We review recent progress in understanding non-perturbative instanton corrections to the hypermultiplet moduli space in type II string compactifications on Calabi-Yau threefolds.

Instanton corrections recently played a prominent role both in the context of string cosmology (KKLT models) and particle physics model building. A setup where one might hope to understand these corrections using string dualities and supersymmetry constraints on the low energy effective action (LEEA) is type II string theory compactified on a Calabi-Yau threefold (CY3). In this case the LEEA is a N = 2, d = 4 supergravity action which generically receives quantum corrections from the world sheet conformal field theory (α'-corrections), from higher genus world sheets (perturbative gs-corrections), and from instanton corrections due to Euclidean branes wrapping supersymmetric cycles of the CY3 (non-perturbative gs-corrections). While the α'-corrections to the classical LEEA are well understood, the non-perturbative gs corrections [1] still pose many open questions.

Besides the supergravity gravity multiplet these LEEA contain nV vector- and nH hypermultiplets. Supersymmetry implies that the total moduli space M of these theories factorizes into a local product M = MVM ⊗ MHM where MVM and MHM are parameterized by the scalars of the VM and HM, respectively. This factorization has the profound consequence that only those subsectors which contain the dilaton (volume modulus) receive gs (α') corrections. Furthermore, supersymmetry dictates that MVM be a special Kähler manifold, and MHM a quaternion-Kähler manifold.

For both IIA and IIB compactifications the dilaton sits in a hypermultiplet. Thus determining the gs corrections to the LEEA requires understanding MHM. In this context it turns out to be useful
that in the presence of \( n_H + 1 \) commuting shift symmetries\(^1\), \( \mathcal{M}_{HM} \) can equivalently be described by rigidly superconformal tensor multiplet (TM) actions [2]

\[
\mathcal{L} = \oint_{\mathcal{C}} \frac{d\zeta}{2\pi i\zeta} H(\eta^I) \, .
\]

(1)

Here \( \eta^I = v^I/\zeta + x^I - \zeta \bar{v}^I \) denotes a \( \mathcal{N} = 2 \) tensor superfield, \( H(\eta^I) \) is a weakly homogeneous function of degree 1 (i.e., logarithmic terms are allowed) with no explicit \( \zeta \)-dependence, and \( \mathcal{C} \) is an arbitrary contour in the complex \( \zeta \)-plane. Taking \( \mathcal{L} \) as a function of the scalar fields \( v^I, \bar{v}^I, x^I \), one can define a “tensor potential” [3]

\[
\chi = -\mathcal{L} + x^I \frac{\partial \mathcal{L}}{\partial x^I} \, .
\]

(2)

Coupling the rigidly superconformal TM Lagrangian to conformal supergravity and carrying out the superconformal quotient both \( \mathcal{L} \) and \( \chi \) can be used to determine \( \mathcal{M}_{HM} \) [3]. Working with \( \chi \) thereby has the virtue that (discrete) symmetries of the supergravity theory, as e.g., the \( \text{SL}(2, \mathbb{Z}) \) invariance of the type IIB string, are reflected by the invariance of \( \chi \).

Building on the off-shell description [4] of the c-map [5] the perturbatively corrected hypermultiplet moduli space has been determined in [6]

\[
\mathcal{L}(v, \bar{v}, x) = \text{Im} \oint_{\mathcal{C}} \frac{d\zeta}{2\pi i\zeta} \left[ \frac{F(\eta^A)}{\eta^0} \pm \frac{i\chi_E}{(2\pi)^3} 2\zeta(2) \eta^0 \ln(\eta^0) \right] \, .
\]

(3)

Here \( F(\eta^A) \) is the holomorphic prepotential underlying the dual vector multiplet geometry, \( \eta^0 \) is an additional TM acting as a conformal compensator and the contour encloses the branch cut between \( \zeta = 0 \) and one of the zeros of \( \zeta \eta^0 \). The second term encodes the universal one-loop correction with the upper (lower) sign correspond to type IIB (type IIA) strings, respectively, with \( \chi_E \) being the Euler number of the CY\(_3\).

Substituting (3) into (2) yields the tensor potential underlying the perturbatively corrected \( \mathcal{M}_{HM} \). For type IIB the corresponding expression naturally splits into a classical part, \( \chi_{cl} \), perturbative \( \alpha' \) and \( g_s \) corrections, \( \chi_{pt} \), and non-perturbative world-sheet instanton contributions, \( \chi_{ws} \):

\[
\chi_{cl} = 4 \tau_2 \frac{1}{3!} \kappa_{abc} t^a t^b t^c, \quad \chi_{pt} = \frac{1}{(2\pi)^3} \tau_2 \chi_E \left[ \zeta(3) \tau_2^2 + 2\zeta(2) \right], \quad \chi_{ws} = -\frac{\tau_2}{(2\pi)^3} \sum_{k_a} n_{k_a} \left[ \text{Li}_3(e^{2\pi ik_a z^a}) + 2\pi k_a t^a \text{Li}_2(e^{2\pi ik_a z^a}) + \text{c.c.} \right] \, .
\]

(4)

These are determined by the triple intersection numbers \( \kappa_{abc} \), the Euler number \( \chi_E \) and the Gopakumar-Vafa invariants \( n_{k_a} \) of the CY\(_3\). The one-loop correction gives rise to the second term in \( \chi_{pt} \) which is suppressed by \( \tau_2^{-2} = g_s^{\gamma} \) compared to the three-level terms.

