EINSTEIN’S ODYSSEY:
FROM SPECIAL TO
GENERAL RELATIVITY

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"Things should be made as simple as possible, but not any simpler."
- Albert Einstein
Einstein’s Description of the Journey

Like most good plays, it consists of three acts,
To which I add a rather long prolog
1907: Act One

Equivalence Principle

“Basic idea for the general theory of relativity”
1912: Act Two

Metric tensor

“Recognition of the non-Euclidean nature of the metric and its physical determination by gravitation”
1915: Act Three

Correct field equations

“Field equations of gravitation. Explanation of the perihelion motion of Mercury”
“The Perihelion Motion of Mercury”

PERIHELION OF MERCURY

Perihelion moves around the sun

574" per 100 years, and 43" are due to the relativistic effect
How Shall We Characterize The Action?

**Invention**
(Internal Impulse
Plus External Constraints)
**Edison invented the phonograph**

**Discovery**
(External Impulse
External Constraints)
**Columbus discovered the West Indies**

**Creation**
(Internal Impulse
Internal Constraints)
**Tolstoy created Anna Karenina**
1905 Prolog: The Relativity Principle

What Is a Spatial Frame of Reference? (Cheat Sheet)

- **Coordinates** are pairs (2D) or triples (3D) of real numbers that designate the position of a point in a coordinate system.
- A **coordinate system** is a set of rules by which a coordinate can spatially relate a location to a unique coordinate system origin and associated axes.
- A **spatial reference frame** ties a coordinate system's origin to some Object Reference Model, such as a model of the Earth, so that it is no longer arbitrary but tied to the real world.
Inertial Frame of Reference

Take a **body** subject to **no (net) external force**: Consider its **motion relative to some frame of reference**, assumed to obey **Euclidean geometry**.

If the **motion is in a straight line and at constant speed** relative to the Euclidean geometry and (absolute) time of the frame

Then it is an **inertial frame of reference**
Newton’s First Law
Applied to Falling Objects

"Every object persists in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces impressed on it."

Before release:
Object in state of rest, airspeed zero, weight but no drag.

When object is released:
Object accelerates – airspeed increases.
Drag depends on airspeed – Drag increases.

When Drag is equal to Weight:
Object no longer accelerates but holds a constant velocity -- terminal velocity.
Galilean Relativity Principle

First enunciated by Galileo (we’ll see him later), and is summarized by the statement:
"No mechanical experiment can distinguish a state of absolute rest from uniform straight-line motion."

Example: If you hit a parked car at sixty miles per hour, the effect is the same as if you and the other car had a head-on collision, with each of you traveling thirty miles per hour.
An example of the Relativity Principle: In two inertial frames in relative motion, the experimenters measure the same electric force between two charges.
Einstein: Why Restrict It to Mechanics?

All attempts to detect the absolute motion of the earth by optical, electrical, and magnetic experiments also failed.

So Einstein removes the restriction and includes light and all other phenomena in the principle. To do so, he has to invent a new kinematics, in which time is no longer absolute (i.e., the same in all inertial frames), but the speed of light is.
What is the relation between the coordinates and time in these two inertial frames?

Figure 1-1
Old Answer (Newton)

Galilei Transformations:

\[ x' = x - vt \]
\[ y' = y \]
\[ z' = z \]
\[ t' = t \]

Time is absolute!
Einstein's Answer in 1912 Manuscript
Einstein’s 1912 Manuscript (close-up)

\[
\begin{align*}
x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \\
\end{align*}
\]

Dieses System nennt man die ‚spezielle Lorentz-Transformation‘. Es sind dies die Gleichungen, welche nach der Relativitätstheorie an das 4-

gleichungssystem (\text{II}) des \S\text{6} treten müssen.
Lorentz Transformations

\[
x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\[
y' = y
\]

\[
z' = z
\]

\[
t' = t - \frac{v}{c^2} \cdot \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}}
\]
Hermann Minkowski

Minkowski realized that a 4-dim treatment of space-time is particularly suited to geometrical visualization of the Lorentz transformations.
Two-Dimensional Minkowski Space-Time
Minkowski Space-Time

