Week 1 Lecture 1 A.S. Goldhaber

What is physics?

A dictionary definition:

Science dealing with matter and energy in terms of motion and force

Better:

a) Method of prediction — To predict, must first describe. Choice of variables makes it easier.

b) Experimental

c) Mathematical

d) Restricted (not, e.g., biology where things are still too complicated for precise physics methods)

Themes of Physics:

Scale extension - Describe phenomena over a range of distance or energy scales

Earliest perhaps: earth = planet, sun = star?

E+M: earth is a magnet 1600

Newton's gravity 1665 (later)

Conservation laws

E+M: Flux lines ~ electric end on charges; magnetic never end.

Unification

E+M: Magnetism = electric current

Changing $\vec{B} \Rightarrow \vec{E}$

Changing $\vec{E} \Rightarrow \vec{B}$

Light = E.M wave

Relativity: c = universal const. (unifies mechanics and E+M)

Symmetry

E+M: Gauge invariance

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

$$\vec{A}' = \vec{A} + \nabla\Lambda \qquad \varphi' = \varphi - \frac{\partial \Lambda}{\partial t}$$

$$\vec{E}' = \vec{E}, \quad \vec{B}' = \vec{B}$$

We'll explore all these themes

Week 1 Lecture 2

Survey of electrostatics

Force on point charge

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Force density: $\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$

Equation of continuity

(Local conservation of charge)

$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$; $(L+L)\frac{\partial \rho}{\partial t} + \mathrm{div}\, J = 0$

Integral form:

$\frac{dQ}{dt} + I_{out} = 0$

$Q = \int dV \rho$, $I_{out} = \int d\vec{s} \cdot \vec{J}$

Steady electric field: $\frac{\partial \vec{E}}{\partial t} = 0$

$\nabla \times \vec{E} = \text{curl}\, \vec{E} = 0$ (a)

$\nabla \cdot \vec{E} = \text{div}\, \vec{E} = 4\pi k \rho = \frac{\rho}{\varepsilon_0}$

Integral forms:

$\oint \vec{E} \cdot d\vec{l} = \mathcal{E}mf = 0$

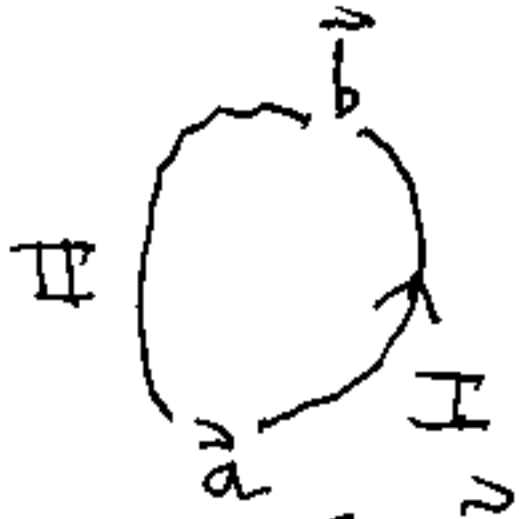
$\oint \vec{E} \cdot d\vec{S} = 4\pi k Q$

If (a) $\vec{E}$ everywhere curl-free

then $\vec{E} = -\nabla \varphi = -\text{grad}\, \varphi$

Proof:

$\int_{\vec{a}}^{\vec{b}} d\vec{l} \cdot \vec{E}$ independent of path



$\int_I - \int_{II} = \oint d\vec{l} \cdot \vec{E} = 0$

$\therefore \int_a^b = \varphi(\vec{x} = \vec{a}) - \varphi(\vec{x} = \vec{b})$

If $\varphi(\vec{x}) \to \varphi(\vec{x}) + C$
no change $\varphi(\vec{a}) - \varphi(\vec{b})$, so
no change in $\vec{E}$.

$$\vec{E} = -\nabla\varphi$$

$$\nabla\cdot\vec{E} = -\nabla^2\varphi = 4\pi k\rho = \frac{\rho}{\epsilon_0}$$

Poisson Equation

$\rho = 0$: $\nabla^2\varphi = 0$ (Laplace eq.)

