$N = 2$ Supersymmetry and Twistors

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Abstract

In this lecture, I describe how twistors arise in $N = 2$ superspace and $N = 2$ σ-model geometry, and apply these ideas to prove a simple theorem about Calabi-Yau supermanifolds

Introduction

As Ulf Lindström explained in his lecture, there is a close link between supersymmetry on the world-volume and target space geometry. I will use this in the context of $N = 2$ supersymmetry on a four dimensional world-volume and stumble upon the twistor space that Claude LeBrun described in his lectures.

The plan of my talk is as follows: After this introduction, I will give

(1) A lightning review of the projective superspace approach to $N = 2$ supersymmetry. (for a fairly complete review, see Appendix B of hep-th/0101161).

(2) A lightning review of hyperkähler geometry.

Combining these leads us to twistors for the first time.

(3) A superficial review of some aspects of $N = 2$ supergravity and a fairly detailed description of Quaternion Kähler (QK) geometry (which is usually not Kähler or even almost complex). Many more details can be found in hep-th/0101161.

(4) A review of some aspects of the conformal formalism for $N = 2$ supergravity and and the hyperkähler cone (HKC) or Swann space of a QK manifold.

Combining these we once again find twistors.

(5) Finally, I will prove a theorem: For any HKC with an equal number of bosonic and fermionic dimensions, the corresponding twistor space is super Calabi-Yau.

1. $N = 2$ supersymmetry in D=4 and projective superspace

In four dimensions, N=2 superspace is characterized by 2 chiral spinor derivatives $D_{a\alpha}$ and their complex conjugates $\bar{D}^a_{\dot{\alpha}}$:

$$\{D_{a\alpha}\,,\,\bar{D}^b_{\dot{\beta}}\} = i\delta_a{}^b\partial_{\alpha\dot{\beta}} \quad , \qquad \{D_{a\alpha}\,,\,D_{b\beta}\} = \{\bar{D}^a_{\dot{\alpha}}\,,\,\bar{D}^b_{\dot{\beta}}\} = 0\ . \tag{1}$$

We may define a maximal; graded Abelian subalgebra analogous to a chiral subspace in $N = 1$ superspace by introducing an *iso*spinor u^a:

$$\nabla_\alpha(u) \equiv u^a D_{a\alpha}\ ,\quad \bar{\nabla}_{\dot{\alpha}}(u) \equiv \epsilon_{ba}u^a\bar{D}^b_{\dot{\beta}} \quad \Rightarrow \quad \{\nabla_\alpha\,,\,\bar{\nabla}_{\dot{\beta}}\} = 0\ ; \tag{2}$$

it is natural to think of the two components of u^a as homogeneous coordinates on $\mathbb{P}^1$. Focusing on the group theoretic properties of this sphere leads to harmonic superspace; we instead focus on the analytic properties, and are led to projective superspace. Choosing the inhomogenous coordinate $\zeta = -u^1/u^2$, we introduce the projectivized spinor derivatives

$$\nabla_\alpha(\zeta) = D_{2\alpha} - \zeta D_{1\alpha} \quad , \qquad \bar{\nabla}_{\dot{\alpha}}(\zeta) = \bar{D}^1_{\dot{\alpha}} + \zeta\bar{D}^2_{\dot{\alpha}}\ ; \tag{3}$$

these define the graded Abelian subspace of $N = 2$ superspace that Ulf Lindström and I have studied with a variety of collaborators for the last 20 years. Note that ∇ and $\bar{\nabla}$ are related (projectively) by the real structure that consists complex conjugation composed with the antipodal map on the sphere:

$$\nabla = -\zeta\bar{\nabla}^\dagger\left(-\frac{1}{\zeta}\right)\ . \tag{4}$$

We can choose superfields analogous to $N = 1$ chiral superfields that are annihilated by the full graded Abelian subalgebra generated by $\nabla, \bar{\nabla}$; these give off-shell realizations of $N = 2$ supersymmetric systems whose precise physical content depends on the details of their ζ-dependence; we can get vector multiplets, tensor multiplets, or various off-shell variants of a hypermultiplet. This is a fascinating subject in its own right, but that is a different lecture; all I really need is the crucial statement that $N = 2$ superspace in four dimensions has a maximal graded Abelian subspace parametrized by a $\mathbb{P}^1$.

