Physical limits for quantum gates with ions

Gaussian entanglement and symmetry graphs

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Multiparticle entanglement

1. Shareability: (Wootters/Schumacher)

If A is very entangled with B, then it is weakly entangled with C.

\[ |\Psi\rangle_{ABC} = |\Phi^{-}\rangle_{AB} |0\rangle_C \]

To increase the entanglement AC one has to decreases that of AB.

\[ |\Psi\rangle_{ABC} = |001\rangle + |010\rangle + |100\rangle \]

(Dür, Vidal and Cirac)

One may consider more particles and different configurations:

- Entanglement nearest neighbors
- Entanglement every partner
- One and the rest
II. Entanglement in quantum statistical models: (Nielsen, Fazio, Vedral, Vidal, Korepin)

\[ H = U \sum_n A_n \otimes A_{n+1} + t \sum B_n \]

Entanglement between two particles:

\[ E_{i,j}(t, U, T) = E_F \left[ Tr_{(i,j)} \left( e^{-H/T} \right) \right] \]

Which is the Hamiltonian for which this entanglement is maximal (at \( T=0 \))?
This talk: Gaussian states

Consider an undirected simple graph with $N$ vertices.

**Adjacency matrix $A$:**

$$A_{k,l} = \begin{cases} 1 & \text{if } k,l \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

**Symmetry of the graph: automorphism group**

$$G = \{ g \mid A = gAg^{-1} \} \subseteq S_N$$

Consider two points, $k,l$, such that for some $g \in G$, $g(k) = l$ and $g(l) = k$

**We determine the Gaussian state, invariant under $G$, which maximizes EdF for $(k,l)$**

$$\Psi$$

**gaussian with $T_g |\Psi\rangle = |\Psi\rangle$, such that $E_F \left[ \text{Tr}_{(k,l)} (|\Psi\rangle\langle\Psi|) \right]$ is maximal**
• **Gaussian states:**

\[
\text{dim}(H_n) = \infty \\
[X_n, P_n] = i \\
H = \bigotimes_n H_n
\]

\(\rho\) is Gaussian if it can be written as:

\[
\rho = ke^{-Q(X_n, P_n)} \\
Q \geq 0 \text{ is a polynomial of degree 2}
\]

• **Optimal state**

1. Build a Hamiltonian

\[
H = \sum_{g \in G} T_g \left[ (x_k - x_l)^2 + (p_k + p_l)^2 \right] T_g^\dagger
\]

2. Determine ground state \(\Psi\)

3. Determine ground state energy \(E_0\).

\(E_F = f(E_0)\)

Everything can be easily determined starting from the Adjacency matrix \(A\).

• **Extensions:**

- General group (including undirected graphs, etc) if \(\text{Index}(k, k) = \text{Index}(k, l)\)

\[
\text{add} \ g \in G, \ g(k) = l \text{ and } g(l) = k
\]

- Qubits
Examples

1. Platonic solids:

<table>
<thead>
<tr>
<th>Platonic Solid</th>
<th>EoF x 100</th>
<th>N adjacent</th>
<th>N nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>19.74</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Cube</td>
<td>19.74</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>11.12</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>Octahedron</td>
<td>10.75</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>5.37</td>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

Tendencies:
- $E$ decreases with the number of adjacent nodes.
- $E$ decreases with the total number of nodes.
- $E$ is suppressed in loops with an odd number of nodes.
### II. Infinite lattices

<table>
<thead>
<tr>
<th>Lattice</th>
<th>$E_{\text{dF}} \times 100$</th>
<th>$N_{\text{adjacent}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hexagonal (2d)</td>
<td>10.61</td>
<td>3</td>
</tr>
<tr>
<td>Square (2d)</td>
<td>6.31</td>
<td>4</td>
</tr>
<tr>
<td>Trigonal (2d)</td>
<td>2.69</td>
<td>6</td>
</tr>
<tr>
<td>Cubic (3d)</td>
<td>2.62</td>
<td>6</td>
</tr>
</tbody>
</table>

**Tendencies:**
- $E$ decreases with the number of adjacent nodes.

### III. Finite lattices

#### Nearest neighbors

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_{\text{dF}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3 ebits</td>
</tr>
</tbody>
</table>

#### Separation

<table>
<thead>
<tr>
<th>$N/2$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Diagram of finite lattices with nearest neighbors and separation plots]
Key points of the derivation

1. Entanglement of formation of symmetric Gaussian states:

\[(\text{Giedke, Wolf, Krüger, Werner and Cirac})\]

\[
\begin{align*}
\rho_{00} &= \begin{pmatrix}
    n & 0 & k_x & 0 \\
    0 & n & 0 & -k_p \\
    k_x & 0 & m & 0 \\
    0 & -k_p & 0 & m
\end{pmatrix}
\end{align*}
\]

Symmetric Gaussian states: \(m = n\)

\[E_F(\rho) = f(\Delta)\]

with \(\Delta = \sqrt{(n - k_x)(n - k_p)}\) and \(f(\Delta)\) a monotonic function.

