M-theory on Complex $\mathbb{P}^1$ bundles and Calabi–Yau Relatives

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Abstract

These are written up notes of the talk I gave at Simons Workshop 2004, based on preprints `hep-th/0402153,0403002,0403038` co-authored by J. Gauntlett, J. Sparks, and D. Waldram. I review the construction of some supersymmetric solutions of 11d supergravity of the type $AdS_5 \times M_6$ where M_6 are complex $\mathbb{P}^1$ bundles over Kähler four-manifolds, closely resembling twistor spaces. Then I discuss the dualization of some of these solutions yielding new Sasaki–Einstein metrics on $S^2 \times S^3$. In addition, I briefly review basic facts about Sasaki–Einstein geometry and discuss general features of the field theory duals of these geometries.

1 Supersymmetric $AdS_5 \times M_6$ solutions

Our first goal is to construct supersymmetric solutions of eleven dimensional supergravity which contain a (warped) AdS_5 factor in the metric. The strategy will be to consider the most general ansatz for the four-form flux G *and* Killing spinor, compatible with the AdS_5 symmetry.

Recall that the bosonic fields of 11d supergravity are a metric g_{MN} and a four-form G_{MNPQ}. A supersymmetric solution of this theory is a configuration obeying the condition

$$\delta\psi_M = \hat{\nabla}_M \eta - \frac{1}{288}\left(G_{NPQR}\hat{\Gamma}^{NPQR}{}_M - 8G_{MNPQ}\hat{\Gamma}^{NPQ}\right)\eta = 0 \tag{1.1}$$

which sets to zero the variation of the gravitino field, the G equation of motion and Bianchi identity

$$\begin{aligned} \mathrm{d}\,\hat{*}G + \frac{1}{2}G\wedge G &= 0 \\ \mathrm{d}G &= 0 \end{aligned} \tag{1.2}$$

and the Einstein equations. Our metric ansatz is the following

$$d\hat{s}^2_{11} = \mathrm{e}^{2\lambda(x)}[ds^2(AdS_5) + ds^2(M_6)] \tag{1.3}$$

where the warp-factor λ is a function on M_6. The G field has arbitrary components (to be determined) along M_6, while the spinorial supersymmetry parameter is of the form

$$\eta = \psi\otimes\xi \tag{1.4}$$

where, *crucially*, ξ is a non-chiral spinor on M_6 and ψ is a Killing spinor in AdS_5, namely it obeys

$$\nabla_\mu\psi = \tfrac{\mathrm{i}}{2}m\gamma_\mu\psi\ . \tag{1.5}$$

Of course a non-chiral spinor in six dimensions can be always decomposed in its chiral components, which are irreducible representations of $Spin(6)$

$$\xi = \xi_+ + \xi_- \tag{1.6}$$

while ξ constitutes the minimal representation on which the Clifford algebra Cliff$(6,0)$ acts in the usual way. So, although naively we seem to have more than minimal supersymmetry, it turns out that the presence of non-trivial flux imposes a relation between ξ_+ and ξ_- giving back minimal supersymmetry in $d=6$[1]. One can easily show that if ξ is a chiral spinor, then $G=0$, λ is constant, and the geometry degenerates to $\mathbb{R}^{1,4}\times\mathrm{CY}_3$.

[1] Thanks to Martin Rocek for asking clarifications on this point.

In order to analyse systematically the geometries obeying these equations, we utilize the formalism of $G-$ *structures*, introduced in [1]. The idea is to consider certain $p-$forms arising as spinorial bilinears, and use the supersymmetry equations to constrain them. In the present situation, this will allow us to make contact with Kähler geometry.

A (complex) non-chiral spinor ξ is equivalent to a "local $SU(2)$-structure" in six dimensions[2]. This means that $SU(2)$ is the stabilizer group of ξ in $Spin(6) \simeq SU(4)$. Using the concrete correspondence

$$\xi \qquad \leftrightarrow \qquad Y_{i_1\dots i_p} = \bar{\xi}\gamma_{i_1\dots i_p}\xi \tag{1.7}$$

where $\gamma_{i_1\dots i_p} \in \text{Cliff}(6,0)$, one is lead to considering the following set of forms on M_6

$$\sin\zeta = \bar{\xi}\gamma_7\xi, K^1, K^2, J, \Omega \tag{1.8}$$

where J, Ω are $(1,1)$ and $(2,0)$ forms respectively and K^A are one-forms "orthogonal" to them, namely $i_{K^A}J = 0$, etc.

After some work (see [2] for details), we are able to constrain the geometry as follows. The metric on M_6 can be cast in the form

$$ds_6^2 \;=\; e^{-6\lambda}[ds^2(M_4) + \sec^2\zeta dy^2] + \tfrac{1}{9m^2}\cos^2\zeta(d\psi+\rho)^2\ . \tag{1.9}$$