2. Hyperkähler geometry

Freedman and Alvarez-Gaumé showed that σ-models with $N = 2$ supersymmetry that describe maps from a four dimensional Minkowski space exist only if the target space $\mathcal{T}$ is hyperkähler. That means that $\mathcal{T}$ admits three globally defined integrable complex structures obeying the algebra of the quaternions: $IJ = -JI = K$ and cyclic permutations, with $I^2 = J^2 = K^2 = -1$.

Furthemore, the metric is hermitian with respect to I, J, K and the connection preserves all three complex structures: $\nabla I = \nabla J = \nabla K = 0$; that is, the metric is Kähler with respect to all three complex structures. Then for any three numbers a, b, c obeying $a^2 + b^2 + c^2 = 1$, the combination $aI + bJ + cK$ is also a complex structure and defines a Kähler structure on $\mathcal{T}$: $(aI + bJ + cK)^2 = -1$, etc. Notice that a, b, c lie on a sphere $S^2 \simeq \mathbb{P}^1$. As Claude LeBrun explained in his talk, the space $\mathcal{T} \times \mathbb{P}^1$ (in the hyperkähler case, this is really a product) is the twistor space $\mathcal{Z}$ of the hyperkähler manifold $\mathcal{T}$; a point in $\mathcal{Z}$ consists of a point in $\mathcal{T}$ and a choice of complex structure. The twistor space $\mathcal{Z}$ is naturally a complex manifold, but it has no particularly canonical metric (we shall see why later).

Remarkably, the $\mathbb{P}^1$ of graded Abelian subspaces of $N = 2$ superspace is the same as the $\mathbb{P}^1$ of complex structures on a hyperkähler manifold $\mathcal{T}$, and hypermultiplet actions are naturally defined in the twistor space $\mathcal{Z}$ of $\mathcal{T}$. Explaining this in detail would bring me back to the lecture that I unfortunately don't have time for, but this is the first way that $N = 2$ supersymmetry "knows" about twistor space.

3. $N = 2$ supergravity and Quaternion Kähler geometry

In 1983, Bagger and Witten discovered a surprising fact: When hypermultiplets ($N = 2$ supersymmetric σ-models) are coupled to $N = 2$ supergravity, their target space geometry is *not* hyperkähler, but rather, it is Quaternion Kähler (QK). Since we haven't really discussed QK manifolds, let me spend a bit more time on them. In four dimensions, they are ASD (anti-self dual) Einstein manifolds, that is, their Weyl Tensor is anti-self dual. In general, they are *not* Kähler, or even complex or almost complex manifolds. They are characterized by having a quaternionic structure:

$$IJ = -JI = K \quad , \quad I^2 = J^2 = K^2 = -1 \; , \tag{5}$$

but I, J, K are not *globally* defined as tensors; rather, they are defined only locally as a section of a nontrivial bundle. Concretely, for $\vec{J} \equiv (I, J, K)$, we have[1]

$$\nabla \vec{J} = \vec{A} \times \vec{J} \; , \tag{6}$$

where $\vec{A}$ is an $SU(2)$ connection with curvature

$$\vec{F} = d\vec{A} + \vec{A} \times \vec{A} = -\frac{1}{2}\vec{\omega} \; , \tag{7}$$

and $\omega = g(\, . \, , \vec{J}) \;\Leftrightarrow \omega_{\mu\nu} = g_{\mu\rho} J^{\rho}{}_{\nu}$. This is *not* the same obstruction as the Nijenhuis tensor; an almost complex structure exists as a tensor on the whole manifold, it just doesn't define complex coordinates on the whole manifold, whereas no linear combination of I, J, K exists as a tensor on the whole QK manifold.

QK manifolds may have positive or negative cosmological constant:

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \; . \tag{8}$$

For $\Lambda > 0$, in four dimensions, the only complete compact examples are $S^4 \simeq \mathbb{H}P^1$ and $\overline{\mathbb{P}^2}$; the latter happens to be Kähler. There are many orbifold examples due to Galicki.

For $\Lambda = 0$, the manifold is hyperkähler, as discussed above.

