**Goal:**
1. Find a B-model formalism to compute unambiguously all open/closed amplitudes for toric CY$_3$
   ["mirror formalism" to top. vertex]
   open/closed
2. Expand the amplitudes at other points in the moduli space (such as orbifold points),
   (B-model well suited for that)

**Refs:**
Maniño '04, VB-Klemm-Maniño-Pasquetti (in progress), ...

**Plan:**
1. Mirror Symmetry (closed)
2. Top. Strings and orbifold GW invariants
3. Open geometry
4. Formalism
5. Examples: - fram-d vertex
   - CS theory on lens space
   - $\mathbb{C}^3/\mathbb{Z}_3$?
0 MIRROR SYMMETRY

1 $X = \text{toric } CY_3$

$$Y = \begin{cases} \text{Conic} \\ \downarrow \\ \mathbb{C}^* \times \mathbb{C}^* \end{cases} \text{ where fiber deg.} \to 2 \text{ lines over } R.S. \quad \Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$$

$$w w' = 1 + x + y,$$

$w, w' \in \mathbb{C}, \, x, y \in \mathbb{C}^*$

$v v' = y^2 + y + 3y + e^{-t} x^3$

$\Sigma = \{ 1 + x + y = 0 \} \subset \mathbb{C}^* \times \mathbb{C}^*$

$\text{genus 0 } (\mathbb{P}^1 \setminus \{0,1,\infty\})$

$\Sigma', \text{ genus 1, 3 punctures}$

$1$-param. c.s. moduli space $[z = e^t]$

2 local $\mathbb{P}^2$, $Q = (-3 \ 1 \ 1 \ 1)$

2 local $\mathbb{P}^1 \times \mathbb{P}^1$, $Q = (-2 \ 1 \ 1 \ 0 \ 0)$

$w w' = x^2 y + xy^2 + xy + e^{-t_1} y + e^{-t_2} x$

$\Sigma = \text{genus 1, 4 punctures}$

2-param. c.s. mod. space

$[z_1 = e^{t_1}, \ z_2 = e^{-t_2}]$
Some features of mirror symmetry

- Isomorphism between moduli spaces

Stringy Kähler m.s. described by secondary fan

Example: local $\mathbb{P}^2$

2 Kähler cones

KEY: $M_X$ contains Kähler cones of topologically distinct manifolds, including sing. ones like orbifolds

Mirror Map(s): - Local iso. in each patch, mapping Kähler $n$ to a basis of periods

* Involves choice of periods, different for each Kähler cone

* Basis of periods are related by symp transform.
A-MODEL (X) $\leftrightarrow$ B-MODEL (Y)

(K"ohler) (c. struct.)

part. fact$^b$: $Z_A[X] = Z_B[Y]$ (moduli space)

- define $Z = \exp (\sum_{g=0}^{\infty} g_3 2g-2 F_g)$ (string coupling)

* the phys. top. string part. fact$^a$ $Z$ is defined all over

the moduli space

$\rightarrow$ it is not holomorphic [Walcher's talk]

$\rightarrow$ non-hol. encoded by HA eqs [BCOV]

INSTEAD

- take the phys. $Z_A$, and take its limit at a LR pt deep inside a K"ahler cone, i.e.

$$\tilde{Z}_A = \lim_{LR} Z_A$$

then $\tilde{F}_g \mapsto$ GW theory of CY$_3$ in this K"ahler cone

$\subset$ holomorphic

EXAMPLE:

- local $\mathbb{P}^2 \mapsto$ 2 K"ahler cones

$\tilde{Z} \mapsto$ GW theory of $\{ \Theta(-3) \rightarrow \mathbb{P}^2 \cap \mathbb{C}^3/\mathbb{Z}_3 \}$

$\tilde{Z}^\times \quad \tilde{Z}_{\text{orb}}$

* in this talk I will always work w/ the limiting

hol. objects $\tilde{F}_g \quad [\text{not the physical ones}]$
F0: prepotential of Spec. Geometry in the good basis of periods, \( t_0 = \frac{2\bar{F}_0}{2t} \).

\( \hat{Z} \): wave-fct\(^b\) in geometric quantization of \( H^3(Y, \mathbb{C}) \) in real polarization corresponding to this basis of periods.

\[ \Rightarrow \hat{Z} \text{ transforms in metaplectic rep}^{b}\text{ (Bogoliubov transform)} \]
under change of basis of periods.