2. Express it in a linear form

\[\Delta(\rho) = \inf_{s > 0} \text{Tr} \left[ \gamma \left( sX + \frac{1}{s} P \right) \right]\]
3. Optimization problem

\[
\Delta_{i,j} = \sup_{\Psi \rightarrow \gamma_{i,j}} \inf_{s > 0} \text{Tr} \left[ \gamma_{i,j} \left( sX + \frac{1}{s} P \right) \right] \quad \text{with } \Psi \text{ invariant under } G.
\]

4. Use symmetry:

\[
\Delta_{i,j} = \sup_{s > 0} \inf_{\Gamma} \text{Tr} \left[ \Gamma \left( sH_X + \frac{1}{s} H_P \right) \right]
\]

where \( H_X = \sum_{g \in G} T_g \left( X \Theta^0 \right) T_g^\dagger \)

5. Algebra + properties of covariant matrices:

\[
\Delta_{i,j} = \inf_{\Psi} \langle \Psi \mid H \mid \Psi \rangle
\]

where \( H \) is a Hamiltonian of harmonic osc. which are coupled according to the graph.
2. Limits on gates for trapped ions

Quantum information processing with trapped ions:
- Internal states manipulated with lasers.
- Two-qubit gates by exciting motional states.
- Difficult to scale up.

Scalable proposals

[J. Cirac and P. Zoller, Nature 2000]
(Wineland and col., Nature 2001)
Two-qubit gates

- Most of the errors occur during two-qubit quantum gates.
- We would like to design the most efficient quantum gates.
- One just has consider two ions in a trap.

Current proposals; limitations
(Mølmer and Sørensen, Plenio, Monroe, Milburn and James, Leibfried and Wineland, Cirac and Zoller,...)

- Time: $\nu T \gg 1$ (typically, of the order of 100 to achieve $F=0.99$).
- Addressability: Different lasers acting on different ions.
- Low temperatures: $\eta \sqrt{N+1} \ll 1$
New scheme based on a different concept

- No time limitation by the trap frequency.
- Insensitive to temperature.
- No addressability.

Idea:

- Use on-resonant lasers to kick the atoms → No spectroscopic limitations.
- Laser pulses must be short, and come from two different directions.

Limitation is laser control.

(as in Poyatos, Cirac, and Zoller, 97)
**Basic principle (with one ion):**

(i) Use lasers to kick both ions.

(ii) Free evolution in the harmonic trap.

Kick depends on the internal state of the ion and laser direction.

\[
\begin{align*}
|g\rangle|\text{mot}\rangle & \xrightarrow{(1,2)} |g\rangle e^{2ikx_1}|\text{mot}\rangle \\
|e\rangle|\text{mot}\rangle & \xrightarrow{(1,2)} |e\rangle e^{-2ikx_1}|\text{mot}\rangle
\end{align*}
\]

If \( \sum_{k=1}^{N} n_k e^{i\phi T_k} = 0 \) then \( U = e^{i\phi \sigma_z} \otimes 1 \)
We return to the initial state. The wavefunction acquires a phase. 
\[ \phi = A \]

- The angle \( \phi \) depends on the path.
- The path depends on the initial state.
- Can be made independent of the initial coherent state.
- The phase is also independent.

If \( \sum_{k=1}^{N} n_k e^{i\theta_k} = 0 \) then \( |\alpha\rangle \rightarrow e^{i\phi} |\alpha e^{-i\nu t}\rangle \)

The angle \( \phi \) depends on the path.
The path depends on the initial state. Quantum gate.
With two ions

Two modes

Conditions

\[ \sum_{k=1}^{N} n_k e^{i\sqrt{3}\nu T_k} = 0 \]

\[ \sum_{k=1}^{N} n_k e^{i\nu T_k} = 0 \]

\[ U = e^{i\phi\sigma_z} \otimes 1 \]

- For any chosen time \( T \), it is always possible to find a sequence of pulses.

There is not limitation by the trap frequency!

- Completely independent of the motional state (no LDL restriction).

- No addressability.

- The number of required pulses increases if the time decreases.

\[ N_p \sim \frac{8}{(\nu T)^{3/2}} \]

- Limitations:
  - Length of the laser pulses.
  - Laser stability.
Conclusions

Entanglement of Gaussian states

- Solved the problem of shareability for arbitrary graphs.
- Connection with quantum statistics: ground state of a symmetric Hamiltonian.

New concepts for quantum gates with ions

- No limitation by trap frequency.
- New scheme
  - New concept (resonant interaction).
  - Fast (not limited by trap frequency).
  - Robust (arbitrary temperatures).
  - Simple (no addressability).