$\nabla^2\varphi = (\nabla\cdot\nabla)\,\Delta\varphi = \text{laplacian}(\varphi)$

Point charge $\vec{E} = \frac{kq\hat{r}}{r^2} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2}$.

$$\varphi = \frac{kq}{r} = \frac{q}{4\pi\epsilon_0 r}$$

$$\nabla^2\varphi = -\frac{q}{\epsilon_0}\,\delta^{(3)}(\vec{r})$$

↖ Dirac delta fn

$\delta^{(3)}(\vec{r}) = 0$, $\vec{r} \neq 0$
= undefined $\vec{r} = 0$
NOT A FUNCTION!!

Def: $\int d^3r\, \delta^{(3)}(\vec{r})\, f(\vec{r}) = f(0)$

$\delta^{(3)}$ is a generalized f'n or distribution
It is not defined, but
$\int \delta f$ is defined as a map from the function $f(\vec{r})$ to the number $f(\vec{r}=0)$. Obviously this assumes f is reasonably smooth, e.g., continuous.

Thought question:
Is this the precise criterion?

Look at this another way using the divergence theorem

$$\int_V dV\, \nabla\cdot\vec{E} = \int_{S \text{ of } V} d\vec{S}\cdot\vec{E}$$

Let V = ball centered on $\vec{r}=0$, w/ radius R.

$$\int_S d\vec{S}\cdot\vec{E} = \int d\Omega\, R^2\, \hat{r}\cdot\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{R^2}$$

$$= \frac{q}{\epsilon_0}.$$ However, outside $\vec{r}=0$ $\nabla\cdot\vec{E}=0$ (check this), so

$\int_V \nabla \cdot \vec{E} = -\int_V \nabla^2 \varphi = 0$ unless V includes $\vec{r} = 0$. $\int_V \nabla \cdot \vec{E} = -q/\epsilon_0$ if V includes $\vec{r} = 0$.

Equivalent to the previous discussion, where now the function $f(\vec{r}) = 1$ for $\vec{r}$ inside V, $f = 0$ otherwise.

Potential energy in electrostatics:

$U(\vec{r}) = q\varphi(\vec{r})$ for point charge q at $\vec{r}$.

Collection of charges

$$U = \sum_{i<j} q_i \varphi_j(\vec{r}_i)$$

Continuous density —

$$U = \frac{1}{2}\int dV\, \rho(\vec{r})\varphi(\vec{r})$$

Reciprocity theorem

$$q_i \varphi_j(\vec{r}_i) = q_j \varphi_i(\vec{r}_j)$$

Obvious for point charges in empty space:

$$\varphi_i(\vec{r}_j) = \frac{q_j}{4\pi\epsilon_0 |\vec{r}_i - \vec{r}_j|}$$

So

$$q_i \varphi_i(\vec{r}_j) = \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_i - \vec{r}_j|}$$

$$= q_j \varphi_j(\vec{r}_i)$$

Reciprocity is more general, e.g. in presence of conductors.

Earnshaw's thm:

No static arrangement of point charges can be in stable equilibrium for finite separation of the charges

Proof: First, a lemma:

For charge q_i the potential at $\vec{r}_i$ from other charges obeys

Laplace eq. $\nabla^2 \varphi(\vec{r}) = 0$.

