Generalized complex geometry and supersymmetric non-linear sigma models

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ABSTRACT

After an elementary presentation of the relation between supersymmetric non-linear sigma models and geometry, I focus on $2D$ and the target space geometry allowed when there is an extra supersymmetry. This leads to a brief introduction to generalized complex geometry, a notion introduced recently by Hitchin which interpolates between complex and symplectic manifolds. Finally I present worldsheet realizations of this geometry,

1 Introduction

A year ago at the previous Simons workshop Marco Gualtierei presented part of what later became his thesis entitled "Generalized Complex Geometry" (GCG) [2]. It was based in work done by his supervisor, Nigel Hitchin , who had been mainly motivated by describing generalized Calabi Yau manifolds, e.g., when including an antisymmetric B-field [1]. Another interesting aspect, however, is the relation to supersymmetric non-linear sigma models. The geometry he described includes the bi-hermitean geometry with a B-field found by Gates, Hull and Roček 20 years ago [9]. Indeed, it is an interesting fact that still 25 years after the original classification by Zumino [3] there are still some open problems in this area. Both the hyperkähler geometry discussed by Alvarez-Gaumé and Freedman [4], and the bi-hermitean geometry of [9] correspond to additional supersymmetries that in general only close on-shell. This prevents formulation with all the supersymmetries manifest, except in special cases. In addition, although the B-field does have a geometrical role in the $\mathcal{N} = (2, 2)$ sigma model as a potential for the torsion, it is only locally defined and the model really only depends on its field strength. It would be nice to have a geometrical setting where the B-field itself aquires a geometrical meaning.

It seemed that the GCG of Hitchin may help shed light on some of these questions, and this is a report on some subsequent development in that direction partly in collaboration with R. Minasian, A.Tomasiello and M.Zabzine [7], [5]. Originally my talk was to be the first of two on this subject, the second was to be delivered by Maxim Zabzine. Unfortunately, due to the vagaries of the US consular system, he was unable to attend the workshop.

2 Sigma models

A non-linear sigma model is a theory of maps

$$X^\mu(\xi) : \mathcal{M} \to \mathcal{T} , \qquad (2.1)$$

where ξ^i are coordinates on $\mathcal{M}$ and $X^\mu(\xi)$ coordinates on the target space $\mathcal{T}$. Classical solutions are found by extremizing the action.

$$S = \int d\xi \; \partial_i X^\mu G_{\mu\nu}(X) \partial^i X^\mu , \qquad (2.2)$$

where the symmetric $G_{\mu\nu}$ is identified with a metric on $\mathcal{T}$, a first sign of the intimate relation between sigma models and target space geometry. Extremizing S results in the X^μ's being harmonic maps involving the pull-back of the covariant Laplacian:

$$\nabla^i \partial_i X^\mu = 0 , \qquad (2.3)$$

where ∇ is defined w.r.t. the Levi-Civita connection for G, another indication of the relation to geometry. The geometry is Riemannian for the bosonic model, but typically becomes complex when we impose supersymmetry.

Supersymmetry is introduced by replacing the X^μ's by superfields:

$$X^\mu(\xi) \rightarrow \phi^\mu(\xi, \theta) , \tag{2.4}$$

with component expansion

$$\begin{aligned} X^\mu &= \phi^\mu| \\ \Psi^\mu_\alpha &= D_\alpha \phi^\mu| \\ &\cdots\cdots \ , \end{aligned} \tag{2.5}$$

where D_α are the superspace spinorial covariant derivatives generating the supersymmetry algebra, $|$ denotes "the θ indepedent part of and "....indicates additional components depending on the dimension of $\mathcal{M}$. To be more concrete, in $4D$, using Weyl spinors, the supersymmetry algebra is

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\partial_{\alpha\dot{\alpha}} , \tag{2.6}$$

and the smallest representation containing a scalar field is a chiral superfield $\bar{D}_{\dot{\alpha}}\phi = D_\alpha\bar{\phi} = 0$. This means that the target space will naturally have complex coordinates. The most general supersymmetric action is determined by an arbitrary function K

$$S = \int d^4\xi d^2\theta d^2\bar{\theta} K(\phi, \bar{\phi}) = \int d^4\xi \left(\frac{\partial^2 K}{\partial\phi^\mu \partial\bar{\phi}^\nu} \partial^i X^\mu \partial_i \bar{X}^\nu + ... \right) , \tag{2.7}$$

and a comparison to ([**?**]) shows that K is a Kähler potential. The geometry is thus Kähler with metric $g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K$.