This is naturally adapted to the Killing vector dual to K^2, i.e. $K^{2\#} = \sec\zeta\partial/\partial\psi$, while $K^1 \sim \mathrm{d}y$ does *not* give rise to a second Killing vector. At any fixed y, we also have

$$\begin{aligned} \mathrm{d}_4 J &= 0 \\ \mathrm{d}_4\Omega &= iP\wedge\Omega \end{aligned} \tag{1.10}$$

which implies that $ds^2(M_4)$ is a family (parameterized by y) of *Kähler metrics*. Note that P is the connection on the canonical line bundle of M_4 and it is related to the one-form ρ as follows

$$P \;=\; \rho - i(\partial_4 - \bar{\partial}_4)\log\cos\zeta\ . \tag{1.11}$$

The non-trivial part of the problem is contained in some first order "dynamical" equations, governing the evolution of the geometry along y. These are

$$\begin{aligned} \frac{\partial}{\partial y}J &= -\frac{2}{3}y\mathrm{d}_4\rho \\ \frac{\partial}{\partial y}\mathrm{vol}_4 &= F[\zeta, \frac{\partial\zeta}{\partial y}] \end{aligned} \tag{1.12}$$

[2]Note that although the mathematical definition of a G-structure requires that relevant $p-$forms, or equivalently spinors, are globally defined, this is by no means true for the notion of G-structure which is useful in String Theory (supergravity). As one is interested in solving supersymmetry equations, which are *local*, one does not have to require a priori that the corresponding spinors/forms be globally defined.

where F is a specific function whose form we don't need here. Finally, the flux G is completely determind in terms of the geometrical data, namely in terms of ζ, λ, K^A, J and their first derivatives (see [2] for more details).

We have characterized the conditions for supersymmetry in terms of the data on a family of Kähler four-manifolds and their evolution along y. At this point one is typically stuck, since imposing the Bianchi identity for G gives rise to nasty second order PDE's which one can't solve. Luckily, we can use the following fact

Lemma [2]: for all $AdS_5 \times M_6$ geometries of 11 dimensional supergravity, supersymmetry implies the equations of motion *and* $\mathrm{d}G = 0$.

Note that typically one can use integrability of supersymmetry equations to show that most components of the equations of motion hold. However, these arguments do not give any information on the Bianchi identities, which one has to impose separately. It is a remarkable fact that in our case, these are also implied by supersymmetry[3].

2 Complex $\mathbb{P}^1$ bundles

The conditions discussed so far have reduced the problem to a small number of coupled first order equations, with clear geometrical meaning. We didn't try to find the general solution to this system — this indeed might still be rather complicated. Instead, with one additional assumption, it will be possible to construct in closed form all the solutions to the system. The assumption that we are going to make is

Assumption: $\mathrm{d}s^2(M_6)$ is a Hermitian metric on a complex manifold M_6, with respect to the natural complex structure inherited from the spinorial bilinears.

This in turn simplifies the conditions considerably, namely

$$\mathrm{d}_4\zeta = 0 \qquad \mathrm{d}_4\lambda = 0 \qquad \partial_y\rho = 0 \tag{2.1}$$

and

$$\rho = P \tag{2.2}$$

implying that ρ is identified with a connection on the canonical line bundle of M_4. Thus at fixed generic y, the resulting 5-manifold is the total space of a $U(1)$ bundle over M_4, which is just the $U(1)$ bundle associated to the canonical line bundle $\mathcal{L}$ of M_4. Recalling the expression for the metric (1.9), and using the explicit form of the function $\cos\zeta(y)$, one has to check that the full six dimensional metric is smooth (complete) when $\cos\zeta$ has zeros or poles. One then finds that taking y to lie in a suitable interval $[y_1, y_2]$, all the necessary requirements are met. In this way the coordinates (y, ψ) parameterize a smooth $\mathbb{P}^1$: this is the $\mathbb{P}^1$ in the title of the talk. In fact, this $\mathbb{P}^1$ is fibered over M_4, and all complex (non-singular) solutions are indeed

$$\mathbb{P}^1 \to M_4 \ . \tag{2.3}$$

[3]We do not have a proof that this is a generic phenomenon, for flux compactifications to AdS. Note however that the same fact is true for the M-theory $AdS_3 \times M_8$ geometries analyzed in [3].

One can give a more detailed description of these bundles – quite intriguely these are very closely related to the twistor spaces, as reviewed by C. LeBrun in his lecture [4].

In particular, M_6 can be viewed as the total space of the bundle of unit self-dual two-forms over M_4. Here one thinks of each $\mathbb{P}^1$ fibre as being a unit sphere in $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$ with the factor of $\mathbb{R}$ being the polar direction on the $\mathbb{P}^1$. The $\mathbb{P}^1$ bundle may then be viewed as the unit sphere bundle in an $\mathbb{R}^3$ bundle, with the transition functions acting only in the $\mathbb{R}^2 = \mathbb{C}$ part of the fibre. The rank 3 real bundle thus splits into a direct sum $\mathcal{O} \oplus \mathcal{L}_{\mathbb{R}}$ of a trivial real line bundle $\mathcal{O}$, and (the realisation of) the complex canonical line bundle $\mathcal{L}$. Recall that the two-forms on M_4 decompose into self-dual and anti-self-dual two-forms:

$$\Lambda^2 M_4 \cong \Lambda^+ M_4 \oplus \Lambda^- M_4 \ . \tag{2.4}$$

and these further decompose (since M_4 is Kähler) as

$$\begin{aligned} \Lambda^+ M_4 &\cong \mathbb{R}[J] \oplus \mathcal{L}_{\mathbb{R}} \\ \Lambda^- M_4 &\cong \Lambda_0^{1,1} M_4 \ . \end{aligned} \tag{2.5}$$

Here $\Lambda_0^{1,1} M_4$ denotes the bundle of primitive $(1,1)$-forms *i.e.* 2-forms which are orthogonal to the Kähler form J, and are invariant under the action of the complex structure. Thus we see that the bundle of self-dual two-forms splits as $\Lambda^+ M_4 \cong \mathcal{O} \oplus \mathcal{L}_{\mathbb{R}}$ where $\mathcal{O}$ is a trivial real line bundle generated by the Kähler form on M_4. It is clear that the $\mathbb{R}^3$ bundle over M_4 associated with our metrics is in fact the bundle of self-dual two-forms.

