Heterotic Fluxes and Non-Kähler Geometries

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This talk is based on work with K. Becker, M. Becker, J.X. Fu, and S.T. Yau [1].

Introduction

In connecting ten (or eleven) dimensional string theory to our low-energy four dimensional observable world, a standard geometrical approach is to wrap the extra dimensions around an internal compact manifold $X_6$. The four dimensional physical observables are then determined by the characteristics of $X_6$. For many years, $X_6$ was commonly taken to be a Calabi-Yau (CY) manifold, mainly because a CY preserves minimal four dimensional supersymmetry without turning on any additional background fields. However, a CY manifold typically has many scalar moduli which result in four dimensions unwanted massless scalars. Thus, in the past few years, there have been much work in incorporating fluxes, i.e. non-zero $p$-form background fields, into the compactification. Adding fluxes while preserving supersymmetry typically requires the geometry to be non-Kähler and hence non-CY.

Below, I shall describe flux compactification in heterotic theory and present a smooth geometrical model that preserves N=1 supersymmetry in four dimensions. Before proceeding, let me mention some of the advantages of studying flux compactification in heterotic theory as compared to type II theories:

1. The only flux present is the three-form $H$. Hence, the supersymmetry conditions incorporating the fluxes are "simpler" and the possible types of geometry are still rather constrained and more tractable.

2. The internal geometry $X_6$ can be smooth since no sources or branes need to be present. In type II theories, supergravity no-go theorems for compactification to four dimensional
Minkowski spacetime stipulate that fluxes can only be non-zero if the compact geometry has singularities (which can arise from the presence of branes). The heterotic theory bypasses the no-go theorem with the anomaly cancellation condition which modifies the Bianchi identity for the three-form $H$.

3. Gauge fields are naturally present. This allows for the possibility of constructing models with interesting phenomenology and possibly reproducing the standard model.

**Review of $N=1$ SUSY Constraints**

The background fields of the heterotic theory are $\{g_{mn}, H_{mnp}, \phi, F_{mn}\}$. In the string frame, preserving supersymmetry requires the ten dimensional geometry to be the product space $M = M^{3,1} \times X_6$.

Preserving $N=1$ supersymmetry is the requirement that there exists a no-where vanishing spinor $\eta$ that satisfies the following equations

$$
\delta \psi_m = \nabla_m \eta + \frac{1}{8} H_{mnp} \gamma^{np} \eta = 0,
$$

$$
\delta \lambda = \gamma^m \partial_m \phi \eta + \frac{1}{12} H_{mnp} \gamma^{mnp} \eta = 0,
$$

$$
\delta \chi = \gamma^{mn} F_{mn} \eta = 0.
$$

With a non-vanishing spinor $\eta$, we can write down a complex structure $J^{(1,1)}_{mn} = -i \eta^\dagger \gamma^m_{\eta} \eta$ such that $J^2 = -1$ and the Nijenhuis tensor, $N_{mn}^p = 0$. Furthermore, the hermitian $(1,1)$-form and the holomorphic $(3,0)$-form which together define the geometry of $X_6$ can also be simply expressed as fermion bilinears as follows.

$$
J^{(1,1)}_{mn} = -i \eta^\dagger \gamma_{mn} \eta,
$$

$$
\Omega^{(3,0)}_{mnp} = e^{-2(\phi - \phi_0)} \gamma_{mnp} \eta.
$$

We now list the supersymmetry constraints on the background fields.

**A. Geometry**

The constraints on the six dimensional geometry can be expressed as differential equations on $J$ and $\Omega$. We compare the case of CY versus non-Kähler case in the following.

<table>
<thead>
<tr>
<th>CY</th>
<th>Non-Kähler</th>
</tr>
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<tbody>
<tr>
<td>$dJ = 0$ (Kähler condition)</td>
<td>$d(|\Omega| \ast J) = 0$ (conformally balanced)</td>
</tr>
<tr>
<td>$d\Omega^{(3,0)} = \partial \Omega^{(3,0)} = 0$</td>
<td>$d\Omega^{(3,0)} = 0$</td>
</tr>
<tr>
<td>$H = 0$</td>
<td>$H = i(\partial - \partial) J \neq 0$</td>
</tr>
<tr>
<td>$\phi = \phi_0 =$ constant</td>
<td>$e^{-2(\phi - \phi_0)} = |\Omega|$</td>
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</table>
Note that since $H^{(1,2)} = i \bar\partial J$, if $H \neq 0$, then the metric $J_{\bar{a}b} = ig_{\bar{a}b}$ must be non-Kähler. However, not any non-Kähler geometry is allowed, the metric with non-zero H-flux turned on must be conformally balanced. The balanced condition is defined to be

$$d(*J) = \frac{1}{2} d(J \wedge J) = 0 .$$

Notice that Kähler metrics are also balanced since $dJ = 0$ implies $d(J \wedge J) = 0$. Indeed, the balanced condition can be thought as a relaxation of the Kähler condition. Consider the number of constraint equations. For Kähler, $dJ = 0$ gives 9 complex constraint equations. For balanced, $*d(*J) = 0$ gives only 3 complex constraint equations. Heuristically, the additional "degrees of freedom" in a conformally balanced geometry can be thought of as being parametrized by $(H, \phi)$. Or conversely, one can just forget about the $H$-flux and $\phi$ and simply consider the compactification geometry as being defined by $J$ and $\Omega$ which are required to satisfy the above constraints.

B. Gauge Field
The background gauge field satisfies the Hermitian-Yang-Mills condition:

$$F_{(2,0)} = F_{(0,2)} = 0, \quad F_{mn}J^{mn} = 0.$$

The first part is the condition that the gauge bundle is holomorphic. The second is a primitivity condition. Together, the Hermitian-Yang-Mills condition is known from the work J. Li and Yau [2] to correspond to the gauge bundle being "stable."

