-manifold M is automatically spin, in the sense than $w_2 = 0$. Equivalently, the first Chern class c_1 of Z is even, in the sense that $c_1 = 2a$ for some $a \in H^2(Z, \mathbb{Z})$. This says that there is a complex line bundle $K^{1/2}$ on Z such that $K = K^{1/2} \otimes K^{1/2}$. On the face of it, this is just a topological statement, but since the transition functions of $K^{1/2}$ are square-roots of the transition functions of K, there is a natural way of making $K^{1/2}$ into a holomorphic line bundle.

Example. Even though $M = \overline{\mathbb{CP}}_2$ is not spin, its twistor space $Z \subset \mathbb{CP}_2 \times \mathbb{CP}_2$ really *is* spin. Indeed, the canonical line bundle K of Z is the restriction of $\mathcal{O}(-2,-2)$ from $\mathbb{CP}_2 \times \mathbb{CP}_2$, and this is the square of the line bundle $K^{1/2}$ obtained by restricting $\mathcal{O}(-1,-1)$ to Z. ♢

Example. When $M = S^4$, we have $Z = \mathbb{CP}_3$, the canonical line bundle of which is $\mathcal{O}(-4)$. Thus we have $K^{1/2} = \mathcal{O}(-2)$ when M is the (spin) manifold S^4. Moreover, $K^{1/4}$ also makes sense in this case. This reflects an interesting general fact: the canonical line bundle of Z has a *fourth* root if and only if M is spin. ♢

3.3 Super-Twistor Spaces

Now let Z be the twistor space of any ASD 4-manifold, or more generally just some complex 3-fold which is *spin*. We can then cook up a super-symmetric version of Z, of complex bi-dimension $(3|4)$, by setting $E = J^1 K^{-1/2}$, meaning the bundle of 1-jets of of holomorphic sections of $K^{-1/2}$. An element of E encodes both the value and the first derivative of a section of $K^{-1/2}$ at some point of Z. More precisely, we have an exact sequence

$$0 \to \Omega^1 \otimes K^{-1/2} \to J^1 K^{-1/2} \to K^{-1/2} \to 0.$$

Note, however, by a theorem of Atiyah, this sequence never splits if Z has $c_1 \neq 0$, as will hold whenever Z is a twistor space – the case of particular interest to us.

Proposition 3.1 *With the above choice,* $(Z, \mathcal{O}(\Lambda^\bullet E^*))$ *is formally super-Calabi-Yau.*

Proof. The adjunction formula tells us that

$$\begin{aligned} \Lambda^4 E &= K^{-1/2} \otimes \Lambda^3(\Omega^1 \otimes K^{-1/2}) \\ &= K^{-1/2} \otimes \Omega^3 \otimes K^{-3/2} \end{aligned}$$

$$= \quad K^{-1/2} \otimes K \otimes K^{-3/2}$$
$$= \quad K^{-1}$$

so that the Berezianian line bundle

$$B = K \otimes \Lambda^4 E = K \otimes K^{-1}$$

is trivial. ■

The interest of this recipe is that in the particular case of $\mathbb{CP}_3$, one has

$$J^1 K^{-1/2} = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1),$$

so that the recipe produces $\mathbb{CP}_{(3|4)}$ essentially out of thin air. More generally, $J^1 K^{-1/2}$ naturally arises in twistor theory in the context of ambitwistors and Yang-Mills fields on ASD spaces, so I strongly suspect that this choice of E will turn out to be the right generalization for twistor string theory.

4 Existence of ASD Metrics

Several natural questions are probably bothering you by now.

- Are there many compact ASD 4-manifolds, or only a few?
- Are the associated twistor spaces usually Kähler manifolds, like the examples we have seen so far?
- Do the examples we have seen so far fit into some broader algebraic pattern?

The short answer to the first question is that ASD manifolds exist in great abundance. The deepest result in this direction is as follows:

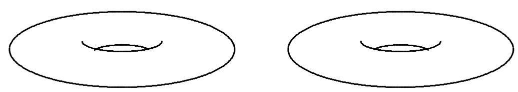
Theorem 4.1 (Taubes) *Let M^4 be any smooth compact 4-manifold. Then for all sufficiently large integers $k \gg 0$, the connect sum*

$$M \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k}$$

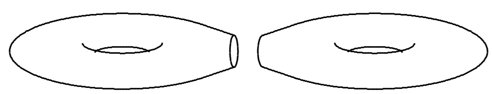
admits ASD metrics.

The connect sums referred to in this statement are defined by iterating the following construction:

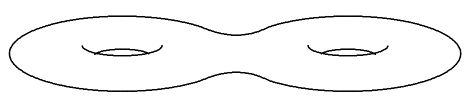
Definition 4.2 *Let M_1 and M_2 be smooth connected compact oriented n-manifolds.*



Their connect sum $M_1 \# M_2$ is then the smooth connected oriented n-manifold obtained by deleting a small ball from each manifold



and identifying the resulting S^{n-1} boundaries



via a reflection.

Taubes' result is very powerful, but it does not by any means allow one to write down formulas for the relevant metrics. Indeed, it does not even tell us, for a given M, how large k should be for such a metric to exist — although the flavor of the proof suggests that the answer might be something like the numbers of protons in the universe!