In a 1907 lecture (*Das Relativitätsprinzip*), he adapted a 4-dimensional coordinatisation similar to that introduced by Poincaré (we’ll see him in a moment), and carried its geometrical interpretation much further. He seems to have been the one who introduced the term “space-time”:

“Ein einzelnes Wertsystem $x, y, z, t$ ... soll ein Raum-Zeitpunkt heißen” (1908)

He rather pretentiously designated the four-dimensional relativistic space-time “die Welt” (“the universe” or “world”)
“Four-Dimensional” Minkowski Space
Global Time versus Proper Time

He also realized that, in contrast to the global time coordinates \( t, t' \) in the Lorentz transformations, the interval between two events is invariant and, in the case that it is time-like, named it the proper time ("Eigenzeit")—the only time that is physically significant!
This is What Can Happen If You Forget That!
Back to Henri Poincaré & His Fourth Geometry

Poincaré had developed a (2-dimensional) “4th geometry as coherent as those of Euclid, Lobachewsky and Riemann” (1887)
Poincaré’s Fourth Geometry (cont’d)

In this geometry, the parallel postulate is preserved, but the isotropy of space is abandoned and the concept of rotation is replaced by that of pseudo-rotation.
Although neither Poincaré in his two 1905-1906 papers *Sur la dynamique de l’électron* nor any of his contemporaries seems to have noted it at the time, today we can recognize that 2-dimensional Minkowski space-time satisfies the axioms of Poincaré’s “4th geometry”
1907-08: Act One

Einstein has to write a review article on *The Principle of Relativity*, and starts to grapple with the problem of how to incorporate gravitation into his theory.
Sir Isaac Newton: Mr. Gravitation
Earth’s Gravitational Field

Gravitational force $\sim \frac{1}{d^2}$
What Creates Gravitation?

Newton: Mass via inverse square law

\[ f_G = \frac{G m_g}{r^2} \]

Laplace: can write it as a field law:

\[ \Box^2 \phi = 4\pi G \rho \]
\[ \rho = \text{matter density} \]
\[ \phi = \text{grav potential} \]
\[ f_G = -\Box \phi \]

Note everything here is calculated at the same absolute time

Gravitational interactions are instantaneous
Problems for Special Relativity

Problems for everyone:

What takes the place of the **scalar mass**?
What takes the place of **instantaneous interactions**?

Problem for **Einstein**:

What is the **deeper significance** of $m_I = m_g$
Laue & The Stress-Energy Tensor

As Max Laue (later von Laue) stressed, in special relativity, the mass density is only one component of a ten-component entity called the stress-momentum-energy tensor or stress-energy tensor for short.
Finite Speed of Propagation

Guess:

If gravitational action is not instantaneous, it propagates something like an electromagnetic wave (light), and like light obeys some sort of generalized wave equation(s)
But Just What Is It Mathematically That Propagates Gravitationally?

Four guesses:

1) **Scalar field** like Newtonian gravity*

2) **4D vector field**

3) **Two 3-vector fields** like electricity and magnetism, forming a "six-vector" = **anti-symmetric 4D tensor**

4) **Ten component 4D symmetric tensor field** to match the stress-energy tensor source

* A scalar can be formed naturally from the SET (its trace) called the **Laue scalar**
Einstein’s Rivals

- Max Abraham
- Gustav Mie
- Gunnar Nordstrøm
1912-1913: Attempts to generalize the special theory to include gravitation—first to write down a non-Euclidean line element for space-time, but quickly drops the idea.
Gustav Mie

Develops a special-relativistic non-linear electrodynamics to explain stability of the electron; later attempts to include gravitation. His ideas motivate Hilbert’s study of *The Foundations of Physics*.
Special-relativistic scalar gravitational theory that includes the equivalence principle. Einstein and Fokker show it can be reformulated as a scalar theory in curved but conformally flat space-time. Only rival theory AE takes seriously. Big difference from GR: no gravitational deflection of light
What Made Einstein Unique?

