From last lecture we concluded that:

\[ \begin{align*}
    x' &= a(x + vt) \\
    t' &= a(t + \frac{1}{\sqrt{k}}x)
\end{align*} \]

The most general case.

Note that if \( t' = t \) then \( a \) must be 1 and \( \frac{1}{\sqrt{k}} = 0 \), which reduces the system to the Galilean case of:

\[ \begin{align*}
    x' &= x + vt \\
    t' &= t
\end{align*} \]

Going back to general case, let's see what restriction will a won't work:

1. Try \( k < 0 \). Let's say at some time, \( t_0 > 0 \), in rest frame A and position \( x = 0 \), a horse gets surprised by a firecracker and immediately takes off with velocity \( v \), along the \( x \)-axis. After the horse has run a distance \( x_1 \) (in the rest frame) what time will the horse think it is (if it could tell time)?

\[ t'_{horse} = a(t_0 + \frac{1}{\sqrt{k}}x_1) \]

It would be possible for the horse to run such that \( v \cdot x_1 \geq |t₀ - k| \).
Assuming \( a > 0 \), then \( t_0 + \frac{1}{k} X_1 \) would be a negative quantity (\( k < 0 \), but to, \( v, a X > 0 \)) and so \( t' < 0 \). This would mean that the horse got scared and ran to \( X_1 \) before the firecracker exploded. This in turn violates the principle of causality. (If \( a < 0 \), then it would have to choose \( v X \), \( > 1 < a k \) and some situation would occur.)

\[
\begin{bmatrix} k > 0 \end{bmatrix}
\]

For this reason let's denote \( k \) by:

\[
\begin{bmatrix} k = \frac{1}{S^2} \end{bmatrix}
\]

where \( S \) must be a real number and have units of speed (so then \( \frac{v}{k X} \) has units of time).

\( 2) \) Consider \( y' \) as a function of time.

Let's look at the horse again, only this time after it has run to \( X_1 \) with velocity \( v' \), it turns around and runs back to \( X_0 \) with velocity \( -v' \).
Since the horse exactly reversed its path it must end in the exact place it started. This means \( y'(t_{\text{start}}) = y'(t_{\text{end}}) \). However, we can again choose \( (x/y) \) such that \( t_{\text{end}} \) is anything. This implies \( y' = \text{constant} \). \( \therefore y' \) is not a function of time, and similarly \( z' \) is not a function of time.

3) Could \( y' \) be a function of \( x' \)? Consider rotating the coordinate system from \( \vec{v} \parallel x \to \vec{v} \parallel -x \).

\[
\begin{align*}
y' &= y \\
\text{For any given point in rotation } y' &= f(\cos(\theta)\vec{v}) \\
\text{\( y' \) must be constant, } |\vec{v}| \text{ is constant,} \\
\text{but } \cos(\theta) \text{ is not. } \therefore y'
\end{align*}
\]

\( y' \) can not be a function of \( x' \) which may vary.

4) Could \( y' \) be a function of \( y' \)? Consider two reference frames, \( A \) & \( B \), where \( A \) is at rest while \( B \) moves in \( x \) at velocity \( V \), or \( B \) is at rest while \( A \) moves with \(-V\). Both of these situations should be equivalent.
In case 1) let \( \gamma_B = f(\gamma_A) \). In case 2),
\[ \gamma_A = f(\gamma_B) \] .
But \( \gamma_B = f(\gamma_A) \) so \( \gamma_A = f(f(\gamma_A)) \).

This only works if \( f(f(x)) = x \) generally.

\[ \gamma_B = f(\gamma_A) = \gamma_A \quad \text{and} \quad \gamma_A = f(\gamma_B) = \gamma_B. \]

Thus, \( \boxed{\gamma' = \gamma} \). Similarly can prove \( \boxed{Z' = Z} \).

Finally, let's consider what 'on' must be when we impose the restriction that the speed of light must be same in all inertial frames. Let \( K'(x';y';z') \) be an inertial frame moving with respect to \( K(x,y,z) \) at velocity \( v \), in \( x = x' \) direction. Let a flash bulb go off at the origins when \( t = t' = 0 \). The speed of the spherical wave-front
of the light, traveling at a constant speed, \( c \), can be given by:

\[
x^2 + y^2 + z^2 = c^2 t^2
\]

\[
x'^2 + y'^2 + z'^2 = c^2 t'^2.
\]

As noted before, \( y' = y \) and \( z' = z \) at \( t' \parallel x' \).

We also know that \( x' = a(x - ut) \) and by symmetry \( x = a(x' + ut) \), depending on which system we decide is moving: \( x' \) with velocity \( v \)

or \( x \) with velocity \( (-v) \). By symmetry then \( ta = a' \).

Along the \( x \) or \( x' \) axes:

\[
x' = a(x - ut) = a(ct - ut) \quad \text{and} \quad x' = ct'
\]

\[
x' = a(x' + ut) = a(ct + ut) \quad \text{and} \quad x = ct
\]

\[ i) \quad ct' = a(ct - ut) \Rightarrow t' = at \left( 1 - \frac{v}{c} \right) \]

\[ ii) \quad ct = a(ct' - ut') \Rightarrow t = at' \left( 1 + \frac{v}{c} \right) \]

Using \( t \) from ii) by plugging into i):

\[
t' = a \left[ (at')(1 + \frac{v}{c}) \right] \left( 1 - \frac{v}{c} \right) = a^2 t' \left[ 1 - \left( \frac{v}{c} \right)^2 \right]
\]

Dividing by \( t' \) \( \left[ 1 - \left( \frac{v}{c} \right)^2 \right] \Rightarrow a^2 = \frac{1}{1 - \left( \frac{v}{c} \right)^2} \]
\[ a = \frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \]

where we pick the positive root of \( \sqrt{1 - \left(\frac{V}{c}\right)^2} \)

so that the principle of causality is not violated (as it was for \( a < 0 \)).