C. Anomaly Cancellation
In the heterotic theory, this is the modified Bianchi identity

$$dH = 2i\partial\bar{\partial}J = \frac{\alpha'}{4}(trR \wedge R - trF \wedge F) .$$

This condition further relates the gauge bundle with the geometry.

A Smooth Compact Model

Now we present a compactification model that satisfy all of the above conditions. The smooth geometry has been called the FSY geometry (after Fu, Strominger, and Yau) [3]. The manifold is a $T^2$ bundle over a CY two-fold. The metric ansatz is

$$ds^2 = e^{2\phi} ds_{\text{CY}^2}^2 + |dz + \alpha|^2 .$$

We shall let $\phi$ and $\alpha$ depend only on the base CY$_2$ coordinates. Further, $\theta = dz + \alpha$ will be taken to be a $(1,0)$-form and we define $\omega = d\theta = \omega^{(2,0)} + \omega^{(1,1)}_A$.

Now in order for the metric to be globally defined, $\omega$ is "quantized," that is it must be an element of the integer class $H^2(\text{CY}_2, \mathbb{Z})$. If one treats $\alpha$ as the gauge field of a Kaluza-Klein $T^2$ reduction, then $\omega$ is the field strength and the quantization of $\omega$ is just the Dirac quantization.
The conformally balanced condition can be easily satisfied as long as we require
\[ \omega \wedge J_{CY_2} = 0 , \]
i.e. \( \omega \) is primitive with respect to the \( CY_2 \) base. Interestingly, the conformally balanced condition then holds for \textit{any} function \( \phi \).

The \((3,0)\)-form can be simply taken to be \( \Omega^{(3,0)} = \Omega^{(2,0)}_{CY_2} \wedge \theta \). It is holomorphic since
\[ d\Omega^{(3,0)} = \Omega^{(2,0)}_{CY_2} \wedge d\theta = \Omega^{(2,0)}_{CY_2} \wedge (\omega_S^{(2,0)} + \omega_A^{(1,1)}) = 0 , \]
recalling that \( \omega^{S,A} \) are two-forms on the \( CY_2 \) base.

Now we must impose the anomaly cancellation. But before doing so, let us take note of the parameters currently present in the model.

1. The \( CY_2 \) base can be either \( T^4 \) or \( K3 \).

2. There are an infinite number of solutions since \( \omega \) is only required to be in the integer class \( H^2(CY_2, \mathbb{Z}) \) but any non-zero integer is allowed.

3. We have an infinite number of scalar moduli due to \( \phi \). This is because \( \phi \) can be any smooth function, and therefore, we can expand \( \phi \) in a basis of functions i.e. \( \phi = \sum_{n=0}^{\infty} a_n \phi_n \). But even though the space is compact, the basis would be infinite dimensional.

The anomaly cancellation equation
\[ 2i\partial \bar{\partial} J = \frac{\alpha'}{4} (\text{tr} R \wedge R - \text{tr} F \wedge F) \]  
(1)

will constrain each of the above freedom.

1’. Integrating \( J \wedge [\text{eq.}(1)] \) over \( X_6 \), we obtain the condition when \( \omega \neq 0 \)
\[ \int_{X_6} \text{tr} R_{CY_2} \wedge R_{CY_2} \wedge J > 0 . \]
This implies that \( \text{tr} R_{CY_2} \wedge R_{CY_2} \neq 0 \), or that \( CY_2 = K3 \).

2’. Since all fields depends on the base coordinates, we can integrate the four-form anomaly equation over \( K3 \) and obtain a condition (dropping factors of \( 2\pi \) and \( \alpha' \))
\[ -\frac{p_1(F)}{2} + \int_{K3} (||\omega_S||^2 + ||\omega_A||^2) = 24 . \]
This is a topological condition that relates the twisting of the \( T^2 \) bundle to that of the gauge bundle (as given by \( p_1(F) \)). Furthermore, since both terms on the left hand side is positive semi-definite, the number of possible solutions becomes finite.
3’. The anomaly equation is also a differential equation. Here specifically, it becomes a second-order highly non-linear differential equation for $\phi$

$$D_2(\phi) = \psi,$$

where $\psi$ can be considered as a source term. Thus an important question is whether this difficult differential equation allows for a solution for $\phi$ at all. As was proved by Fu and Yau [4], there exists a smooth solution for $\phi$ if the topological condition mentioned in (2’) is satisfied. The existence proof assumes an elliptic condition which effectively guarantees that the moduli of space of solution for $\phi$ is finite. (It is not known whether a solution can exist that doesn’t satisfy the elliptic condition.)

We see that the anomaly condition severely constrains the parameters of the model.

Finally, we describe the Hermitian-Yang-Mills gauge bundle, or equivalently, the stable gauge bundle on $X_6$. The primitivity condition $F_{mn}J^{mn} = 0$ is equivalent to $F \wedge J \wedge J = 0$ in six dimensions and $F \wedge J = 0$ in four dimensions. It is not hard to show that any stable bundle on $K3$ can be lifted to a stable bundle on $X_6$. The stable bundle on $K3$ has been studied by Mukai [5] and for SU($r$) group, they satisfy $c_2(F) = -p_1(F)/2 \geq r - \frac{1}{r}$. Therefore, one can easily turned on stable SU($r$) gauge bundle for $r = 4, 5$ that breaks one of the $E_8$ gauge group down to the phenomenologically interesting gauge groups $SO(10)$ and $SU(5)$, respectively.

References