\( \Rightarrow \) we can extract \( \hat{Z}^{ab} \) from knowledge of \( \hat{Z}^{00} \).

[Aganagic, VB, Kleene]

EX: local \( \mathbb{P}^2 \rightarrow GW [\mathbb{C}^3/\mathbb{Z}_3] \)

(proven at genus 0 by Coates et al., some higher genus, VB-Cavaliere, etc.)

Many questions:

- Integrality à la Gopakumar-Vafa?
  (counting BPS states?)

- rel\(^b\) between GW and DT?
  (different than for smooth CY\(_3\))

- "OSV" and rel\(^w\) w/ black holes?

etc.
Other point of view:

- from wave-fct behavior under monodromy [or hol. anomaly]
  extract recursive rels directly at one pt in moduli space
  \( \hat{F}_g = \) lower genus data + hg
  a modular fct which is not fixed
- equivalent to HA eqs, but holomorphic
- same eqs at any LR pt, and hg is same everywhere
- need \( \tilde{F}_0, \tilde{F}_1 \rightarrow \) all \( \tilde{F}_g \) up to amb. hg \( \{ \) for any \( \) LR point \( \) \}

NEXT STEP:

- find new recursive rels for \( \hat{F}_g \) which are completely fixed \( \) [no modular ambiguity hg] 
- extend to open amplitudes

need new formalism
"Open Geometry"

- We need to fix a special Lagrangian (SLAG) submanifold of $X$ on which open strings end.

**Example:**

- Topology $\mathbb{C} \times S^1$, ends on a toric leg.

  [Aganagic-Vafa]

**Open amplitudes:** Maps with boundaries ending on this SLAG.

- From CS, each brane comes with a framing integer $f \in \mathbb{Z}$ which must be fixed.

  \[
  \text{[different } f \leftrightarrow \text{ different amplitudes]} \Rightarrow \text{choice of "location" on toric diagram and } f \in \mathbb{Z}
  \]

**Mirror:**

- A-brane $\Rightarrow H(x,y) = 0 = \omega \omega'$

  - Moduli space parameterized by R.S.: $\Sigma_i : \{ H(x,y) = 0 \} \subset \mathbb{C}^* \times \mathbb{C}^*$

  \[
  \Rightarrow \text{location + framing A-brane}
  \]

  \[
  \text{choice of parameterization of } \Sigma_i \text{ [choice of open string parameter]}
  \]

**But (crucial):**

- $\Sigma_i \subset \mathbb{C}^* \times \mathbb{C}^*$

  \[
  = \text{ } SL(2,\mathbb{Z}) \text{ acts as } (x,y) \mapsto (x^\alpha y^\beta, x^\gamma y^\delta)
  \]

  - Group of reparameterizations of a curve $\subset \mathbb{C}^* \times \mathbb{C}^*$

  \[
  \text{[think of } e^u, u \mapsto au + bv] \]

- \[
  (a,b) \in SL(2,\mathbb{Z})
  \]
RESULT:
- location and framing

A-Model

GEOMETRY:
- toric CY3
- SLAG ending on a toric leg w/ a choice of \( f \in \mathbb{Z} \)

B-Model
- choice of \( SU(2, \mathbb{Z}) \) parameterization

WARNING: - this \( SU(2, \mathbb{Z}) \) is very different than
- symp. transfer of periods
- reparameterizations of \( \Sigma \) (open string amplitude)
- moving in moduli space (changes everything)
- R.S. (mirror marriage)
  \( \Sigma = \text{H}(x, y) = 0 \) \( \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \)
- choice of \( SU(2, \mathbb{Z}) \) parameterization of \( \Sigma \)
### Idea:
Eynard-Orantin:
- Matrix model $\rightarrow$ Spectral curve $E \subset \mathbb{C}^2$
- Free energies $F_g$ & correlation functions $W_k^{(g)}$ defined recursively
  (solution of loop eqs., see Klemm's talk)

* Everything is geometric on $E$, and can be defined whether $E$ is the spectral curve of a MM or not.

### Recall:
- B-model: we have a mirror curve $\Sigma \subset \mathbb{C} \times \mathbb{C}$

### Strategy:
- Use EO to generate open/closed top. string amplitudes ($F_g$ and $W_k^{(g)}$), suitably modified for curves in $\mathbb{C}^2 \times \mathbb{C}^2$ rather than $\mathbb{C}^2$.