For $\Lambda < 0$, there are many examples; coupling to $N = 2$ supergravity requires $\Lambda < 0$, as otherwise the sign of Newton's constant is wrong.

[1] To include the hyperkähler case, one may rescale the $SU(2)$ connection so that $\nabla \vec{J} = \Lambda \, \vec{A} \times \vec{J}$.

4. $N = 2$ Conformal formalism

Before describing the conformal formalism for $N = 2$ supergravity, let me explain some general features of compensators or Stückelberg fields.

Introduction to compensators

Compensators are fields that allow one to realize gauge symmetries in a system that doesn't actually have the symmetry. A familiar example is the scalar field used by Stückelberg to restore gauge invariance to massive QED (which is really the simplest example of the Higgs mechanism): he rewrote the mass term A_μ^2 as $(\partial_\mu\phi - A_\mu)^2$, which is gauge invariant under the gauge transformations $A_\mu \to A_\mu + \partial_\mu\lambda$ and $\phi \to \phi + \lambda$. Choosing the gauge $\phi = 0$, one recovers the mass term.

The most relevant examples for us are conformal compensators. In four dimensions, the Einstein action $\int \sqrt{g}R$ is not conformally invariant; under a Weyl rescaling $g_{\mu\nu} \to \phi g_{\mu\nu}$,

$$\int \sqrt{g}R \to \int \sqrt{g}(\phi^2 R - 6\phi\nabla^2\phi) \ . \tag{9}$$

Of course, this action is tautologically Weyl invariant under the transformation

$$g_{\mu\nu} \to \lambda g_{\mu\nu} \quad , \qquad \phi \to \lambda^{-1}\phi \ . \tag{10}$$

Here ϕ is the conformal compensator.

In $N = 1$ supergravity, there are different choices of compensator superfields. The most common and useful choice is a chiral superfield. It compensates super-Weyl transformations–these include component Weyl rescalings as well as component $U(1)_R$ axial rotations and conformal S-supersymmetry transformations. This gives rise to an $N = 1$ conformal formalism. The target space of σ-models in this formalism is not arbitrary, but is rather a cone over a Hodge manifold. This has direct parallels in N=2 supergravity.

$N = 2$ compensators

In $N = 2$ supergravity, the compensator is a hypermultiplet (there is also a vector multiplet compensator, which compensates $U(1)_R$ transformations and provides the physical graviphoton of the Poincaré theory, but it does not affect the geometry of the hypermultiplet σ-model target space). The compensating hypermultiplet has 4 physical scalar components, and these compensate component Weyl and $SU(2)_R$ transformations (actually, the Weyl transformations also act on the vector multiplet compensator, but this plays no role in our discussion). These compensators give rise to the $N = 2$ conformal formalism.

The symmetries of this formalism are extremely restrictive, and hypermultiplets that couple to it must have a very particular target space geometry: they must lie on a hyperkähler manifold that is a kind of cone above a Quaternion Kähler (QK) space. The mathematician Swann discovered this construction in 1991 starting from the QK, but for us it is more natural to write down the hyperkähler cone (HKC) and discover the QK space that lies beneath it. Along the way, we will discover the twistor space of the QK space.

An HKC or Swann space is a characterized as follows: It is a hyperkähler variety with a homothetic conformal killing vector χ:

$$D_A\chi^B = \delta_A^B \ . \tag{11}$$

This implies that the three vectors $\vec{X}^A \equiv \vec{J}^A{}_B\chi^B$ (*i.e.*, $\vec{X} \equiv \vec{J}(\chi)$ in index free notation) are killing vectors. They are also the vector fields that generate rotations of the three complex structures $\vec{J}$. Such vector fields exist (locally) on any hyperkähler variety, but usually they are not killing, that is, in general, rotations of the complex structures $\vec{J}$ are not generated by isometries; they are on an HKC (however, there are examples, such as the Taub-Nut metric, that do have such isometries but do not have a homothety and hence are not HKC's).