A similar analysis in other dimensions leads to a classification for dimensions [1]$D \leq 6$ which may be summarized as [2]

D	6	4	2	GEOMETRY
N	1	2	4	Hyperkähler
		1	2	Kähler
			1	Riemannian

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[1]Higher D's will necessarily have multiplets with vector components.

[2]For brevity only even D's are included.

The interesting part, where this table is "incomplete" is for $D = 2$, the dimension relevant for string theory. There are (at least) two special features in $D = 2$. First, there can be different amounts of supersymmetry in the left and right moving sectors denoted $\mathcal{N} = (p, q)$ supersymmetry. Second, if parity breaking terms are allowed, the background may contain an antisymmetric $B_{\mu\nu}$-field. For $\mathcal{N} = (2, 2)$, the supersymmetric action written in terms of real $\mathcal{N} = (1, 1)$ superfields reads

$$S = \int d^2\xi d^2\theta d^2 D_+ \phi^\mu E_{\mu\nu}(\phi) D_- \phi^\nu \ , \tag{2.8}$$

where $E_{\mu\nu}(\phi) \equiv G_{\mu\nu}(\phi) + B_{\mu\nu}(\phi)$. This action has manifest $\mathcal{N} = (1, 1)$ supersymmetry without any additional restrictions on the target space geometry. Gates, Hull and Roček showed that it has an additional non-manifest supersymmetry,

$$\delta\phi^\mu = \varepsilon^+ D_+\phi^\nu J_\mu^{(+)\nu} + \varepsilon^- D_-\phi^\nu J_\mu^{(-)\nu} \ , \tag{2.9}$$

provided that the following conditions are fulfiled[3].:
• Both the J's are *almost complex structures*, i.e. $J^{(\pm)2} = -1$.
• They are *integrable*, i.e., their Nijenhuis-tensors vanish

$$\mathcal{N}_{\mu\nu}^{(\pm)\ \rho} \equiv J_\lambda^{(\pm)\mu} \partial_{[\lambda} J_{\nu]}^{(\pm)\rho} - (\mu \leftrightarrow \nu) = 0 \tag{2.10}$$

• The metric is *hermitean* w.r.t. both complex structures, i.e. they both preserve the metric $J^{(\pm)t} G J^{(\pm)} = G$
• The J's are *covariantly constant* with respect to a torsionful connection:

$\nabla^{(\pm)} J^{(\pm)} = 0$ with $\nabla^{(\pm)} \equiv \nabla^0 \pm H$, the Levi-Civita connection plus completely antisymmetric torsion in form of the field-strength $H = dB$[4].

The above conditions represent a bi-hermitean target space geometry with a B-field, and result from requiring invariance of the action (2.8) under the transformations (2.9) as well as closure of the algebra of these transformations. Closure is only achieved on-shell, however. Only under the special condition that the two complex structures commute does the algebra close off-shell. In that case there is a manifestly $\mathcal{N} = (2, 2)$ action for the model, given in terms of chiral and twisted chiral $\mathcal{N} = (2, 2)$ superfields [9]. An interesting question is thus: What is the most general $\mathcal{N} = (2, 2)$ sigma model (with off-shell closure of the algebra) and what is the corresponding geometry? In asking this we have in mind an extension of the model to include additional fields to allow off-shell closure in the usual "auxiliary field" pattern and a geometry that includes these fields.

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[3]There is a further relation between the torsion and the complex structures. which we left out
[4]Strictly, the torsion $T = g^{-1}H$.

As mentioned in the introduction, the GCG does contain the bi-hermitean geometry as a special case and thus seems a promising candidate. We therefore turn to a brief description of the GCG.

3   Generalized Complex Geometry

To understand the generalization, let us first briefly look at some aspects of the definition of the ordinary complex structure. The features we need are that an almost complex structure J on a d-dimensional manifold $\mathcal{T}$ is a map from the tangent bundle $J : T \to T$ that squares to minus the identity $J^2 = -1$. With these properties $\pi_\pm \equiv \frac{1}{2}(1 \pm iJ)$ are projection operators, and we may ask when they define integrable distributions. The condition for this is that

$$\pi_\mp[\pi_\pm X, \pi_\pm Y] = 0 \tag{3.11}$$

for $X, Y \in T$ and $[,]$ the usual Lie-bracket on T. This relation is equivalent to the vanishing of the Nijenhuis tensor $\mathcal{N}(J)$, as defined in (2.10).