It is quite tempting now to identify the bundles as twistor spaces. We will show momentarily that this is not quite correct. However, first we have to recall a couple of useful theorems.

Lemma[2]: at any fixed y the Ricci tensor $R_{ij}(M_4)$ on M_4 has two pairs of constant eigenvalues.

Proof: Let $\Re = \mathrm{d}_4 P$ be the Ricci form of M_4. Using the "dynamical" equations (1.12), we have that

$$\Re = -\frac{3}{2y} \partial_y J \qquad\qquad \Re^+ = 3m^2 \mathrm{e}^{-6\lambda} \sec^2 \zeta \, (1 + 6y \partial_y \lambda) \, J \tag{2.6}$$

hence the scalar curvature of M_4 ($R \equiv R_{ij} g^{ij}$) is constant

$$\mathrm{d}_4 R = 0 \ . \tag{2.7}$$

Moreover, using the fact that $\partial_y P = \partial_y J_i{}^j = 0$ we find

$$R_{ij} = -\frac{3}{2y} \partial_y g_{ij} \ , \qquad \partial_y R_{ij} = 0 \ . \tag{2.8}$$

Note that the first equation is called a "Ricci-flow" in the maths literature. Collecting all these simple facts, it is trivial to check that

$$R_{ij}R^{ij} = \frac{3}{2y}\partial_y R \tag{2.9}$$

and hence

$$\mathrm{d}_4(R_{ij}R^{ij}) = 0\ . \tag{2.10}$$

Thus concluding the proof.

We can now feed the result of this lemma into the following theorem, in order to list all the possibilities for the Kähler base four-manifold. We have

Theorem [Th. 2 of [5]]: A compact Kähler four-manifold whose Ricci tensor has two distinct pairs of constant eigenvalues is locally the product of two Riemann surfaces of (distinct) constant curvature. If the eigenvalues are the same the manifold is Kähler–Einstein.

The compactness in the theorem is essential[4], since there exist non-compact counterexamples. However, for AdS/CFT purposes, we are most interested in the compact case (for example, the central charge of the dual CFT is inversely proportional to the volume).

Using this, we can write down simple ansatze for the dependence of the Kähler metric on y. Moreover, quite remarkably, it turns out that the resulting equations are simple enough that one can integrate them in closed form. The metric functions we obtain in this way will depend on certain integration constants. The last step we have to carry out is to check for which (if any) ranges of these integration constants the metrics are complete, and non singular metrics on the total space of the $\mathbb{P}^1$ bundles described above.

In summary, we have found that the following cases yield non-singular metrics:

i) **Kähler–Einstein case:**

The metric takes the form

$$\mathrm{d}s^2 = \tfrac{1}{3}\mathrm{e}^{-6\lambda}\left(1-y^2\right)\mathrm{d}\tilde{s}^2(M_4) + \mathrm{e}^{-6\lambda}\sec^2\zeta\mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta(\mathrm{d}\psi + \tilde{P})^2 \tag{2.11}$$

where $\mathrm{d}\tilde{s}^2(M_4)$ is a *y-independent* KE metric on the four-dimensional base satisfying $\tilde{\mathfrak{R}} = \mathrm{d}_4\tilde{P} = \tilde{J}$, with e.g.

$$\mathrm{e}^{6\lambda} = \frac{2m^2(1-y^2)^2}{cy+2+2y^2}\ . \tag{2.12}$$

We find that:

> *For $0 \leq c < 4$ we have a one-parameter family of completely regular, compact, complex solutions with the topology of a $\mathbb{P}^1$ fibration over a positive curvature KE space.*

[4]It is also assumed that the Goldberg conjecture on almost Kähler manifolds is true.

Since four-dimensional compact Kähler-Einstein spaces with positive curvature have been classified [6, 7], we have a classification for the above solutions. In particular, the base space is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$, or $\mathbb{C}P^2 \#_n \mathbb{C}P^2$ with $n = 3, \ldots, 8$. For the first two examples, the KE metrics are of course explicitly known and this gives explicit solutions of M-theory when fed into the above solutions.

ii) **Product base**

Now we consider the case where the base is a product of two constant curvature Riemann surfaces. Here the metric has the following form

$$\begin{aligned} \mathrm{d}s^2 = \tfrac{1}{3}\mathrm{e}^{-6\lambda}\left(a_1 - k_1 y^2\right)\mathrm{d}\tilde{s}^2(C_1) + \tfrac{1}{3}\mathrm{e}^{-6\lambda}\left(a_2 - k_2 y^2\right)\mathrm{d}\tilde{s}^2(C_2) \\ + \mathrm{e}^{-6\lambda}\sec^2\zeta \mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta\left(d\psi + \tilde{P}\right)^2 \end{aligned} \quad (2.13)$$

where the *y-independent* metrics $\mathrm{d}\tilde{s}^2(C_i)$ describe constant curvature Riemann surfaces with curvature $k_i \in \{0, \pm 1\}$. In other words the metrics on C_i are the standard ones on either tori T^2, spheres $\mathbb{P}^1$ or hyperbolic spaces[5] H^2.