On an HKC, the homothety χ^A has many interesting properties: there exists a hyperkähler potential

$$\chi \equiv \frac{1}{2}\chi^A g_{AB}\chi^B \tag{12}$$

such that

$$\chi_A \equiv g_{AB}\chi^B = \partial_A\chi \quad \Rightarrow \quad g_{AB} = D_A\partial_B\chi \ , \tag{13}$$

which is like the Kähler condition but much stronger: In complex coordinates, for a Kähler metric,

$$g_{a\bar{b}} = \partial_a\partial_{\bar{b}}K \ , \tag{14}$$

with $g_{ab} \equiv g_{\bar{a}\bar{b}} \equiv 0$ by definition, and hence implying no restriction on K, whereas on an HKC

$$g_{a\bar{b}} = \partial_a\partial_{\bar{b}}\chi \quad , \quad g_{ab} = \nabla_a\partial_b\chi \equiv 0 \ . \tag{15}$$

As the HKC has an $SU(2)$ isometry, it can be thought of as an $SU(2)$ bundle. Furthermore, the relation $\vec{X}^A \equiv \vec{J}^A{}_B\chi^B$ implies $\chi^A = -J^{(\Pi)A}{}_B X^{(\Pi)B}$ for any particular complex structure $J^{(\Pi)A}{}_B$ and corresponding killing vector $X^{(\Pi)B}$; that is, the homothety and any of the $X^{(\Pi)B}$ define the complexified action of a $U(1)$ subgroup of the $SU(2)$ generated by all the $\vec{X}$; this is precisely the situation that arises in symplectic reduction (Kähler quotients).

At the request of a some participants, a brief review of Kähler and hyperkähler quotients is attached in an Appendix.

Thus, if we take the Kähler quotient of the HKC, we get a Kähler manifold $\mathcal{Z}$ with one complex dimension less than the HKC. The manifold $\mathcal{Z}$ has the structure of a $\mathbb{P}^1$ bundle, where the $\mathbb{P}^1$ is the $SU(2)/U(1)$ that remains from the $SU(2)$ bundle after the Kähler quotient; it parameterizes the choices of Kähler reductions from the HKC to $\mathcal{Z}$, but because of the $SU(2)$ isometry of the HKC, we get the same manifold $\mathcal{Z}$ for any choice of $\{X^{(\Pi)}, J^{(\Pi)}\}$. The manifold $\mathcal{Z}$ carries a natural Einstein metric induced by the Kähler quotient of the HKC, and is the twistor space of a QK with positive cosmological constant Λ; let's see how this arises.

If we choose a complex structure, at a generic point (not the tip of the cone) we can pick coordinates such that the corresponding killing vector has the form $X^A\partial_A = i(\partial_z - \partial_{\bar{z}})$ for some holomorphic coordinate z; then

$$\chi^A\partial_A = \partial_z + \partial_{\bar{z}} \ . \tag{16}$$

In these coordinates, the tip of the cone has been sent off to $-\infty$, and the hyperkähler potential χ takes the form

$$\chi = e^{z+\bar{z}+K(u^i,\bar{u}^j)} \ , \tag{17}$$

where the u^i are the remaining complex coordinates other than z. The the HKC metric is given by

$$g_{a\bar{b}} \ = \ \partial_a\partial_{\bar{b}}\chi \ = \begin{pmatrix} K_{i\bar{j}} + K_iK_{\bar{j}} & K_i \\ K_{\bar{j}} & 1 \end{pmatrix} \ . \tag{18}$$

As the HKC is hyperkähler, χ satisfies the Mong-Ampère equation:

$$\det\left(g_{a\bar{b}}\right) = e^{f(z,u)+\bar{f}(\bar{z},\bar{u})} \tag{19}$$

for some (locally) holomorphic function $f(z,u)$. Evaluating this using the explicit form (18), we obtain:

$$\chi^{2N}\ \det\left(K_{i\bar{j}}\right)\ =\ e^{f+\bar{f}}\ , \tag{20}$$

where $2N$ is the complex dimension of the HKC; hence

$$\det\left(K_{i\bar{j}}\right)\ =\ e^{-2NK+(f-2Nz)+(\bar{f}-2N\bar{z})}\ . \tag{21}$$

The term $(f-2Nz)+(\bar{f}-2N\bar{z})$ can be dropped, as it is just a Kähler transformation of K, and hence we finally arrive at:

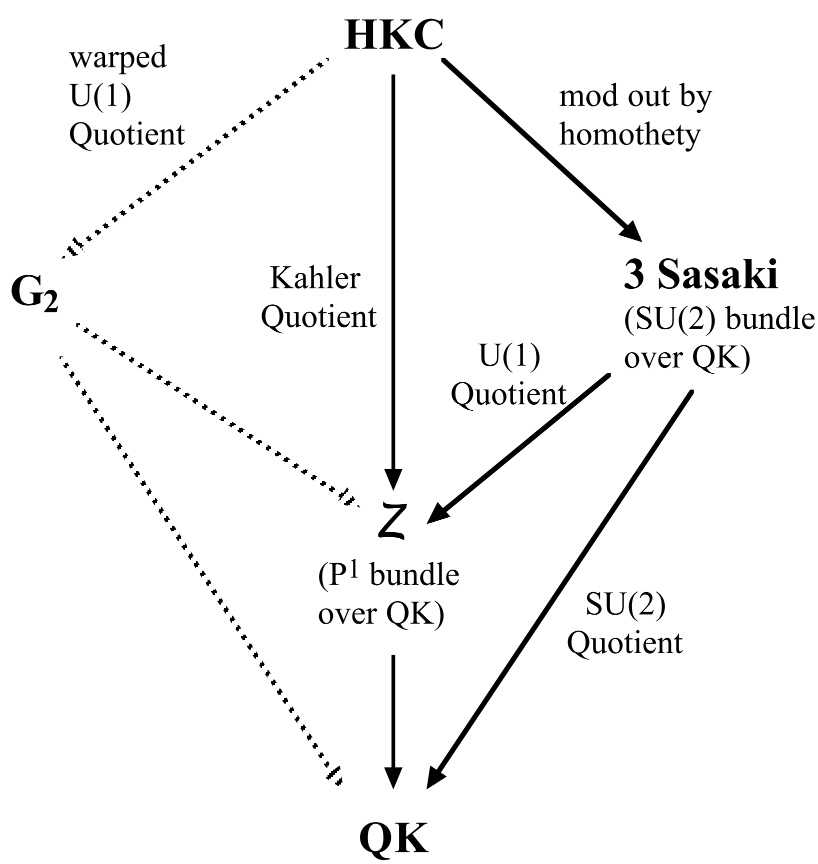
$$\det\left(K_{i\bar{j}}\right)\ =\ e^{-2NK(u,\bar{u})}\ . \tag{22}$$

Thus we find that the metric $K_{i\bar{j}}$ on $\mathcal{Z}$ is Einstein with cosmological constant $2N$.

We can proceed and find the QK by projecting down from the $\mathbb{P}^1$ bundle, as described in great detail in hep-th/0101161, but our main interest here is the twistor space $\mathcal{Z}$.

Diagrammatic summary and some observations

It is useful to have a picture of the relations between the various spaces that we have discussed, as well as a few other spaces.



The HKC, the twistor space $\mathcal{Z}$, the QK, and some related spaces.

The HKC can be reduced to the twistor space $\mathcal{Z}$ by a Kähler quotient; it can be reduced to a 3-Sasakian manifold by projecting along the homothety, and, for an eight real dimensional HKC, to a G_2 holonomy cone by a warping a $U(1)$ quotient. The 3-Sasakian manifold is the obvious $SU(2)$ bundle over the QK, and can be reduced by a $U(1)$ quotient to the twistor space $\mathcal{Z}$. Claude LeBrun has also told me that there exists a spin(7) holonomy metric on an eight real dimensional HKC, but its significance is unclear.

To get a QK space with negative cosmological constant $\Lambda < 0$, we need to an HKC with an indefinite signature metric (one negative signature quaternionic dimension $\Leftrightarrow$ two negative signature complex dimensions). This gives rise to a twistor space with one negative signature complex dimension and a positive definite signature QK with negative cosmological constant. Thus this construction seems to break down when the base is hyperkähler case ($\Lambda = 0$); a twistor space exists, but no natural metric comes with it. Perhaps one could consider HKC's and twistor spaces with degenerate metrics.