To define GCG, we turn our attention from the tangent bundle $T(\mathcal{T})$ to the sum of the tangent bundle and the co-tangent bundle $T \oplus T^*$. (Note that the structure group of this bundle is $SO(d,d)$, the string theory T-duality group, an important fact that will not be further pursued in this lecture). We write an element of $T \oplus T^*$ as $X + \xi$ with the vector $X \in T$ and the one-form $\xi \in T^*$. The natural pairing $(X + \xi, X + \xi) = -\imath_X \xi$ gives a metric I on $T \oplus T^*$ as $X + \xi$, which in a coordinate basis (∂_μ, dx^ν) reads

$$\begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} . \tag{3.12}$$

In the definition of a complex structure above we made use of the Lie -bracket on T. To define a generalised complex structure we will need a bracket on $T \oplus T^*$. The relevant bracket is the the skew-symmetric *Courant bracket* [8] defined by

$$[X + \xi, Y + \eta]_c \equiv [X, Y] + \pounds_X \eta - \pounds_Y \xi - \frac{1}{2} d(\imath_X \eta - \imath_Y \xi) . \tag{3.13}$$

This bracket equals the Lie-bracket on T and vanishes on T^*. [5] The most important property for us in the context of sigma-models is that its group of automorphisms is not only $Diff(\mathcal{T})$ but also *b-transforms* defined by closed two-forms b,

$$e^b(X + \xi) \equiv X + \xi + \imath_X b , \tag{3.14}$$

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[5] It does not in general satisfy the Jacobi identity; had it satisfied the Jacobi identity $(T \oplus T^*, [,]_c)$ would have formed a Lie algebroid. It *does* satisfy the Jacobi identity on subbundles $L \subset T \oplus T^*$ that are Courant involutive and isotropic w.r.t. $\mathcal{I}$, but fails to do so in general. It fails in an interesting way which leads to the definition of a Courant algebroid [2].

namely,

$$[e^b(X + \xi), e^b(Y + \eta)]_c = e^b[X + \xi, Y + \eta]_c \ . \tag{3.15}$$

A *generalized almost complex structure* is an endomorphism $\mathcal{J} : T \oplus T^* \to T \oplus T^*$ that satisfies $\mathcal{J}^2 = -1_{2d}$ and preserves the natural metric $\mathcal{I}$, $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$. The projection operators $\Pi_{\pm} \equiv \frac{1}{2}(1 \pm i\mathcal{J})$ are then used to define integrability (making $\mathcal{J}$ a generalized complex structure) as

$$\Pi_{\mp}[\Pi_{\pm}(X + \xi), \Pi_{\pm}(Y + \eta)]_c = 0 \tag{3.16}$$

In a coordinate basis $\mathcal{J}$ is representable as

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix} , \tag{3.17}$$

where $J : T \to T, \quad P : T^* \to T, \quad L : T \to T^*, \quad K : T^* \to T^*$. The condition $\mathcal{J}^2 = -1_{2d}$ will impose conditions

$$\begin{aligned} J^2 + PL &= -1_d \\ JP + PK &= 0 \\ KL + LJ &= 0 \\ LP + K^2 &= -1_d \ , \end{aligned} \tag{3.18}$$

and (3.16) will impose differential conditions on J, P, L and K.

The ordinary complex structure is given by

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} , \tag{3.19}$$

and a symplectic structure ω corresponds to [6]

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} . \tag{3.20}$$

A b-transform acts as follows

$$\mathcal{J}_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} . \tag{3.21}$$

The general situation is illustrated in the following diagram:

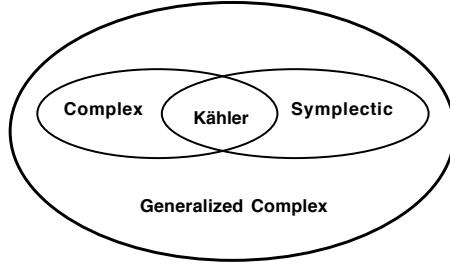[6]For a generalized complex structure to exist T has to be even-dimensional.

Fig1. The relation between the different geometries discussed.