He focused on two questions that soon became intertwined:

1) Can we extend the principle of relativity to accelerated frames of reference?
2) How to include gravitation? Equivalence principle shows we must go beyond special relativity. How far do we have to go?
Accelerated Frames & Inertial Forces
Galileo Galilei (here he is at last) and the Leaning Tower of Pisa
Because inertial and gravitational mass are equal, there is no (unique) way to separate the effects of inertia and gravitation on a falling body.

THE EQUIVALENCE PRINCIPLE
Take a Ride on the Einstein Elevator

Equivalence Principle

A
accelerated upwards

B
downwards gravitational field

\[ g \]
This looks just like gravity
Free fall “annihilates” gravity
1911- Solar Deflection of Starlight

The beam of light from the star is deflected by the gravitational field of the sun. Consequently, for the observer on Earth, the position of the star appears to have shifted from its true position.
1912- Gravitational Lensing (*Science* 1936)
Gravitational Lensing (cont’d)
Mach’s *Mechanics*, Einstein’s “Principle”

As a student, Einstein read Mach’s “historico-critical study” of mechanics, and was impressed by Mach’s suggestion that the inertia of each material body was due to the influence of all the other matter in the universe. He wanted to incorporate his version of “Mach’s principle” into his theory of gravitation: “The gravitational field is entirely determined by the matter in the universe”
Einstein starts to look for field equations for his theory

After developing a scalar theory for the static case (like electric field),

he turns to the stationary case (like magnetic field)

and looks at the gravitational field in a rotating disc

he realizes that gravitation can curve space
The Rotating Disc
Geometry on the disc

The Circumference of the Disk is:
2πr, according to K, but
2πr/√1-(v^2/c^2), according to K'.
Thus the geometry according to K' is "curved", and non-Euclidean.
Gravitation Curves Space-Time

• But the equivalence principle says:
• A *rotating* frame of reference *without* a gravitational field is equivalent to a *non-rotating* frame *with* a gravitational field
• Conclusion: this *stationary* gravitational field must produce a curvature of space
• Guess: A *non-stationary* gravitational field will *curve* space-time
SOS Marcel!!

• Einstein knows a bit about **curved surfaces** from his course in differential geometry at the Poly

• But **Einstein was no Newton when it came to mathematics**, so

• he turns to his old schoolmate **Marcel Grossmann**, now his colleague at the ETH, for help with the **mathematics of curved space-times**
Einstein and Friends, 1899:
Marcel Grossmann on the Left
Grossmann Tells AE About Tensors

Gregorio Ricci Curbastro (left) and Tullio Levi Civita (picture coming) published a formalism in 1901 they called “the absolute differential calculus” (a.k. a. tensor calculus) that could be applied to Riemannian geometry.
Space is **locally flat** (Euclidean) but **globally non-flat**: it has a curvature that varies from point to point.

What does curvature mean here?
Karl Friedrich Gauss & Gaussian Curvature

What are the radii $R_1$, $R_2$, .. of the circles that best fit the cross sections of the surface?

The inverses $1/R_1$, $1/R_2$, … are the Gaussian curvatures
Gaussian Curvature (cont’d)
Gaussian Curvature (cont’d)
Higher Dimensions

Riemann extended Gauss’ ideas about curvature of a 2-D surface to higher dimensions:

3-D: What is the radius of the best-fitting sphere ....?

4-D: What is the radius of the best fitting hypersphere ....?
Another Kind of Curvature

But there is another kind of curvature, more important for geometrizing gravitation.

To understand it, we must learn about Grassmann and affined space.
Affine Space—Parallism is All!

Forget about distance, keep concept of parallel lines.