First suppose that one of the Riemann surfaces is the flat torus. In this case, non-singular solutions arise only if the second Riemann surface is a $\mathbb{P}^1$. The general solution for the warp factor is then:

$$\mathrm{e}^{6\lambda} = \frac{2m^2(a - y^2)}{1 - cy} \quad (2.14)$$

where a, c are integration constants. A detailed analysis reveals that:

> *For $0 < a < 1$ and $c \neq 0$ we have a one-parameter family of completely regular, compact, complex solutions that are topologically trivial $\mathbb{P}^1$ bundles over $\mathbb{P}^1 \times T^2$. A single additional solution of this type is obtained when $c = 0$ and $a \neq 0$.*

These will be the focus for the second part of the talk. The remaining non-singular solutions that we have found are $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \to \mathbb{P}^1 \times H^2$. More details can be found in [2].

Before turning to the second main topic of the talk, let us come back to an issue risen earlier, namely how do the $\mathbb{P}^1$ bundles that we constructed differ from the seemingly identical twistor space construction. Recall the following classical theorem, as reviewed in C. LeBrun's lecture [4]:

Theorem [8]: If $(M_4, [g])$ is a self-dual (respectively, anti-self-dual) conformal Riemannian manifold, its *twistor space* is a complex 3-manifold Z whose underlying

[5]Note that although we assumed compactness, H^2 factors naturally arise as solutions of the equations. These are clearly non-compact. However we can in principle quotient by some group Γ to obtain compact higher genus Riemann surfaces. In this case one needs to check that this operation doesn't break supersymmetry.

6-manifold is the total space of the sphere bundle of the rank-three real vector bundle of anti-self-dual (respectively, self-dual) 2-forms.

Recall that by definition, a self-dual (SD) Riemannian manifold is such that its Weyl tensor obeys $W = *W$. Conversely, for an anti-self-dual (ASD) manifold $W = -*W$.

There is another theorem which we need to recall here:

Theorem [8]: A Kähler four-manifold M_4 is ASD (with respect to the standard orientation) iff its scalar curvature vanishes.

Now it's easy to see that, since in our solutions the scalar curvature does not vanish, the corresponding M_4 four-manifolds can not be ASD, in particular they may be SD[6]. Thus we conclude that the $\mathbb{P}^1$ bundles in question are not twistor spaces. Notice that we can't simply reverse the orientation of M_4, as this operation would swap the meaning of self-duality for the Weyl tensor and the two-forms simultaneously.

3 Dualization to type IIB

For the rest of the talk we will concentrate on the solutions in the second class, where M_4 is taken to be $T^2 \times \mathbb{P}^1$. The metric on M_6 is now, schematically

$$ds_6^2 \sim ds^2(T^2) + ds^2(\text{base } \mathbb{P}^1) + ds^2(\text{fiber } \mathbb{P}^1) \tag{3.1}$$

and its isometry group is clearly

$$U(1)^2 \times SU(2) \times U(1)_R \tag{3.2}$$

where each factor acts on the corresponding metric block, and we have denoted $U(1)_R$ the isometry corresponding to the $\partial/\partial\psi$ Killing vector, which makes up the azimutal coordinate on the fiber $\mathbb{P}^1$.

We would like now to perform a reduction to type IIA and subsequently a T-duality to type IIB theory, thus we have to decide which $U(1)$'s to pick. The point here is that the choice is not arbitrary, and it is dictated by requiring supersymmetry to be preserved by these two operations.

The following example illustrates why one should be concerned with this issue.

Example: $AdS_5 \times S^5$ is notoriously a (maximally) supersymmetric solution of type IIB supergravity, when supplemented with appropriate self-dual five-form flux. Now, the five-sphere can be viewed as an S^1 bundle in the following way $S^1 \to S^5 \to \mathbb{C}P^2$, where the fibered $S^1 \simeq U(1)$ is an isometry. One can then T-dualize along this direction, thus "untwisting" the fibration, and obtaining a type IIA solution of the type $AdS_5 \times \mathbb{C}P^2 \times S^1$, with appropriate fluxes. However, this is *not* a supersymmetric solution of type IIA supergravity.

[6] For instance $\mathbb{P}^2$ with its Einstein metric is SD, while $\mathbb{P}^1 \times \mathbb{P}^1$ is neither SD nor ASD.

The reason why one breaks supersymmetry in this example is that the Killing spinors are not invariant along the S^1 we are using in the T-duality. In more physical language, we say that they are "charged" with respect to the corresponding $U(1)$. Now, the $U(1)$ in question is part of the field theoretical R-symmetry group, which in this case is $SO(6) \simeq SU(4)$, and the reason why the Killing spinors are charged is because they correspond to the supercharges of the dual $\mathcal{N} = 4$ superconformal field theory.