Examples

The simplest example of an HKC is $\mathbb{C}^4$; we write the hyperkähler potential as

$$\chi = z^a \bar{z}^a \equiv e^{z+\bar{z}+\ln(1+u^i\bar{u}^i)} \ , \quad a = 1 \cdots 4 \ , \quad i = 1 \cdots 3 \ , \tag{23}$$

and $z \equiv z^4$, $u^i = z^i/z^4$; we immediately recognize the Kähler potential of the twistor space $\mathcal{Z}$ as the Fubini-Study metric on $\mathbb{P}^3$. In this case, the QK is the four-sphere (quaternionic projective space) $S^4 \equiv \mathbb{HP}^1$.

A second simple example of and HKC is the hyperkähler quotient of $\mathbb{C}^6$ with respect to a $U(1)$ at zero level (see Appendix). This gives a twistor space that is a quadric in $\mathbb{P}^2 \times \mathbb{P}^2$, and a QK that is Kähler: $\overline{\mathbb{P}^2}$.

A large class of models are given by homogeneous QK spaces, or Wolf spaces; here we list the classical examples: In the table, $/\!/\!/_0$ is the hyperkähler quotient at level 0, and the twistor space

HKC	$\mathcal{Z}$	**QK**
$\mathbb{H}^{n+1}$	$\mathbb{P}^{2n+1}$	$\mathbb{HP}^n \equiv \frac{Sp(n+1)}{Sp(n)\times Sp(1)}$
$\mathbb{H}^{n+2}/\!/\!/_0\, U(1)$	$\mathbb{P}^{n+1}\times\mathbb{P}^{n+1}/(u_+\cdot u_- = 0)$	$X(n) \equiv \frac{SU(n+2)}{Su(n))\times U(2)}$
$\mathbb{H}^{n+4}/\!/\!/_0\, SU(2)$	$\mathcal{Z}[Y(n)]$	$Y(n) \equiv \frac{SO(n+4)}{SO(n))\times SO(4)}$

Table 1: HKC's and $\mathcal{Z}$'s of the classical Wolf Spaces

$\mathcal{Z}[Y(n)]$ of the orthogonal Wolf space $Y(n)$ is the intersection of three holomorphic quadrics in the Grassmannian described by the Kähler quotient $\mathbb{C}^{2n+8}/\!/U(2)$ (at positive level). Note that because of relations between the first few classical groups, $\mathbb{HP}^1 = Y(1)$ and $X(2) = Y(2)$; their corresponding twistor spaces and HKC's are the same as well.

5. Super-HKC's, supertwistor spaces, and super-CY metrics

Super-Calabi-Yau manifolds were first discussed by Sethi as candidates for the mirrors of rigid CY manifolds. The recent work of Witten and others involving strings on such spaces and the relation to $N = 4$ supersymmetric Yang-Mills theory has revived interest in such manifolds. I would like to end my lecture with a description of some recent and as yet unpublished work with Rikard von Unge. We considered super-HKC's, which are hyperkähler cones with some fermionic as well as some bosonic coordinates. Because super-HKC's are hyperkähler, they obey the super-Monge-Ampère equation

$$\text{sdet} g_{a\bar{b}} = 1 \ , \tag{24}$$

where the superdeterminant of a matrix with bose-bose components A, bose-fermi components B, fermi-bose components C, and fermi-fermi components D obeys

$$\text{sdet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det A}{\det(D - CA^{-1}B)} = \frac{\det(A - BD^{-1}C)}{\det D} \ . \tag{25}$$

Direct calculation of the superdeterminant using the super version of the metric (18) gives

$$\text{sdet}(K_{i\bar{j}}) = e^{-2(N_b - N_f)K(u,\bar{u})} \ . \tag{26}$$

Here N_b is the number of bosonic dimensions of the HKC, and N_f is the number of fermionic dimensions. Thus the cosmological constant $\Lambda = 2(N_b - N_f)$, and if $N_b = N_f$, the supertwistor space is super-Ricci flat, and hence Calabi-Yau. We may state this as a theorem:

Theorem: The induced metric of a supertwistor space derived from an HKC with an equal number of bosonic and fernionic dimensions is always super-Calabi-Yau.

This gives a powerful technique for obtaining interesting super-CY manifolds, and research is underway applying these ideas to the study of deformations of $\mathbb{P}^{3|4}$.