A useful property for calculations is that locally (in an open set around a non-degenerate point) a manifold which admits a generalized complex structure may be brought to look like an open set in $\mathbb{C}^k$ times an open set in (R^{2d-2k}, ω), where ω is in Darboux coordinates and $\mathbb{C}^k$ in complex (holomorphic and anti-holomorphic) coordinates (using diffeomorphisms and b-transform) [7].
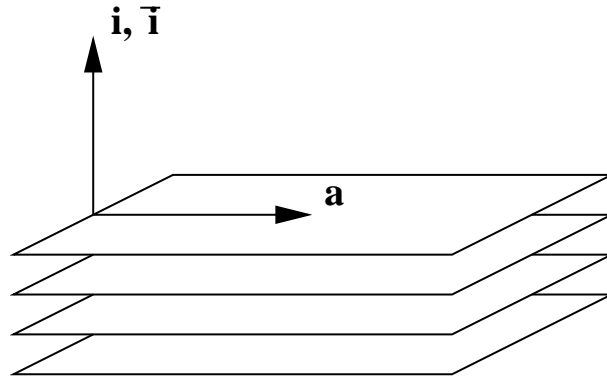

Fig 2. A naive picture of the local foliation in complex coordinates $z^i, \bar{z}^i$ and Darboux coordinates x^a.

The generalized complex geometry is said to be generalized Kähler if there exist two commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $T \oplus T^*$. For a Kähler manifold (J, g, ω), using (3.19) and (3.20) one finds the metric

$$G = -\mathcal{J}_J\mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{3.22}$$

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[7]The proof of this, generalizing the Newlander-Nirenberg and the Darboux theorems, may be found in Gualtieri's thesis, [2], Sec. 4.7.

Finally, it is worth mentioning that it is possible to twist the above structure by a closed three-form.

We now turn to the question of how this geometry may be realized in sigma models.

4 Sigma model realization

In the sigma model action (2.8) $D\phi \in T(\mathcal{T})$. Clearly we shall need a formulation with additional fields $S \in T^*$ to be able to realize the GCG. We thus consider the following first order action

$$S = \int d^2\xi d^2\theta \left(S_{\mu+} E^{\mu\nu}(\phi) S_{\nu-} - S_{\mu(+} D_{-)}\phi^\mu + D_+\phi^\mu (B - b)_{\mu\nu} D_-\phi^\nu \right) , \tag{4.23}$$

where $E_{\mu\nu} \equiv g_{\mu\nu} + b_{\mu\nu}$ and its inverse may be thought of as open string data:

$$E^{(\mu\nu)} = G^{\mu\nu} , \quad E^{[\mu\nu]} = \theta^{\mu\nu} . \tag{4.24}$$

In (4.23) $S_{\mu\pm}$ acts as an auxiliary field which extends the model to a sigma model on $T \oplus T^*$ and b is a globally defined two-form which allow us to display the b-transform. (Note that the original model (4.23) depends only on $H = dB$, and B is thus typically only locally defined). Eliminating $S_{\mu\pm}$ we recover the action in (2.8). The b-transform is the statement that if in two actions of the form (4.23) $E_{\mu\nu}$ and $\tilde{E}_{\mu\nu}$ differ by a closed two-form $\tilde{b}$ the two actions are equivalent.

The action (4.23) has many interesting limits. For example, if the metric is set to zero it is a supersymmetric version of a Poisson sigma model [10]. In what follows we shall not be interesting in the difference between the B and b-fields but set them equal each other, so the $\mathcal{N} = (1, 1)$ action we study is

$$S = \int d^2\xi d^2\theta \left(S_{\mu+} E^{\mu\nu}(\phi) S_{\nu-} - S_{\mu(+} D_{-)}\phi^\mu \right) , \tag{4.25}$$

We shall also be interested in it's N $= (1, 0)$ reduction

$$S = \int d^2\xi d^2\theta \left(S_{\mu+} E^{\mu\nu}(\phi) S_{\nu=} - S_{\mu+} \partial_=\phi^\mu + D_+\phi^\mu S_{\mu=} \right) , \tag{4.26}$$

and the purely topological $\mathcal{N} = (1, 0)$ model

$$S = \int d^2\xi d^2\theta \left(S_{\mu+} \partial_=\phi^\mu \right) . \tag{4.27}$$