Coming back to our solutions, one can convince oneself that the Killing spinors are by construction charged with respect to the $\partial/\partial\psi$ direction, which was indeed denoted $U(1)_R$ above. As these solutions correspond to $\mathcal{N} = 1$ supersymmetry in the dual field theory, this is indeed the full R-symmetry group. In conclusion, if we want to preserve supersymmetry in the process, we must perform the dualization along the T^2, whose crucial role now becomes clear. Following the standard rules, we are lead to the following chain:

$$
\begin{array}{ccccc}
\text{M-theory} & \text{reduction} & \text{Type IIA} & \text{T-duality} & \text{Type IIB} \\
AdS_5 \times M_6 & \longrightarrow & AdS_5 \times X_4 \times S^1 & \longrightarrow & AdS_5 \times X_5 \\
G \text{ flux} & & \text{RR and NS flux} & & \Phi\text{=const, } F_5 \sim \text{vol}(AdS_5)
\end{array}
$$

Let us write down the resulting IIB metric:

$$
\begin{aligned}
ds^2(X_5) &= \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}[d\psi - \cos\theta d\phi]^2 \\
&+ w(y)\,[d\alpha + A]^2
\end{aligned} \tag{3.3}
$$

with

$$
\begin{aligned}
w(y) &= \frac{2(a-y^2)}{1-cy} \\
q(y) &= \frac{a-3y^2+2cy^3}{a-y^2} \\
A &= \frac{ac-2y+y^2c}{6(a-y^2)}[d\psi - \cos\theta d\phi] \; .
\end{aligned} \tag{3.4}
$$

Recall that we started from a $\mathbb{P}^1$ bundle over $\mathbb{P}^1 \times T^2$ and we came down along the T^2. On the first line of (3.3) one recognizes the remnant of this bundle, namely a $\mathbb{P}^1 \to \mathbb{P}^1$, while the second line arises from T-dualizing one circle in T^2. We will turn to a detailed discussion of these metric momentarily.

We now face a potential "puzzle": if the reduction hasn't broken any supersymmetry, as we claimed, then we seem to have a supersymmetric solution of type IIB supergravity, of the type $AdS_5 \times X_5$ – for these, it is well known that X_5 must be a Sasaki–Einstein manifold. So, before proceeding further, it will useful to review some basic aspects of Sasaki–Einstein geometry.

4 Sasaki–Einstein geometry

There are a few different ways to define a Sasaki–Einstein structure on an odd dimensional manifold. The original definition involved a specific type of contact structure – we will come back to it shortly. The handiest definition is perhaps the following

Definition/proposition: A manifold X admits a Sasaki–Einstein structure if and only if its metric cone $\mathcal{C}(X)$ is a (non-compact) Calabi–Yau manifold.

A metric cone is simply the naive metric

$$ds^2(\mathcal{C}) \;=\; dr^2 + r^2 ds^2(X) \; . \tag{4.1}$$

Examples:

i) X is a sphere S^{2n-1} with its round metric $\rightarrow \mathcal{C}(X) = \mathbb{R}^{2n}$

ii) $X = T^{11} \simeq S^2 \times S^3 \rightarrow \mathcal{C}(X)$ =singular conifold (defined by $uv = xy$)

Both these metric cones are well known Calabi–Yau metrics. In general, there is always a natural CY metric associated to any SE manifolds. (These are the CY's in title of the talk).

The geometric data of a SE structure are summarized as follows:

- a unit-norm Killing Vector $V = \frac{\partial}{\partial \phi}$
- its dual one-form η
- a two-form K such that (in a conventional normalization) $\mathrm{d}\eta = 2K$

One important aspect of SE manifolds is that they admit "Killing" spinors[7] obeying (in a conventional normalization)

$$\nabla_m \epsilon \;=\; \frac{i}{2}\gamma_m \epsilon \; . \tag{4.2}$$

Note that this condition follows easily from the reduction (along r) of the covariantly constant spinors on the associated Calabi–Yau. The existence of these spinors is the reason why, when combined with AdS_5 and appropriate five-form flux, Sasaki–Einstein manifolds provide supersymmetric backgrounds of type IIB supergravity

Fact: $AdS_5 \times X_5$ (with non-trivial RR F_5) where X_5 is a Sasaki–Einstein manifold, is a supersymmetric solutions of type IIB supergravity, preserving 8/32 supersymmetry.

[7]Note that in the mathematical literature "Killing" spinors are by definition those obeying equation (4.2) above, whereas in supergravity one denotes loosely as "Killing" spinors any spinor obeying a given supersymmetry equation.

4.1 "Transverse geometry"

In order to appreciate the results presented in [9, 10], we will need to introduce a finer characterization of Sasaki–Einstein manifolds. This is achieved using the Killing vector V, which naturally induces a $(2n)+1$ split of the geometry. The properties of this vector, or equivalently, of the $2n$-dimensional part, then characterize more finely the geometry.