Only the ratio of parallel intervals has meaning.
Equal Parallel Intervals
Ratio of Parallel vectors
Affine Transformations
Affine Curvature

• Levi Civita did for affine space what Riemann had done for Euclidean space:

• He went from global to local, and this enabled a new interpretation of curvature
Tullio Levi Civita and Parallel Displacement
Riemannian Curvature
Geodesic Deviation
Geodesic Deviation (cont’d)

Is there any relative acceleration between two nearby freely-falling bodies (I.e., each following a geodesic)

• The amount of such relative accelerations in various directions is a measure of the components of the Riemannian curvature tensor
But Geodesic Deviation

Physically =

Tidal Forces!!
FIG. 13.3 (a) Directions of centripetal force per unit mass (CF) and moon's gravitational force per unit mass \((F_x, F_y, F_z, F_d)\) at points on earth (not to scale), (b) directions and relative magnitudes of residuals of \(CF\) and \(F\) at various points on the earth's surface (correct to first order—see text), (c) form of the horizontal tractive forces over the earth's surface, (d) tidal "bulges" (much exaggerated) at \(A\) and \(C\) according to the equilibrium theory for an ocean covering the entire earth.
Parallel Displacement: A Post-Mature Concept

Riemann went from global Euclidean to local Euclidean with Gaussian curvature (1854)

Grassmann developed global affine geometry (1844, 1862)

Someone should have gone from global affine to locally affine with Riemannian curvature by 1880 at the latest– but no one did!
But all of Levi Civita’s work and its subsequent generalization was done after the completion of general relativity and in response to it.

Einstein had to make do with the tools developed by Gauss and Riemann, and this accounts for most of the detours in his Odyssey between late 1912 and mid-1915.
The Metric Tensor

• By the end of 1912, Einstein is convinced that the ten-component metric tensor $g_{\mu\nu}$ is the correct representation of the potentials for the inertio-gravitational field, and with Grossmann’s help has succeeded in writing the effects of this gravitational field on all other physical processes in a satisfactory way.
Riemann's Metric Tensor in Four Dimensions
Pythagoras’ Theorem

Characterizes Euclidean geometry

Pythagoras’ Theorem in a right angled triangle is:

The square on the hypotenuse = the sum of the squares on the other two sides

\[ a^2 + b^2 = c^2 \]
Two Dimensional Surface (Gauss)

Distance between two neighboring points:

\[ ds^2 = g_{11}(dx^1)^2 + 2g_{12}(dx^1)(dx^2) + g_{22}(dx^2)^2 \]

Local Euclidean geometry:
Just Pythagoras’ Theorem with curvilinear coordinates
Remaining Problem

What are the correct equations describing the effect of the rest of physics (matter, electromagnetic field, etc.) on gravitation?
What Are Christoffel Symbols?

Working definition: Christoffel symbols are coefficients that encapsulate the manifold curvature, coordinate system, and metric.

The name symbol is historical, not descriptive. Actually they are $n \times n^2$ matrix-valued functions of position. For $n = 2$, a typical example is

$$\Gamma^i_{jk} = \begin{bmatrix} -1.22 & -0.21 & -0.21 & 9.52 \\ 1.82 & 0.63 & 0.63 & 7.01 \end{bmatrix}$$

Symmetric with six distinct elements.

Formal definition:

$$\Gamma^i_{jk} \equiv \frac{1}{2} \left( g_{jk,i} + g_{ki,j} - g_{ij,k} \right).$$
Einstein Splits apart what God– and Christoffel Had Joined Together

What is the **correct representation of the inertio-gravitational field?**:

**Answer** (the affine connection) **unavailable** to Einstein (not yet invented)

**Next best thing:** **The Christoffel Symbols**

But Einstein decomposes them to get the **derivatives of the metric tensor**
1913-mid 1915: Adrift at Sea

- I shall not go into the numerous steps—later seen to be mis-steps-- that took Einstein away from the formulation of his field equations in terms of the Riemann tensor, a step he almost took in 1912-13, for over two years.

- In addition to the absence of the concept of affined connection, the other main conceptual problem was that:
Unlearn a Lesson Well Learned

- To succeed in formulating the special theory, Einstein had to attach physical significance to the coordinate system.

- To succeed in formulating general relativity, Einstein had to learn that coordinates have no inherent physical significance.
Late 1915: Act Three

• Finally, as problems piled up with his attempts to formulate field equations not based on the Riemann tensor, Einstein re-examined his whole approach, returned to the Riemann tensor, and gave the final formulation of his field equations:

  \[ G_{\mu\nu} = 8\pi G \ T_{\mu\nu} \]
At Journey’s End Einstein Looks Back

The years of anxious searching in the dark, with their intense longing, their alternations of confidence and exhaustion, and the final emergence into the light—only those who have experienced this can understand it.
1919– Einstein Becomes Famous
What is the Big Difference Between SR and GR?