For the $\mathcal{N} = (1, 1)$ model, the form of ansats for the second supersymmetry $(\delta = \delta^{(+)} + \delta^{(-)})$ is determined by a dimensional analysis to be

$$\delta^{(\pm)}\phi^\mu = \varepsilon^\pm \Lambda_\pm^A A_A^{(\pm)\mu}$$

$$\delta^{(\pm)} S_{\mu\pm} = \varepsilon^\pm \left(D_\pm \Lambda_\pm^A B_{\mu A}^{(\pm)} + \Lambda_\pm^A \Lambda_\pm^B C_{\mu AB}^{(\pm)} \right)$$

$$\delta^{(\pm)} S_{\mu\mp} = \varepsilon^\pm \left(D_\pm \Lambda_\mp^A M_{\mu A}^{(\pm)} + D_\mp \Lambda_\pm^A N_{\mu A}^{(\pm)} + \Lambda_\pm^A \Lambda_\mp^B X_{\mu AB}^{(\pm)} \right) , \tag{4.28}$$

where $L_\pm^A \equiv (D_\pm \phi^\mu, S_{\mu\pm})$ and all the coefficient are fuctions of ϕ. The conditions which follow from invariance of the action and closure of the algebra are of two kinds, algebraic and differential. The two (A-type) index coefficients typically turn out to be given as derivatives of the one-index coefficients, just like the generalized complex structures contain are given in terms of J, P, L and K which subsequently obey differential conditions via the integrability requirement.

For the topological model, considering only the left-moving sector, the relations corresponding to (4.28) simplify considerably. They are

$$\delta^{(+)} \phi^\mu = \varepsilon^+ \left(D_+ \phi^\lambda J^\mu_\lambda - S_{\lambda+} P^{\mu\nu} \right)$$

$$\delta^{(+)} S_{\mu+} = \varepsilon^+ \left(i\partial_{+\!\!+} \phi^\lambda L_{\mu\lambda} - D_+ S_{\lambda+} K_\mu{}^\lambda + ... \right) , \tag{4.29}$$

where "...." indicates the higher coefficients determined by the differential conditions. In [5] we show that the algebraic and differential conditions in this case are satisfied if and only if

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix} ,$$

is a generalized complex structure.

Similar results hold for the full $\mathcal{N} = (2,0)$ sigma model, but there we were not yet able to find the most general solution to the differential constraints. Under certain assumptions, however, we found a solution which is the geometry given by the following generalized complex structure

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ L & -J^t \end{pmatrix} ,$$

with $L_{\mu\nu} = J^\rho_{[\nu} b_{\mu]\rho}$ and $\nabla_\mu^{(+)} J_{\nu\rho} = 0$. All other components are again determined by $\mathcal{J}$.

The full $\mathcal{N} = (1,1)$ model presents the most challenge as the solutions to the conditions corresponding to $(\mathcal{N} = (1,0))$ are least known. In [7], where the model was introduced, the relation to the bi-hermitean geometry was established. However, some of the assumptions in that paper may be considerably relaxed, and there are reasons to expect that this relaxation will be enough to make the model invariant with the algebra closing off-shell.

8

5 Conclusions

We have presented GCG and shown how it can be realized in the context of supersymmetric nonlinear sigma models. Since we have not yet found the general solution to the constraints on the transformation coefficients for $\mathcal{N} = (2,0)$ and $\mathcal{N} = (2,2)$, we cannot yet say that GCG is the most general target space geometry. In fact we have found hints that the full solution of the constraints may go beyond GCG, but this is yet unclear. Off-shell closure is not yet proven, but seems possible. That would certainly make the whole approach more interesting, since in that case not only would we have an nice geometrical framework, which reduces to the known bi-hermitean geometry when the auxiliary S is integrated out, but also the possibility of extending the action while keeping $\mathcal{N} = (2,2)$ supersymmetry. This would allow, e.g., inclusion of higher derivative terms, in keeping with the sigma model as an effective action. There are many other directions one could investigate starting from the new form of the action. For example, it allows the study of models where $E^{\mu\nu}$ is non-invertible. The question for open sigma models of what are the most general boundary conditions allowed by supersymmetry has proven very useful in understanding geometrical restrictions on D-branes [12, 11] and has a natural extension in to the present case. In fact GCG has already helped in interpreting some of the geometric structures previously found [13].

The question of which criteria would allow the $\mathcal{N} = (2,2)$ sigma models with a B-field to be used for constructing topological strings was raised during the talk. In analogy to the Calabi-Yau case, a natural conjecture is that the criterion is the vanishing of the first Chern-class, as defined by the torsionful Ricci tensors $R^{\pm}_{\mu\nu}$. Since the torsion enters the Ricci tensor quadratically, $R^+ = R^-$, but there are nevertheless two possible Ricci-forms depending on J^+ and J^- respectively. Again, perhaps this question is best addressed within the framework of generalized complex geometry. A recent relevant paper is [14].

Finally, we mention that GCG has been considered in other contexts. Studying generalized Calabi Yau manifolds (the original motivation) and discussing supersymmetrical backgrounds in such manifolds [6] are but two examples.

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