# The Electromagnetic Field Tensor (L&L method)

(In the following all equations are given in international units and including c. I hope anybody will derive the most profit from comparing that to his own notes.)

### 1. Electromagnetic Potentials

Making use of the homogeneous Maxwell equations we want to introduce the electromagnetic potentials.
Since $\nabla\vec{B}=0$ and we know, that the divergence of the curl of any vector is zero $\left[\nabla\left(\nabla\times\vec{A}\right)=0\right]$, it suggests itself to introduce: $\vec{B}=\nabla\times\vec{A}$ The vector field $\vec{A}$ is called "vector potential"

This definition is now inserted into $\nabla\times\vec{E}=-\dot{\vec{B}}$:

$$\nabla\times\vec{E}+\frac{\partial}{\partial t}\left(\nabla\times\vec{A}\right)=0 \qquad\Leftrightarrow\qquad \nabla\times\left(\vec{E}+\dot{\vec{A}}\right)=0$$

Once again we know, that the curl of the gradient of any scalar is zero $\left[\nabla\times(\nabla\varphi)=0\right]$. Therefore we introduce:

$$\vec{E}+\dot{\vec{A}}=-\nabla\varphi$$

The scalar function φ is called "scalar potential"
(The minus sign in this definition will later prove to be useful.)

The two homogeneous Maxwell Equations can now be replaced by our new relations:

$$\boxed{\vec{B}=\nabla\times\vec{A} \qquad \textit{and} \qquad \vec{E}=-\nabla\varphi-\dot{\vec{A}}} \tag{1}$$

These equations may now be inserted into the inhomogeneous Maxwell equations:

$$\nabla\vec{E}=\frac{\rho}{\varepsilon_0}\Leftrightarrow\frac{\rho}{\varepsilon_0}=\nabla\left(-\nabla\varphi-\dot{\vec{A}}\right)=-\Delta\varphi-\nabla\dot{\vec{A}}$$

$$\nabla\times\vec{B}=\mu_0\vec{J}+\frac{1}{c^2}\dot{\vec{E}}\Leftrightarrow\mu_0\vec{J}=\nabla\times\left(\nabla\times\vec{A}\right)-\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\varphi-\dot{\vec{A}}\right)=\nabla\left(\nabla\vec{A}\right)-\Delta\vec{A}+\frac{1}{c^2}\nabla\dot{\varphi}+\frac{1}{c^2}\ddot{\vec{A}}$$

$$\Rightarrow \quad \boxed{\begin{aligned}&-\nabla\dot{\vec{A}}-\Delta\varphi=\frac{\rho}{\varepsilon_0}\\ &\frac{1}{c^2}\ddot{\vec{A}}-\Delta\vec{A}+\nabla\left(\nabla\vec{A}+\frac{1}{c^2}\dot{\varphi}\right)=\mu_0\vec{J}\end{aligned}} \tag{2}$$

So we have reduced the four Maxwell eqs to only two eqs for our potentials.

### 2. Gauge Invariance

$$\boxed{\vec{A}\to\vec{A}'=\vec{A}+\nabla\Lambda \quad ; \quad \varphi\to\varphi'=\varphi-\dot{\Lambda}} \tag{3}$$

with an arbitrary scalar function $\Lambda(\vec{r},t)$ is called "gauge transformation".
The fact, that the em fields (and also the Maxwell eqs) are invariant under such a transformation is called "gauge invariance". Proof:

$$\vec{B}'=\nabla\times\vec{A}'=\nabla\times\left(\vec{A}+\nabla\Lambda\right)=\nabla\times\vec{A}+\nabla\times(\nabla\Lambda)=\nabla\times\vec{A}=\vec{B}$$

$$\vec{E}'=-\nabla\varphi'-\dot{\vec{A}}'=-\nabla\left(\varphi-\dot{\Lambda}\right)-\frac{\partial}{\partial t}\left(\vec{A}+\nabla\Lambda\right)=-\nabla\varphi+\nabla\dot{\Lambda}-\dot{\vec{A}}-\nabla\dot{\Lambda}=-\nabla\varphi-\dot{\vec{A}}=\vec{E}$$

## 3. Lorentz Gauge

One tries to choose the gauge function Λ so, that the derived eqs for the transformed potentials $\left(\vec{A}', \varphi'\right)$ are as easy as possible. The gauge transformations still include no statement about the sources of the vector potential.