### Warning:
This is experimental mathematics:
- We have no matrix model or large $N$ transition to justify our claim (yet!), but it works! :)

[Marino '04, Eynard-Orantin '07, VB-Klemm-Marino-Pasquetti]
FORMALISM

We start w/ \( \Sigma: \{ H(x,y) = 0 \} \subseteq \mathbb{C}^* \times \mathbb{C}^* \).

**Ingredients:**

\( \Sigma \ni q \): ram. pts. \( q_i \in \Sigma \) of projection \( \pi: \Sigma \rightarrow \mathbb{C}^+ \) on x-axis (\( \frac{\partial H}{\partial x}(q_i) = 0 \))

\( \mathbb{C}^* \ni x(q) \): near \( q_i \), there are 1 pts \( q, \bar{q} \in \Sigma \) s.t. \( x(q) = x(\bar{q}) \)

- mwo. diff \( \Phi = \log y \frac{dx}{x} \)
  - coming from symp. form \( \frac{dx \wedge dy}{x} \) on \( \mathbb{C}^4 \times \mathbb{C}^4 \)
  - governs comp. struct. transformations

- Bergmann kernel \( B(p,q) \):
  - unique mwo. diff \( \Phi \) w/ double pole at \( p=q \)
  - and no other pole, and normalized by \( \int_{\mathbb{C}^4} B(p,q) = 0 \)
  - in canonical basis of cycles.

\( dE_{q,\bar{q}}(p) = \frac{1}{2} \int_{\mathbb{C}^4} d\Phi B(p,q) \) ^{\uparrow} (important choice)

**Ex:** \( \Sigma \) genus 0, \( B(p,q) = \frac{d\overline{z}(p)dz(q)}{(z(p)-z(q))^2} \)
**Recursion:**

**Step 1:**
- Generate new diffs: $W_k^{(g)}(p_1, \ldots, p_k)$
- Fix $W_1^{(g)}(p_1) = 0$, $W_2^{(g)}(p_1, p_2) = B(p_1, p_2)$

\[
W_k^{(g)}(p_0, \ldots, p_{k-1}) = - \sum_{q_i} \text{Res}_{q_i = q_i} \frac{dE_{q, \bar{q}}(p_0)}{\Phi(q) - \Phi(\bar{q})} \left( W_{k+1}^{(g-1)}(q, \bar{q}, p_1, \ldots, p_{k-1}) \right)
\]

\[
+ \sum_{e=0}^{g} \sum_{n=0}^{k-1} \sum_{s \in S(k-1)} \frac{1}{n!} W_{m+1}^{(g-e)}(q, p_{o(1)}, \ldots, p_{o(m)}) \cdot W_{k-m}^{(e)}(\bar{q}, p_{o(m+1)}, \ldots, p_{o(k-1)})
\]

with $s \in S(k-1)$ permutations of $k-1$ elements.

**Step 2:**
- Generate fact: $F_g$

\[
F_g = \frac{1}{2-2g} \sum_{q_i} \text{Res}_{q_i = q_i} \Phi(q) W_1^{(g)}(q)
\]

with $d\phi(p) = \Phi(p)$

any anti-derivative

---

**Claim:**
- $F_g + \text{minu map} \to \text{closed amplitudes}$
- $A_k^{(g)} = \int W_k^{(g)} + \text{minu map} \to \text{open amplitudes}$
Comments

- gluing procedure, e.g.
  \[ F_1 = W_1^{(1)} \cdot \text{disk} \]

  etc.

- fund. objects: disk & annulus
  \[ \left[ \text{closed bulk} \right] \] \[ \left[ \text{from open} \right] \]

- \( F_g \) invariant under \( \text{SL}(2,\mathbb{Z}) \), but \( A_k^{(g)} \) depend on parameterization of \( \Sigma \): framing + location of the brane

- recalling previous discussion, this generates hol. \( \tilde{F}_g \) and conv. fct\( s \) \( \tilde{W}_k^{(g)} \), i.e.
  the limits of the physical amplitudes at a given point in moduli space

  * no ambiguity in recursion rels

  \[ \Rightarrow \] if we know disk + annulus at a point, we can generate unambiguously all open/closed amplitudes at this point.