Consider a sphere centered on $\vec{r}_i$ of radius R.

$$\langle \varphi(|\vec{r}-\vec{r}_i| = R) \rangle = \varphi(\vec{r}_i).$$

(Mean value theorem)

Proof:

$$\int_{|\vec{r}-\vec{r}_i| \le R} dV \left(\nabla^2 \varphi \underbrace{\left[\frac{1}{|\vec{r}-\vec{r}_i|} \right]}_{0} - \varphi \nabla^2 \frac{1}{|\vec{r}-\vec{r}_i|} \right)$$

$$= 4\pi \varphi(\vec{r}_i) \underbrace{\left[-\frac{1}{R} \right]}_{0} - 4\pi \varphi(\vec{r}_i)$$

$$= \int_{|\vec{r}-\vec{r}_i| = R} d\vec{S} \cdot \left(\nabla \varphi \left(\frac{1}{R} - \frac{1}{R} \right) - \varphi \nabla \frac{1}{|\vec{r}-\vec{r}_i|} \right)$$

$$= \int d\Omega \, R^2 \hat{r} \cdot \varphi \left(-\nabla \frac{1}{|\vec{r}-\vec{r}_i|} \right)$$

$$= \int d\Omega \, R^2 \hat{r} \cdot \varphi(\vec{r}) \frac{\hat{r}}{|\vec{r}-\vec{r}_i|^2}$$

$\because \hat{r} = (\vec{r}-\vec{r}_i)/|\vec{r}-\vec{r}_i|$ $\quad \approx R^2$

$$= \int d\Omega \, \varphi(\vec{r}) = 4\pi \langle \varphi(\vec{r}) \rangle.$$

Done!

This means that if Φ goes up in some directions from $\vec{r}_i$ it must go down in others, and vice versa, so there must be a lower value of $U(\vec{r}_i)$ somewhere else. The only exception would be if Φ were constant, giving neutral equilibrium, but that is not possible if any other charges are present, because Φ actually blows up at the location of each source charge.

Thus Φ is always possible to find a place for q_i with lower potential energy, unless q_i is right on top of an opposite sign charge.

This proves that any classical atom is unstable, but it proves too much, because the same holds for the planets in the solar system!

What is missing?

For an electron near a nucleus in an orbit, electromagnetic radiation rapidly takes it in to collapse, so a classical atom would be unstable. For a sufficiently big orbit the radiation has little effect for a very long time. Not only are the planetary orbits very big, but also the radiation is less efficient because gravity is weak and the strongest radiation is quadrupole, not dipole (discussed later in the course).

Magnetostatics

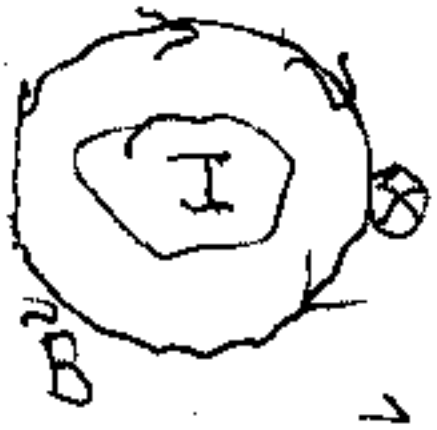
$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = 4\pi k' \vec{J} = \mu_0 \vec{J}$$

Stokes' Theorem

$$\oint_{\text{Border of } S} \vec{B} \cdot d\vec{l} = \int_S d\vec{S} \cdot \nabla \times \vec{B}$$

$$= 4\pi k' I = \mu_0 I$$



By symmetry, $\vec{B} = B(r)\,\hat{\varphi}$

$$\int \vec{B}\cdot d\vec{l} = 2\pi B(r) = 4\pi k' I$$

$$B(r) = \frac{2k'}{r} I = \frac{\mu_0 I}{2\pi r}$$

Homework

1. Find $\nabla \cdot \vec{f}$ in spherical polar coordinates.
2. Find $\nabla^2 g$ in spherical polar coordinates.

Background:

$$dg = dr\,\partial_r g + d\theta\,\partial_\theta g + d\varphi\,\partial_\varphi g$$

$$= d\vec{r}\cdot\nabla g$$

$dl_r = dr$, $dl_\theta = r\,d\theta$, $dl_\varphi = r\sin\theta\,d\varphi$

So $\nabla g_r = \hat{r}_r g$, $\nabla g_\theta = \frac{1}{r}\partial_\theta g$

$\nabla g_\varphi = \frac{1}{r\sin\theta}\partial_\varphi g$