We make use of that freedom and postulate: $$\boxed{\nabla\vec{A} + \frac{1}{c^2}\dot{\varphi} = 0} \qquad (4)$$

This is the socalled "Lorentz condition". Eqs (3) and (4) together are called "Lorentz gauge" of the potentials. By using this we uncouple the two eqs (2) and get two inhomogeneous wave eqs for φ and $\vec{A}$:

$$\frac{1}{c^2}\ddot{\varphi} - \Delta\varphi = \frac{\rho}{\varepsilon_0} \qquad ; \qquad \frac{1}{c^2}\ddot{\vec{A}} - \Delta\vec{A} = \mu_0\vec{J}$$

These eqs can be shortened by introducing the four dimensional "D'Alembert operator"

$$\boxed{\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta}$$

$$\Rightarrow \qquad \boxed{\Box\varphi = \frac{\rho}{\varepsilon_0} \qquad ; \qquad \Box\vec{A} = \mu_0\vec{J}} \, (5)$$

At given potentials $\vec{A}$ and φ it is always possible to find a gauge transformation Λ, that satisfies the Lorentz condition. By inserting the gauge transformed potentials (3) in (4) we obtain:

$$0 = \nabla\vec{A}' + \frac{1}{c^2}\dot{\varphi}' = \nabla\left(\vec{A} + \nabla\Lambda\right) + \frac{1}{c^2}\left(\dot{\varphi} - \ddot{\Lambda}\right) = \frac{1}{c^2}\dot{\varphi} + \nabla\vec{A} + \Delta\Lambda - \frac{1}{c^2}\ddot{\Lambda} = f - \Box\Lambda$$

where $\boxed{f(\vec{r},t) = \frac{1}{c^2}\dot{\varphi} + \nabla\vec{A}}$ (6) is a scalar function, that is unequal zero for at least one pair of $\left(\vec{r}_0, t_0\right)$.

$$\Rightarrow \qquad \boxed{\Box\Lambda = f} \, (7)$$

This eq for Λ can be solved $\left[\Lambda(\vec{r},t) = \frac{1}{4\pi}\int\limits_V \frac{f\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} d^3r'\right]$, but not unique, for solutions of the

homogeneous differential eq $\Box\Lambda=0$ (free plane waves with speed c) can always be added.

## 4. Covariant Formulation

In the following we try to formulate the above using four vectors from special relativity:

$$x^\mu = (ct, \vec{x}) \qquad ; \qquad x_\mu = (-ct, \vec{x})$$

(Remember: greek indices stand for 0,1,2,3 whereas latin indices stand for 1,2,3)

The electric charge $\left[Q = \int\limits_V \rho d^3x\right]$ is Lorentz invariant , because it is independent of the speed of a particle.

A four dimensional volume element is also Lorentz invariant.

=> $\rho d^3x = \rho' d^3x'$ and $d^4x = d^4x'$

So can be seen, that ρ transforms like t and cρ like $ct = x^0$. Since cρ and $\vec{J}$ have same units we can postulate:

$$\boxed{J^\mu = (c\rho, \vec{J})} \qquad (8) \qquad \text{“\underline{four vector of current density}”}$$

We will also need the four dimensional differential operator:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \quad ; \quad \partial^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \quad \Rightarrow \quad \partial_\mu \partial^\mu = -\Box$$

(In east-coast metric this last minus sign is avoided!)

### 4.1. Equation of Continuity

Now we can deduce the covariant equation of continuity:

$$0 = \frac{\partial \rho}{\partial t} + \nabla \vec{J} = \frac{\partial c\rho}{\partial ct} + \frac{\partial J^k}{\partial x^k} = \partial_0 J^0 + \partial_k J^k = \partial_\mu J^\mu \qquad \text{(written in Einstein convention!)}$$

$$\Rightarrow \qquad \boxed{\partial_\mu J^\mu = 0} \qquad (9)$$

### 4.2. Lorentz Gauge

We can rewrite the eqs (5) as:

$$\Box \frac{\varphi}{c} = \frac{\rho}{c\varepsilon_0} = \mu_0 c\rho = \mu_0 J^0 \quad \text{and} \quad \Box \vec{A} = \mu_0 \vec{J}$$