Anyway, we now see that the world of ASD 4-manifolds is incredibly rich, and as is the world of complex 3-manifolds associated to these objects by the twistor construction. However, these twistor spaces are almost never Kähler!

Theorem 4.3 (Hitchin) *If $(M^4, [g])$ is an ASD 4-manifold for which Z is a Kähler manifold, then either $M = S^4$ or $\overline{\mathbb{CP}}_2$, and $[g]$ is the standard conformal structure of our previous discussion.*

In particular, there are no other ASD 4-manifolds for which the twistor spaces are complex projective varieties! However, there *are* many others for which the twistor spaces can be obtained from singular projective varieties by performing small resolutions of the singularities. Such complex manifolds are called *Moishezon manifolds*, and while they are in some sense very close to being projective algebraic, they are quite different from projective varieties in other important ways. For example, projective algebraic varieties are necessarily Kähler, but the same is generally *not* true of Moishezon spaces.

Theorem 4.4 (LeBrun) *The* 4*-manifolds*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2\# \cdots \#\overline{\mathbb{CP}}_2}_{k}$$

all admit ASD metrics for which the twistor spaces are Moishezon.

In fact, certain ASD metrics on each of these spaces can be written down in closed form, and their twistor spaces are equally explicit.

On the other hand, we do not yet have a full classification of all compact ASD manifolds with Moishezon twistor space. But we do at least have a partial converse to the previous result:

Theorem 4.5 (Campana, Poon) *If M admits an ASD metric for which Z is Moishezon, then M is* homeomorphic *to S^4 or to a connect sum $\overline{\mathbb{CP}}_2\# \cdots \#\overline{\mathbb{CP}}_2$.*

Can one show that M must be *diffeomorphic* to such a connect sum? This is certainly true when k is small, but the problem is completely open when $k \gg 0$.

The difference between homeomorphism and diffeomorphism is very important to mathematicians, but it may be confusing for some physicists. Recall that a homeomorphism between manifolds is a 1-to-1-correspondence which respects the notion of continuous function; by contrast, a diffeomorphism is a 1-to-1 correspondence which respects the notion of *differentiable* function. When mathematicians use the word *topological*, they generally are talking about properties which only depend on homeomorphism type. But when physicists talk about a *topological field theory*, they mean a field theory that depends only on the diffeomomorphism type of a manifold. Still, I'm not sure if physicists are really to blame for this confusion. After all, the mathematical discipline that studies diffeomorphism types of manifolds is known as *differential topology*! And, at any rate, mathematicians themselves systematically overlooked this subtle distinction until the second half of the 20^{th} century.

For 4-manifolds, this distinction is particularly dramatic. Let us henceforth only consider *simply connected* manifolds (that is, manifolds in which any loop can be continuously contracted to a point). Michael Freedman showed that compact, simply connected 4-manifolds can be completely classified up to homeomorphism. In particular, in conjunction with a result of Donaldson, his work implies the following:

Theorem 4.6 (Freedman-Donaldson) *Any smooth compact simply connected* non-spin 4*-manifold is homeomorphic to a connect sum*

$$j\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2 = \underbrace{\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2}_{j}\#\underbrace{\overline{\mathbb{CP}}_2\#\cdots\#\overline{\mathbb{CP}}_2}_{k}.$$

However, for "most" values of j and k, it is now known that there are *infinitely many* distinct differentiable structures on these manifolds; moreover, many of these differentiable structures had gone undetected until only a couple of years ago. In short, while we have a pretty good picture of the "topology" of 4-manifolds, their "differential topology" is an ongoing story, with no end in sight.

Now if M is any 4-manifold, $M\#\overline{\mathbb{CP}}_2$ is non-spin, so the simply connected case of Taubes' theorem concerns manifolds homeomorphic to some $j\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$. Now let's focus on the *standard* differentiable structure, and ask for what values of j and k this connect sum admits ASD metrics. For example, Taubes' theorem tells us that there are enormous values of k for which $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ admits ASD metrics. But k fortunately doesn't really have to be astronomical for this to work. Indeed, here's some relevant late-breaking news:

Theorem 4.7 (Rollin-Singer) *The* 4*-manifold* $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ *admits ASD metrics for every* $k \geq 10$*.*

This improves a decade-old result of LeBrun-Singer, which made the same assertion for $k \geq 14$.

Using a general gluing construction of Donaldson-Friedman, the Rollin-Singer result immediately allows one to read off the following:

Corollary 4.8 *The* 4*-manifold* $j\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ *admits ASD metrics if* $k \geq 10j$.

These metrics are not completely explicit, but the construction does give an approximate geometric picture of them. Things are even better at the twistor-space level, where the Donaldson-Friedman construction really takes place.

I can only hope that this brief discussion has conveyed something of the richness of ASD manifolds and their twistor spaces. We already know some remarkable things about these objects, but I am convinced that we have as yet only scratched the surface of the subject. Taubes' theorem tells us that 4-manifolds can effectively be geometrized, in a way that is natural from the point of view of several complex variables. I can only hope that, following Witten's lead, string theorists will now open our eyes to new and unexpected vistas on this wild world of complex 3-folds and 4-dimensional geometries.