Going back to determine \( S \) in \( t' = a(t - \frac{V}{c}x) \):

Recall \( x = a(x' + vt') \) & \( x' = a(x - vt) \)

which: \( \Rightarrow \) \( x = a(x' + vt') = a(a(x' + vt) + vt') \)

or \( x = a^2(x - vt) + a(vt') \).

Solving for \( t' \) \( t' = \frac{x}{av} - a^2(x - vt) \)

but \( a = \frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \) so \( t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \) \( (\text{Note:} \, s = c) \)

\[ \text{\# Note: proof of \# 5. (ie the Lorentz Transformation) was taken from Modern Physics for Scientists & Engineers, S. Thornton & A. Rex 1993.} \]

Altogether:

\[ \begin{cases} y' = y \\ z' = z \\ x' = \frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} (x - vt) \\ t' = \frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \left( t - \frac{V}{c^2}x \right) \end{cases} \]
Energy and momentum

In Galilean case, $dH = p \cdot \delta V$, where $H$ is the energy of a particle in terms of momentum, $p$. This can be obtained in 2 ways:

1) Since we are only concerned with energy of motion (not potential energy), the $H = \text{kinetic energy} = \frac{1}{2}mv^2$, and $dH = \left( \frac{1}{2} \right) mv \delta v = (mv) \delta v = p \delta V$.

2) Changes in $H$ must be functions of changes in velocity $dH = p \delta V$. This description and have units of energy. For completeness let $dH = p \delta V + T \delta V$, where $T$ is not a function of $v$ (that has already been taken care of by $p \delta V$). $H$ is a scalar and $V$ is a vector, so $T$ must also be a vector. However, $T$ is not a function of $v$, so it would have to be fixed in space. This violates symmetry of space which says space is isotropic and does not have a preferred direction. Thus, $T \delta V$ disappears from $dH$. 
\( dp \) is in the same direction as \( d\vec{v} \), so we can write
\[ dp = M \, d\vec{v} \] (classically \( M \) = mass, but for now, let's simply leave it as some scalar that is not a function of velocity). The reason for this last requirement is that any infinitesimal velocity shifts should be same everywhere, independent of the velocity itself.

Since \( \vec{p} = m \vec{v} \), when \( \vec{v} = 0 \), \( \vec{p}(v=0) = 0 \).

\[
\int_0^v dp = p(v) - p(0) = \vec{p}(v) \quad \Rightarrow \quad \vec{p}(\vec{v}) = M \vec{v}
\]

\[
\int_0^v dp = \int_0^v M \, d\vec{v} = M \vec{v}
\]

Recall \( dH = \rho \, d\vec{v} \), so \( H = H_0 + \int_0^v \frac{M \, v \, d\vec{v}}{2} \)

\[
H = H_0 + \frac{1}{2} M \, v^2
\]

If \( H_0 \) set to zero, then \( H = \frac{1}{2} M \, v^2 = \frac{1}{2} \left( \frac{mv}{m} \right)^2 = \frac{p^2}{2M} \).
For relativistic particle

d\rho is a function of velocity, i.e. it depends on dV and H (also a function of v).

Then, d\rho = M dV (as in classical case) + \frac{\hbar}{c^2} dV

where \frac{1}{\sqrt{2}} is needed to obtain correct units for H-term, and c is the universally invariant speed.

So \frac{d\rho^2}{dV} = M + \frac{\hbar}{c^2} = \frac{\hbar}{c^2}

\frac{dH}{dV} = \frac{d}{dV}(H + \frac{\hbar^2}{c^4}) = \frac{dH}{dV} = \tilde{\rho}.

\frac{dH}{dV} = \rho \text{ and } \frac{d\rho}{dV} = H \text{ (these are two coupled equations with initial conditions } \rho(0) = 0, \text{ \ H}(0) = H_0.)

Finally, dV = d\gamma where \gamma is the rapidity or boost parameter.

(Note 1.) dV = d\gamma, but \nu is not necessarily \gamma.

(2.) \gamma is defined in Jackson as:

\( \gamma = \tanh^{\frac{1}{2}}(\gamma) = (1 - (\frac{\gamma}{c})^2)^{-\frac{1}{2}} = \sinh(\gamma). \)
Solution which satisfies $\frac{\partial H}{\partial v} = p$ and $\frac{\partial P}{\partial v} = H$ is

\[
\begin{align*}
\bar{P} &= H_0 \sinh \gamma \\
\bar{H} &= H_0 \cosh \gamma
\end{align*}
\]

Check:

\[
\frac{\partial \bar{H}}{\partial v} = \frac{\partial H}{\partial y} = \frac{\partial (H_0 \cosh \gamma)}{\partial y} = H_0 \sinh \gamma
\]

\[
\frac{\partial \bar{P}}{\partial v} = \frac{\partial p}{\partial y} = \frac{\partial (H_0 \sinh \gamma)}{\partial y} = \bar{p}.
\]

\[
\frac{\partial \bar{P}}{\partial v} = \frac{\partial \bar{H}}{\partial v} = \bar{H}.
\]