SR is based on **fixed background space-time structures**. Coordinates can be given a fixed physical significance by attaching them to the fixed background structures.

GR is a **background-independent theory**. All space-time structures are dynamic; hence, coordinates can have no fixed physical significance.
General relativity forces one to adopt a relational view of space-time

“On the basis of the general theory of relativity ... space as opposed to ‘what fills space’ ... has no separate existence. If we imagine the gravitational field ... to be removed, there does not remain a space of the type [of the Minkowski space of SRT], but absolutely nothing, not even a ‘topological space’ [i.e., a manifold]... There is no such thing as an empty space, i.e., a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field “

(Einstein, 1952).
Quantum Gravity– The Great Challenge

• How can we combine this background-independent, relational approach to space-time with quantum theory, which is based on a fixed background, absolute space-time?

• That is the basic problem of quantum gravity
BRONSTEIN CUBE

Quantized Newtonian Gravitational Theory

Non-Relativistic Quantum Mechanics

Quantum Field Theory

General Non-Relativity (Newtonian Gravitation)

Special Relativity

Galilei-Newtonian Theory

G

h

1/κ

General Relativity

Quantum Gravity
**BRONSTEIN SQUARE**

Gen. Non-Rel. S-T ($G$) — Gen. Rel. S-T ($c,G$)

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**Inertio-Gravitational Connection**

- G: Chronometry + Geometry
- Combine
- Dynamize

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Galilean S-T: $\frac{1}{c}$

Minkowski S-T ($c$)
The slogan is:

NO KINEMATICS WITHOUT DYNAMICS !!!!!
Die grossen politischen Dinge unserer Zeit sind so entmutigend, dass man sich in der eigenen Generation ganz vereinsamt fühlt. Es ist, wie wenn die Menschen die Leidenschaft für Recht und Würde verloren hätte und nicht mehr schätzen, was bessere Generationen mit unsäglichem Opfermut erworben haben ... Das Fundament aller menschlichen Werte ist eben das Moralische. Das in primitiver Zeit klar gesehen zu haben, ist die einzigartige Grösse unseres Moses. Schau Dir die Heutigen dagegen an!

(Einstein, 1938)
The great political events of our time are so disheartening, that one feels quite isolated in his own generation. It is as if people have lost the passion for justice and honor, and no longer prize what better generations have won through indescribably courageous self-sacrifice. ... The foundation of all human values is precisely morality. To have seen this clearly in primitive times is the peculiar greatness of our Moses. Just look at the contrast with our contemporaries!

(Einstein, 1938 translation by JS)
In its Summer 1947 issue, *The American Scholar* published an article by Albert Einstein entitled "The Military Mentality“, a response to an article by Louis Ridenour in the Spring issue of that magazine entitled "Military Support of American Science, A Danger?" essentially denying that such a danger existed.

In his reply, Einstein soon passes beyond the question raised by Ridenour to what he sees as the issue at the root of all such questions: the military mentality:
“It is characteristic of the military mentality that non-human factors (atom bombs, strategic bases, weapons of all sorts, the possession of raw materials, etc.) are held essential, while the human being, his desires and thoughts -- in short, the psychological factors – are considered as unimportant and secondary…

The individual is degraded to a mere instrument; he becomes 'human materiel.'

The normal ends of human aspiration vanish with such a viewpoint. Instead, the military mentality raises 'naked power' as a goal in itself -- one of the strangest delusions to which men can succumb.”
He points out that:

“The Germans, misled by Bismark's successes in particular, underwent just such a transformation of their mentality— in consequence of which they were entirely ruined in less than a hundred years.”