  \( \Rightarrow \) this works for any point in moduli space
CHECKS:

LR point of:
- framed vertex
- framed inner/outer branes in resolved conifold
- " " " " local $\mathbb{P}^2$
- " " " " local $\mathbb{P}' \times \mathbb{P}'$
- local $F_1$
- local $F_2$

Other points
- blow-down $\mathbb{P}' \times \mathbb{P}' \rightsquigarrow$ CS theory on $S^3/\mathbb{Z}_2$
- blow-down $\mathbb{P}^2 \rightsquigarrow \mathbb{C}^3/\mathbb{Z}_3$
**EXAMPLES:**

1. **FRAMED VERTEX:**

   \[ H(x, y) = x + y + 1 = 0 \]

   **Framing:** \( \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}) \)

   \((x, y) = (x, y^f, y)\)

   \(\rightarrow\) \(H_4(x, y) = x + y^{f+1} + y^f = 0\)

2. **Disk:**

   \[ A_1^{(0)} = \int \log y \, \frac{dx}{x} \]

   solve \( y = y(x) = -1 - (-1)^f x + f x^2 + \ldots \)

   **minn map:** \( X = -x \),

   \(\rightarrow\) \( A_1^{(0)} = \frac{-(-1)^f x}{f} - \frac{1}{4} (1 + 2f) x^2 - \ldots \)

3. **Annulus:** \( \Sigma \) genus 0 \(\Rightarrow\) \( B(x_1, x_2) = \frac{dy_1(x_i) \, dy_2(x_i)}{(y_1(x_i) - y_2(x_i))^2} \)

   \[ A_\Sigma^{(0)} = \int \left( B(p, q) - \frac{dx_1 \, dx_2}{(x_1 - x_2)^2} \right) \]

   \(= \log(-y_1(x_1) + y_2(x_2)) - \log(-x_1 + x_2)\)

   \(= \frac{1}{2} f(f+1)x_1 x_2 + \frac{(-1)^f}{3} \cdot f(1 + 3f + 2f^2)(x_1^2 x_2 + x_1 x_2^2) + \ldots \)
3-HOLE: \text{ ram points: } \frac{2H}{2y} = 0

\implies \text{ one pt at } y = \frac{-f}{f+1} := q_1

A_3^{(0)} = \int \text{Res}_{y=q_1} \frac{x(y) y \ dy_1(x_1) dy_2(x_2) dy_3(x_3)}{(y-y_1(x_1))^2 (y-y_2(x_2))^2 (y-y_3(x_3))^2} \left(\frac{dx}{dy}\right)^{-1}

= \frac{f^2}{f+1} \sum_{i=1}^{3} \frac{1}{f+(f+1)y_i(x_i)}

= -(-1)^f f^2 (1+t)^2 x_1 x_2 x_3 + \ldots

\text{ etc. always works!}

- genus 1 examples much more complicated computationally, but always works

- the procedure is algorithmic and could be implemented in a computer code
(2) **LOCAL $\mathbb{P}^1 \times \mathbb{P}^1$:**

\[
H(x,y) = x^2 y + xy^2 + xy + e^{-t_1} y + e^{-t_2} x = 0
\]

$Z_1 = e^{-t_1}$, $Z_2 = e^{-t_2}$

**LR:** works

**other point:** [Aganagic-Klemm-Mariño-Vafa]

\[
X_1 = 1 - \frac{Z_1}{Z_2}, \quad X_2 = \frac{1}{\sqrt{Z_2} (1 - \frac{Z_1}{Z_2})}
\]

$X_1, X_2 \to 0$ \["orbifold" point, $\mathbb{P}^1 \times \mathbb{P}^1 \to$ zero size\]

---

**THEN:** closed top. strings $\iff$ CS theory on lens space $S^3/\mathbb{Z}_2$

because geometric transition:

local $\mathbb{P}^1 \xleftarrow{\text{blow down $\mathbb{P}^1$}} \text{sing. cm.} \xrightarrow{\text{defang}} T^* S^3$

now

$\mathbb{Z}_2$

blow up extra $\mathbb{P}^1$ (fixed pt under $\mathbb{Z}_2$)

$\mathbb{Z}_2$

local $\mathbb{P}^1 \times \mathbb{P}^1 \xleftarrow{\text{sing.}} T^* (S^3/\mathbb{Z}_2)$
What we now have

(open amplitudes)

(schematically)

framed unknot in

$\text{CS}(S^3/\mathbb{Z}_2)$

IT WORKS!