Specifically, V defines a foliation, whose leaves are the $2n$-dimensional part we alluded to above. Note that this is a *local* concept. To be concrete, let us introduce a metric adapted to the Killing vector $V = \partial/\partial\phi$. In these local coordinates, the metric takes the form

$$ds^2(\mathrm{SE}) \;=\; ds^2_{2n} + (d\phi + \eta)^2 \tag{4.3}$$

and it turns out (in fact just using the geometric data recalled) that ds^2_{2n} is *locally* a Kähler–Einstein metric, with Kähler form K. Recall that the Einstein condition means that the Ricci tensor is proportional to the metric: $R_{ij} = sg_{ij}$, for some constant s. The exact nature of the KE leaves, determines the type of SE manifold, in the following way:

Regular. In this case (ds^2_{2n}, K) is a positive-curvature Kähler–Einstein *manifold* M_{2n}. Moreover, the orbits of the Killing vector V close, so that there is a well-defined $U(1)$ action. This action is also free, so that taking the quotient doesn't give rise to fixed points. We have that $M_{2n} = X_{2n+1}/U(1)$ and the Sasaki–Einstein manifold is then a circle bundle over M_{2n}, that is $U(1) \to M_{2n}$.

Quasi-regular. Here one still has a $U(1)$ action as the orbits of V close. However now taking the quotient one introduces orbifold singularities. In this case (ds^2_{2n}, K) defines a positive-curvature Kähler–Einstein *orbifold* $\tilde{M}_{2n}$. Similarly to the regular case, we have that $\tilde{M}_{2n} = X_{2n+1}/U(1)$ and the total space is the circle orbi-bundle $U(1) \to \tilde{M}_{2n}$.

Irregular. This case is the less intuitive one, however it is also the generic one. Here the orbits of V *do not* close so that there isn't a $U(1)$ action associated to the canonical Killing vector V. In this case (ds^2_{2n}, K) is simply not globally defined and the operation "$M_{2n} = X_{2n+1}/U(1)$" doesn't make sense.

It is important to remark that in all three cases the Sasaki–Einstein spaces are smooth compact manifolds, equipped with complete metrics, irrespective of the transverse geometry. In particular, in the non-regular cases, the singularities of the "base" and of the "fibration" compensate in a non trivial way.

A useful way to understand the nature of the irregular geometries is as follows. One can show [11] that irregular SE manifolds admit a $T^2 \simeq U(1)\times U(1)$ subgroup of isometries. Normalize this to be a square torus, and denote A and B its generators. Now, the canonical Killing vector V must be embedded in the T^2, hence we have

$$V \;=\; rA + sB \tag{4.4}$$

for some coefficients r, s. The point is that these coefficients need not be rational numbers. In particular, if these are irrationals, the orbit of V will densely cover the torus[8].

Let us now have a glance at the state of the art about the construction of Sasaki–Einstein manifolds. We concentrate here on five-dimensional manifolds.

Regular. Using the fact that they are in one-to-one correspondence with four-dimensional KE manifolds, these are completely classified using the results of Tian and Yau [6, 7]. In fact the possibilities are the following:

1) $S^1 \to S^5 \to \mathbb{C}P^2$ (and $S^5/\mathbb{Z}_3$)

2) $S^1 \to T^{11} \to \mathbb{P}^1 \times \mathbb{P}^1$ (and $T^{11}/\mathbb{Z}_2$)

3) S^1 bundles over del Pezzo surfaces P_k with $k = 3 \ldots 8$

The metrics are of course known explicitly only for the first two examples.

Quasi-regular. Until recently there were no examples known. Using algebraic geometric techniques some examples have been constructed recently (see e.g. [12] for a review). These include Sasaki–Einstein metrics on connected sums $l\#S^2 \times S^3$, with $l = 1$ and higher. In particular, it is shown in [13] that there exist 14 inequivalent quasi-regular SE metrics on $S^2 \times S^3$. However none of these metrics are known explicitly.

Irregular. The algebraic geometric techniques mentioned above, can only produce quasi-regular examples. So, *no* examples of irregular SE metrics were known to exist.

So, the natural question arises as to how do the metrics on X_5 that we found fit in this picture. Perhaps the metrics are singular and therefore meaningless? Or perhaps they are diffeomorphic to some of the few examples known?

Luckily enough, after careful analysis, it will turn out that in fact these are *new* explicit, non-regular, Sasaki–Einstein metrics!

5 Back to the type IIB solutions

In the following we will answer to the positive the following question:

Question: Are the metrics $ds^2(X_5)$ "honest" Sasaki–Einstein metrics?

The strategy to understand these solutions splits in two parts: I) local analysis and II) global analysis.

[8] This case is called irregular of rank 2. The rank n is the number of generators of a given square T^n in terms of which V can be written.

The local analysis consists in making sure that the metrics admit local Killing spinors of the type recalled. This is guaranteed by the fact that we started with a supersymmetric solution of M-theory and we have reduced an T-dualized along "supersymmetric" directions, i.e. the flat two-torus. The most direct way to confirm this fact, is to find a change of coordinates which brings the metric in the canonical Sasaki–Einstein form (4.3). This is readily achieved posing $6\alpha = -\beta - \psi'$, $\psi = \psi'$ in (3.3). Moreover, after setting $\rho^2 = 2(1-y)/3$, the local four-dimensional metric looks like

$$\begin{aligned} ds_4^2 &= \frac{1}{\Delta}d\rho^2 + \frac{\rho^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \Delta\sigma_3^2\right) \\ \Delta &= 1 + \frac{4(a-1)}{27}\frac{1}{\rho^4} - \rho^2 \end{aligned} \tag{5.1}$$

where σ_i are the usual one-forms on S^2, i.e. $\mathrm{d}\sigma_1 = \sigma_2 \wedge \sigma_3$, etc. It was shown in [14] that these metrics are indeed locally KE for any $0 < a \leq 1$. However, it was also shown there that they are singular, unless $a = 1$, for which they reduce to the Einstein metric on $\mathbb{C}P^2$.