• **Einstein** knew whereof he spoke. He had witnessed the culmination of German militarism during the First World War (1914-1918) from Berlin, the heart of the German Reich; and from this same vantage point he lived through the first stages of German ruination: military defeat and aborted revolution in 1918-1919 and subsequent rise, decline and fall of the Weimar republic (1919-1932).
• After the Nazi seizure of power in 1933, he severed all ties with Germany and from Princeton, .J., watched the rise and fall of the Third Reich (1933-1945) culminating first in the spiritual ruination of the German people under fascism, and then their physical ruination in the closing phase of World War II.
He now expressed the fear that his new homeland was embarking on the same path:

”I must frankly confess that the foreign policy of the United States since the termination of hostilities [in 1945] has reminded me, sometimes irresistibly, of the foreign policy of Germany under Kaiser Wilhelm II, and I know that independently of me, this analogy has most painfully occurred to others as well.”
With remarkable prescience, only two years into the Cold War, he foresaw where this trend was leading the United States:

“Today, the existence of the military mentality is more dangerous than ever because the offensive weapons have become much more powerful than the defensive ones. [This was written after the development and use of the atomic bomb (1945), but before the development and testing of the hydrogen bomb (1951)]. This fact will inevitably produce the kind of thinking that leads to preventive wars.
The general insecurity resulting from these developments results in the sacrifice of the citizen's civil rights to the alleged welfare of the state.

Political witch-hunting and governmental controls of all sorts (such as control of teaching and research, of the press, and so forth) appear inevitable, and consequently do not encounter that popular resistance that, were it not for the military mentality, might serve to protect the population.

A reappraisal of all traditional values gradually takes place and anything that does not clearly serve the utopian goal of militarism is regarded and treated as inferior.”
By extrapolating the trends in the United States that he saw in the 1940s in the light of his experience of German militarism, Einstein was able to predict with uncanny accuracy the contemporary situation we face in the United States. When reading his words, who can avoid thinking of the elusive "war on terror" or the all-too-concrete wars on Afghanistan and Iraq, with their mounting list of American war crimes; of the unpatriotic "Patriot Acts," so reminiscent of the Alien and Sedition Laws that blighted the lives and liberties of an earlier generation of Americans and their foreign guests; of the careful management of information— and mismanagement of people— the sedulous spread of misinformation by government agencies charged with facilitating our rights to information, to liberty and true security?
Seduced by War

Andrew Bacevich — international relations professor and former Army colonel — argues that Republicans and Democrats, conservatives and liberals, have bought into the new American militarism as a solution to our international problems. And that, he says, is bad for our democracy.

by Taylor McNeil
After 9/11, “we instantly embraced this notion of open-ended global war. This shows the extent to which the political elite in this country has bought into the notion that if you have a big problem, the way to solve it is by going to war.”
“The point is to think realistically of other ways of achieving our purposes in the world, because the military way alone, in my judgment, which I think is supported by recent events, isn’t going to work.”
“I think the beginning of wisdom is to rethink our attitudes and expectations with regard to military power and to come to something that’s more realistic and balanced — and I’d emphasize, more in harmony with our democracy.”
This is NOT a Left-Right Issue!

True radicals (like me, I hope) and True conservatives (like Andrew Bacevich) must come together to oppose the so-called “neo-conservatives,” who are not true conservatives but ADVENTURISTS, willing to stake the fate of their country and of the world on their wild, utopian schemes!
BUT WHAT ABOUT TERRORISM?

We are all against terrorism!
But some of us are against all forms of terrorism,
state terrorism (i.e., terrorism perpetrated by the state) included!

And we suggest that state terrorism used to fight terrorism only compounds the terror
The danger now is that the west's current response to the terrorist threat compounds that original error. So long as the struggle against terrorism is conceived as a war that can be won by military means, it is doomed to fail. The more the west emphasizes confrontation, the more it silences moderate voices in the Muslim world who want to speak up for cooperation. Success will only come from isolating the terrorists and denying them support, funds and recruits, which means focusing more on our common ground with the Muslim world than on what divides us.
In this struggle, we Americans need all the help we can get from the great heroes of our past. We must never forget the lesson taught by Frederick Douglass:

"Without struggle there is no progress."

And we should heed the warning by Thomas Jefferson of what faces us if we fail:

"I tremble for my country when I reflect that God is just; that His justice cannot sleep forever."
Through his writings and the force of his moral example, Albert Einstein stands at our side in this struggle.