(rather involved calculation, 
strong check of validity of 
the formalism)
LR \rightarrow \text{orbifold: S-duality transform}

(see that by looking at ram. points, i.e. cuts)

"exchange" of the cuts

\[ \xrightarrow{\text{S-duality}} \]

\[ \xrightarrow{\text{S-duality}} \]

\[ \xrightarrow{\text{S-duality}} \]

= easy to compute $B(x_1, x_2)$ at orbifold using Akemann's formula for hyperelliptic curves

just anal. continuation, no modularity

\[ \xrightarrow{\text{S-duality}} \]

So we have disk, annulus + Akemann

\[ \Rightarrow \text{generate all other amplitudes recursively at the orbifold point} \]
CS SIDE:

CS \( (S^3/\mathbb{Z}_2) \): 2-Matrix model.

\[
Z(N_1, N_2, g_s) = \int dM_1 dM_2 \exp \left\{ \frac{-1}{2g_s} \text{Tr} M_1^2 - \frac{1}{2g_s} \text{Tr} M_2^2 + V(M_1) + V(M_2) + W(M_1,M_2) \right\}
\]

\[
V(M) = \frac{1}{2} \sum_{k=1}^{\infty} a_k \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \text{Tr} M^s \text{Tr} M^{2k-s}
\]

\[
W(M_1,M_2) = \sum_{k=1}^{\infty} b_k \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \text{Tr} M_1^s \text{Tr} M_2^{2k-s}
\]

\[
a_k = \frac{B_{2k}}{\kappa(2k)!} \quad b_k = \frac{2^{2k} - 1}{\kappa(2k)!} B_{2k}
\]

\[\text{UNKNOWT: } W_k^{(g)}(N_1, N_2, g_s) = \frac{1}{Z(N_1, N_2, g_s)} \text{Tr} \, e^M \langle e^M \rangle \]

\(w\), in terms of eigenvalues \( m_i^1, m_j^2 \) of \( M_1, M_2 \),

\( e^M = \text{diag}( e^{m_1^1}, \ldots, e^{m_{N_1}^1}, -e^{m_1^2}, \ldots, -e^{m_{N_2}^2} ) \)

\[\text{\underline{WINDS: } } W_k^{(g)} \text{ match top. string calculation !} \]
**CRUCIAL POINT**: Mirror map at orbifold pt

- closed mirror map found in [AKMV]
- open mirror map?

→ by watching disk amplitude, very simple open mirror map:

\[ X = \frac{x}{x_1 \cdot x_2} \]

[would be nice to have a better justification]

⇒ good results for \( W_2^{(r)} , W_3^{(0)} , W_1^{(1)} , ... \)

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**3 LOCAL \( \mathbb{P}^2 \)**: - "open" amplitudes for \( \mathbb{C}^3/\mathbb{Z}_3 \)?

"PROBLEMS": • open mirror map?

- transf is not S-duality, but complicated
  
  \[ \text{SL}(2, \mathbb{C}) \text{ transf} \quad [\text{VB-Nagano\-\-Vlemm}] \]

**Disk**: • find open mirror map by requiring monodromy-invariance,

\[ X = \frac{x}{\psi} \quad \leftarrow \text{orbifold parameter} \]

\[ \psi \quad \text{comp. struct. parameter}, \text{orbifold at } \psi = 0 \]

[nothing to compare with]
Annulus: • not clear how to implement SL(2, C) transf.
of B(x1, x2), but should be possible

⇒ all the other amplitudes recursively

Q: • what would these amplitudes compute? [• open orbifold GW invariants?
• large N transition?

SUMMARY

• new B-model formalism for open/closed amplitudes on toric CY3, no ambiguity

• can be used at other pts in moduli space
  ⇒ open/closed amplitudes for sing. CY3 [orbifolds, ...]
OPEN QUESTIONS

• Is there a matrix model (and/or large \( N \) transition) behind this story? 
  \( \Rightarrow \) non-perturbative insight on top. strings

• Compact CY\(_3\) ?
  - need to find something to replace \( \Sigma \),
  or generalize the formalism to higher-dim. manifolds

• rel. w/ Witten’s open hot. anomaly \( \phi \)’s?

• etc...

THANK YOU! ☺️