Let us now turn to the (more interesting) global analysis. As just shown, the metric written in the canonical form, seems to be singular. However, now we have an option, that is to go back to the original coordinates in which the metric was obtained. Recall that it arose from dualizing a $\mathbb{P}^1 \to \mathbb{P}^1 \times T^2$ bundle in M-theory. In fact, it's easy to convince oneself that the metric on the first line of (3.3) obtained "forgetting" the T^2 is just a $\mathbb{P}^1 \to \mathbb{P}^1$ bundle. We can be more precise. Recalling that these bundles are classified by $\pi_1(SO(3)) = \mathbb{Z}_2$, and calculating the corresponding Chern number to be 2, we conclude that it is a topologically trivial bundle. As a complex surface, this is called second Hirzebruch surface, and is denoted

$$\mathbb{F}_2 = P(\mathcal{O} \oplus \mathcal{O}(-2)) \simeq_{\text{topologically}} S^2 \times S^2 \ . \tag{5.2}$$

We have then established the existence of a globally well defined four-dimensional metric hidden in the full five-dimensional one. The hope is now to use this as a base of $U(1)$ fibration.

In fact, α is a coordinate on a circle S^1 and the full space is indeed a $U(1)$ bundle (where α is the fiber coordinate) over $\mathbb{F}_2$ if A can be made into a *connection*. This is true if the integrals of its curvature over a basis of two-cycles on $\mathbb{F}_2$ are *integers*:

$$\frac{1}{2\pi}\int_{\Sigma_i} \frac{\mathrm{d}A}{\ell} = p, q \tag{5.3}$$

with Σ_i a basis for the second homology group $H_2(\mathbb{F}_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, and ℓ a suitable coefficient.

A detailed analysis reveals that it is possible to chose the parameter a appropriately to satisfy this requirement, for any integer $p > q$. We therefore conclude that we constructed infinitely many (labelled by two integers p and q) new Sasaki–Einstein metrics on $U(1)$ bundles

$$U(1)_{p,q} \to Y^{p,q} \to S^2 \times S^2 \ . \tag{5.4}$$

Let us conclude with a few remarks:

The global topology. It is possible to show that if p, q are co-prime, then

$$Y^{p,q} \simeq S^2 \times S^3 \tag{5.5}$$

while a common factor h implies that the topology is simply a quotient of this by $\mathbb{Z}_h$.

A close relative. The attentive reader should have noticed by now the striking similarity with the construction of Einstein metrics on $S^2 \times S^3$, denoted in the physics literature $T^{p,q}$. Indeed these metrics arise as $U(1)$ bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ (with its round direct product metric), and are topologically $S^2 \times S^3$ for p, q coprime. Schematically

$$Y^{p,q} \simeq_{\text{as bundles}} T^{p,q} \ . \tag{5.6}$$

The main difference is that the metrics $Y^{p,q}$ are also Sasaki, therefore admit Killing spinors, while the $T^{p,q}$ metrics are not, except for T^{11}. Correspondingly, the metric cones are Calabi–Yau for $Y^{p,q}$, while they are Ricci-flat, but *not* Kähler for $T^{p,q}$.

Non-regular metrics. It is obvious that these metrics cannot be regular, in the sense reviewed above. They are in fact generically *irregular*. Moreover, it turns out that the metrics are quasi-regular if a certain quadratic Diophantine equation is satisfied, namely

$$4p^2 - 3q^2 \ = \ n^2 \tag{5.7}$$

where n is an integer. It is possible to show that there are infinitely many solutions to this equation.

Volumes. It is straightforward to compute a volume formula in terms of p and q. This reads

$$\mathrm{vol}(Y^{p,q}) \ = \ \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}\pi^3$$

It then turns out that in the general, irregular case, the volumes are given in terms of square roots of rational numbers, while they are rational if and only if the metrics are quasi-regular. We thus uncover[9] a first surprising feature of irregular SE metrics: their volumes are generically irrational!

6 AdS/CFT and $\mathcal{N} = 1$ SCFT

We will close in this section with a brief discussion of the most immediate physical application for the new Sasaki–Einstein metrics we discovered.

[9]Recall that these are the first examples of irregular metrics which have appeared in the literature. For quasi-regular metrics one can prove that the volumes must be rational numbers. This is essentially because of the existence of a four-dimensional base Kähler–Einstein orbifold.

According to the AdS/CFT correspondence, type IIB supergravity on $AdS_5 \times X_5$, where X_5 is Sasaki–Einstein, should be dual, in the large N limit, to a $\mathcal{N} = 1$ super conformal field theory (SCFT), arising on N D3-branes placed at the tip of the Calabi–Yau cone $\mathcal{C}(X_5)$.