The right sides of these eqs include the components of the four vector of current density. Therefore we introduce:

$$\boxed{A^\mu = \left(\frac{\varphi}{c}, \vec{A}\right)} \qquad (10) \qquad \text{“\underline{four potential}”}$$

$$\Rightarrow \qquad \boxed{\Box A^\mu = -\partial_\nu \partial^\nu A^\mu = \mu_0 J^\mu} \, (11)$$

Rewriting the Lorentz condition we get:

$$0 = \nabla \vec{A} + \frac{1}{c^2}\dot{\varphi} = \frac{\partial}{\partial x^k} A^k + \frac{1}{c}\frac{\partial}{\partial t} A^0 = \partial_k A^k + \partial_0 A^0 = \partial_\mu A^\mu$$

$$\Rightarrow \qquad \boxed{\partial_\mu A^\mu = 0} \qquad (13)$$

The gauge transformation (3) is analogous condensed in one eq:

$$A'^k = A^k + \frac{\partial}{\partial x^k}\Lambda = A^k + \frac{\partial}{\partial x_k}\Lambda = A^k + \partial^k \Lambda$$

$$A'^0 = \frac{\varphi'}{c} = \frac{\varphi}{c} - \frac{1}{c}\frac{\partial}{\partial t}\Lambda = A^0 - \frac{\partial}{\partial x^0}\Lambda = A^0 + \frac{\partial}{\partial x_0}\Lambda = A^0 + \partial^0 \Lambda$$

$$\Rightarrow \qquad \boxed{A'^\mu = A^\mu + \partial^\mu \Lambda} \qquad (14)$$

Analogous to above (3.) the gauge invariance is to be used to impose the Lorentz condition:
$\partial_\mu A^\mu = f(\vec{r},t) \quad ; \quad \partial_\mu A'^\mu = 0 \quad \Rightarrow \quad -\partial_\mu \partial^\mu \Lambda = f = \Box\Lambda$
Besides that the product of four vectors is invariant under Lorentz transformation: $\partial'_\mu A'^\mu = \partial_\mu A^\mu = 0$

Thus the Lornetz condition can always be fulfilled in a particular frame and is therefore automatically preserved in all frames for any $A'^\mu = \Lambda^\mu_\nu A^\nu + \partial^\mu \Lambda$ .

## 4.3. The Electromagnetic Field Tensor

Let's have a closer look on eqs (1) and express the kth components of $\vec{E}$ and $c\vec{B}$ by our new means:

$$E^k = -\frac{\partial}{\partial t}A^k - \frac{\partial}{\partial x^k}\varphi = c\left(-\frac{1}{c}\frac{\partial}{\partial t}A^k - \frac{\partial}{\partial x^k}\frac{\varphi}{c}\right) = c\left(\frac{\partial}{\partial x_0}A^k - \frac{\partial}{\partial x_k}A^0\right) = c\left(\partial^0 A^k - \partial^k A^0\right) \equiv F^{0k} = -F^{k0}$$

$$cB^k = c\left(\frac{\partial}{\partial x_i}A^j - \frac{\partial}{\partial x_j}A^i\right) = c\left(\partial^i A^j - \partial^j A^i\right) \equiv F^{ij} = -F^{ji}$$ (i,j,k can be 1,2,3 and cyclic permutations)

It is evident, that $F^{\mu\mu} = 0$

Summary:

$$\boxed{\begin{array}{l} F^{\mu\nu} = -F^{\nu\mu} \quad ; \quad F^{\mu\mu} = 0 \quad ; \quad F^{0k} = E^k \quad ; \quad F^{ij} = c\varepsilon^{ijk}B^k \\ F^{\mu\nu} = c\left(\partial^\mu A^\nu - \partial^\nu A^\nu\right) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & cB^3 & -cB^2 \\ -E^2 & -cB^3 & 0 & cB^1 \\ -E^3 & cB^2 & -cB^1 & 0 \end{pmatrix} \end{array}}$$

The covariant antisymmetric F tensor is called "electromagnetic field tensor".
$F^{00} = $"time – time – component"
$F^{0k}, F^{k0} = $"space – time – components"
$F^{ij} = $"space – space – components"

The change of the field components under boosts can now easily be computet by boosting F!

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