Appendix: Symplectic reduction[2]

Consider a Kähler manifold M with Kähler form ω, and an isometry X that preserves both the metric and ω: $\mathcal{L}_X g = \mathcal{L}_X \omega = 0$. Because the Kähler form is closed, we can define the moment map μ by

$$d\mu^X = \omega(X, \bullet) \tag{27}$$

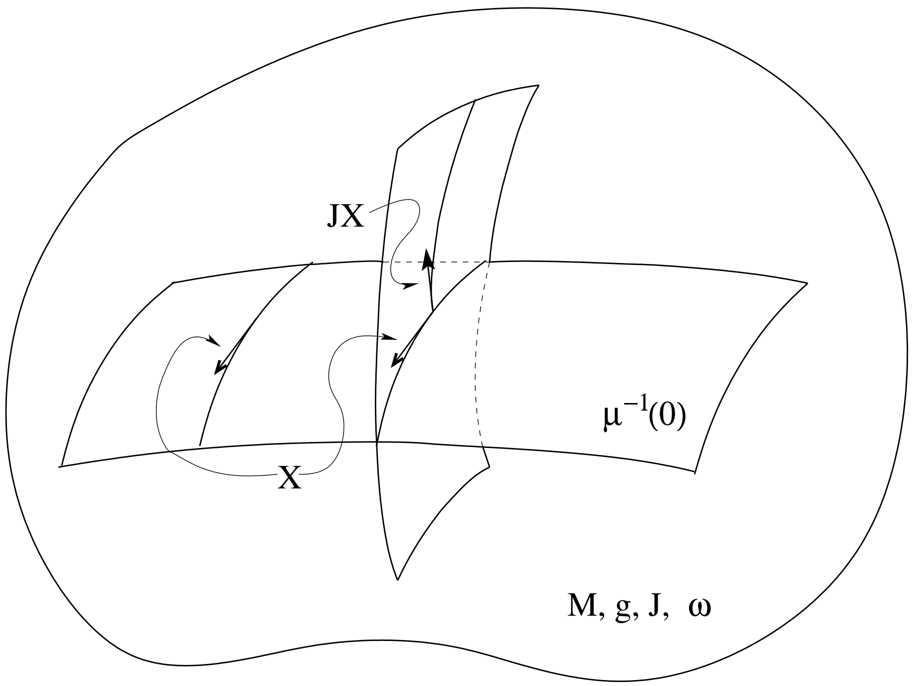
Now we define the Kähler quotient to be the quotient with respect to the isometry generated by the Killing field X of the submanifold defined by the zero set of the moment map $(\mu^X)^{-1}(0)$. An alternative but equivalent way to define the Kähler quotient is as the quotient of (most of) M with respect to the action of the *complexified* vector field $\{X, JX\}$ ($\{X, \chi\}$ in our case). Note that definition of the moment map is ambiguous by a constant. For a nonabelian group, this ambiguity is fixed by equivariance; however, $U(1)$ factors remain ambigous, and

[2] This is largely copied from my lectures at the Srni Winter School, January 2003.

this constant is called the level of the moment map. Because of the definition of μ, the action of X always lies within $(\mu^X)^{-1}(0)$ whereas the action of JX takes us out of this submanifold. This can be written as

$$(\mu^X)^{-1}(0)/G \equiv \text{“}M\text{”}/G^* \ , \tag{28}$$

where G^* is the complexified gauge group and “M” denotes the stable submanifold of M, which consists of all the points in M that can be reached by the action of G^* on $(\mu^X)^{-1}(0)$. In the figure below, we see the orbits generated by X lying in $(\mu^X)^{-1}(0)$ as well as the complexified orbits generated by X and JX:



For hyperähler quotients, the story is much the same; in this case, for an isometry that preserves all three complex structures and hence three Kähler forms, there are three moment maps (the isometries above preserve only one complex structure each, and hence a generic HKC does *not* admit a hyperkähler quotient). The hyperkähler quotient is taken as the ordinary quotient of the intersection of the zero-set of all three moment maps; if we focus on a particular Kähler structure, the remaining Kähler forms and corresponding moment maps can be combined into conjugate holomorphic and antiholomorphic pairs, and hence the hyperkähler quotient can be regarded as a particular complex submanifold of the Kähler quotient.