The isometries of X_5 then correspond to global symmetries of the SCFT. More precisely, the isometry group will be in general

$$\mathcal{F} \times U(1)_R \tag{6.1}$$

where the first factor is a "flavor" symmetry and the second is the R-symmetry. In addition, the SCFT will be characterized by a central charge c which is inversely proportional to the volume of X_5

$$c \sim \frac{1}{\mathrm{vol}(X_5)} \ . \tag{6.2}$$

The simplest example (after $X_5 = S^5$) of this construction is given by the conifold[10] "$uv = xy$", whose dual SCFT was identified in [16]. In fact, it's fear to say that this is the *only* example!

There are many ways to arrive at a guess for the SCFT, which are based on a detailed knowledge of the conifold, as well as of the corresponding Sasaki–Einstein metric T^{11}. In particular, the gauge group and the field content are most easily inferred from the defining equations for the conifold, and it turns out that the complete theory is a quiver SCFT, with gauge group $SU(N) \times SU(N)$ with bi-fundamental chiral fields and a suitable superpotential. Moreover, the theory has a global symmetry group inherited from the isometries of T^{11}, which is $SU(2) \times SU(2) \times U(1)_R$, and a central charge $c = 27/16$.

In order to appreciate the implications of our supergravity results in the context of the AdS/CFT correspondence, it is useful to summarize some recent results on general $\mathcal{N} = 1$ SCFT in four dimensions due to [17]. In [17] the authors presented a method for determining uniquely the R-symmetry of an $\mathcal{N} = 1$ SCFT, using a procedure dubbed "a-maximization". Recall that a SCFT usually arises as the infrared fixed point of some (possibly Lagrangian) field theory, and it is far from obvious to determine the various R-charges, or the anomalous dimensions of the operators in the chiral ring.

Recalling that the central charges are given in terms of the R-symmetry generators by

$$a = \frac{3}{32}(3\mathrm{Tr}\, R^3 - \mathrm{Tr}\, R) \qquad\qquad c = \frac{1}{32}(9\mathrm{Tr}\, R^3 - 5\mathrm{Tr}\, R) \ , \tag{6.3}$$

the procedure advocated in [17] is implemented as follows: consider a "trial" R-symmetry

$$R_t = R_0 + \Sigma_I s^I F_I \tag{6.4}$$

[10] See for instance Vafa's lectures in this workshop for a nice introduction to conifolds [15].

where R_0 is a possible R-symmetry assignment, F_I are generators of $U(1)$ factors contained in $\mathcal{F}$, and s^I are a priori real coefficients.

"a-maximization" [17]: the exact R-symmetry is the one that maximazes (locally) the central charge a. In particular it is determined by solving the equations

$$9\mathrm{Tr}\,(R^2 F_I) = \mathrm{Tr}\,F_I \qquad\qquad \mathrm{Tr}\,(RF_I F_J) < 0\ . \tag{6.5}$$

One of the consequences of this result is that

Fact: all $\mathcal{N}=1$ four-dimensional SCFT's are *"algebraic"*, i.e. their R- and central charges are square roots of rational numbers.

Let us finally come back to the new Sasaki–Einstein metrics $Y^{p,q}$ and discuss them in the light of these results. The full isometry group of the metrics is

$$SU(2)\times U(1)\times U(1). \tag{6.6}$$

Recall that the T^2 action indeed played a fundamental role in the construction of the metrics. In fact, although the geometry naturally picks a Killing vector, the canonical SE vector, which corresponds to the R-symmetry of the SCFT, this generically can not be integrated to a $U(1)$ action. So generically (for irregular metrics) the R-symmetry will be *non compact*, since the corresponding generator has non compact orbits - namely it densely covers a two-torus. Quite remarkably, the corresponding central charges (which are inversely proportional to the volumes) will be generically irrational, and rational iff the R-symmetry group is compact.

So, the CFT predictions of our metrics, using the AdS/CFT dictionary, are in perfect agreement with the results reported in [17]! We can summarize the situation in the following diagram

$$\begin{array}{llcl}
\textbf{Geometry:} & Y^{p,q} \text{ is } \textit{irregular} & \Leftrightarrow & \mathrm{vol}(Y^{p,q}) \text{ is } \textit{irrational} \\
 & \Updownarrow & & \Updownarrow \\
\textbf{SCFT:} & \textit{non-compact } R\text{-symmetry} & \Leftrightarrow & \textit{irrational} \text{ central charges } a, c
\end{array}$$

In conclusion, we have reported on a completely new mathematical construction of (in-homogeneous) Sasaki–Einstein metrics. This generalizes to arbitrary dimensions, as described in [10] and it is conceivable that the general principles underlying it can be applied to the construction of other types of special geometries, like 3-Sasakian, or G_2 metrics.

Moreover, using the AdS/CFT correspondence, the new metrics on $S^2\times S^3$ are expected to correspond to a family of $\mathcal{N}=1$ SCFT's arising at the singularity of the corresponding Calabi–Yau cones. It will be of extreme interest to find an appropriate description of these non-compact Calabi–Yau's, and to shed light on the nature of the singularities.

Of course, from the physical point of view, the ultimate challenge will be to fully uncover the details of dual conformal field theories.

Note:

I originally planned to conclude the talk with a description of how the five-dimensional construction of [9] has been generalized in [10] to any dimension, thus providing at once infinitely many *irregular* Sasaki–Einstein metrics in arbitrary dimensions. Due to lack of time, this topic was only briefly mentioned, and so I will not include it in these notes.

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