\(^1\)In the case of string compactifications these shift symmetries naturally arise from the reduction of the p-forms in the ten-dimensional type II supergravity actions and are preserved in the presence of D(-1) and D1 instanton corrections in type IIB or A-type membrane instanton corrections in type IIA.
In order to determine the D(-1) and D1-brane instanton corrections to the IIB LEA we make use of the non-perturbative SL(2, $\mathbb{Z}$) invariance of the type IIB string. The modular transformation properties of the four-dimensional scalars can be found by tracing the transformation properties of the ten-dimensional IIB supergravity fields through the dimensional reduction [7]

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad t^a \mapsto t^a |c\tau + d|, \quad b^a \mapsto d b^a + c e^a, \quad e^a \mapsto b b^a + a e^a. \quad (5)$$

The modular transformation of the conformal compensator $r^0 \mapsto r^0 |c\tau + d|$ is determined by lifting the action of SL(2, $\mathbb{Z}$) to superspace [8].

Applying the SL(2, $\mathbb{Z}$) transformations (5) to (4) one finds that $\chi_{cl}$ is modular invariant while the $\alpha'$ and $g_s$ corrections break the SL(2, $\mathbb{Z}$) invariance. Restoring SL(2, $\mathbb{Z}$) invariance by a modular completion of $\chi_{pt}$ and $\chi_{ws}$ gives

$$\chi_{\text{IIB}}^{(-1)} = \frac{r^0 \tau_2}{2(2\pi)^3} \chi_{E} \sum_{m,n} \frac{\tau_2^{3/2}}{|m\tau + n|^3}, \quad \chi_{\text{IIB}}^{(1)} = \frac{r^0 \tau_2}{(2\pi)^3} \sum_{k_a} n_{k_a} \sum_{m,n} \frac{\tau_2^{3/2}}{|m\tau + n|^3} \left( 1 + 2\pi |m\tau + n| k_a t^a \right) e^{-S_{m,n}}, \quad (6)$$

and determines the corrections of the D(-1) and D(1) instanton corrections, respectively. Here

$$S_{m,n} = 2\pi k_a (|m\tau + n| t^a - im e^a - in b^a), \quad (7)$$

is the instanton action of a $(m, n)$-string with $m$ units of D(1) and $n$ units of fundamental string charge.

By the SYZ construction of mirror symmetry [9], the instanton contributions (6) are mirror to a certain class of membrane instantons, i.e., Euclidean D2-branes, wrapping so-called A-cycles of the mirror CY$_3$. The corresponding mirror map in the hypermultiplet sector is given by [10]

$$\phi_{\text{IIA}} = \phi_{\text{IIB}}, \quad A^1 = \tau_1, \quad A^a = -(e^a - \tau_1 b^a), \quad z^a_{\text{IIA}} = z^a_{\text{IIB}}. \quad (8)$$

Furthermore, implementing mirror symmetry requires some rearrangements of the various contributions to the type IIB tensor potential which is illustrated in table 1. The resulting corrections due to D2-brane instantons was then given in [10]

$$\chi_{\text{IIA}}^{\Lambda - \text{D2}} = \frac{r^0 \tau_2}{2\pi^2} \sum_{k_{\Lambda}} n_{k_{\Lambda}} \sum_{m \neq 0} \frac{|k_{\Lambda} z^\Lambda|}{|m|} K_1 \left( 2\pi \tau_2 |m k_{\Lambda} z^\Lambda| \right) e^{-2\pi i m k_{\Lambda} A^\Lambda}. \quad (9)$$

Here, $k_{\Lambda} = (n, k_a)$, $z^\Lambda = (1, z^a)$, $A^\Lambda = (A^1, A^a)$, and the sum over $k_a$ now includes the zero-vector $k_a = 0$, but $k_{\Lambda} = 0$ is excluded. The type IIA "instanton numbers" are

$$n_{(n, k_a = 0)} = \frac{1}{2} \chi_{E}(X), \quad n_{(n, k_a)} = n_{k_a} \quad \text{as in type IIB}, \quad (10)$$

with $\chi_{E}(X)$ the Euler number of the CY$_3$ and $n_{k_a}$ the Gopakumar-Vafa invariants of the mirror CY$_3$. This indicates a deep connection between the properties of holomorphic two-cycles (counted by the Gopakumar-Vafa invariants) and special Lagrangian three-cycles of the mirror manifold.
\[
\begin{array}{|c|c|c|}
\hline
\text{IIB HM} & \text{SL}(2, \mathbb{Z})\text{-invariant} & \text{IIA HM composed from IIB terms} \\
\hline
\chi_{\text{cl}} & = \chi_{\text{cl}} & \chi_{\text{tree}} = \chi_{\text{cl}} + \chi_{\text{ws-pert}} + \chi_{\text{ws-inst}} \\
\chi(-1) & = \chi_{\text{ws-pert}} + \chi_{\text{loop}} + \chi_{D(-1)} & \chi_{\text{loop}} = \chi_{\text{loop}} \\
\chi(1) & = \chi_{\text{ws-inst}} + \chi_{D1} & \chi_{A-D2} = \chi_{D(-1)} + \chi_{D1} \\
\hline
\end{array}
\]

Table 1: Rearrangement of contributions of the type IIB tensor potential under mirror symmetry.

### References


- 4 -