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10-8-01 / 10-10-01

Conservation of Energy and Momentum for System of Charged Particles and EM Fields:

- For a single charge q, rate of work done by $\vec{E},\vec{B} = q\vec{v}\cdot\vec{E}$
  [$\vec{B}\perp\vec{v} \Rightarrow$ does no work on the charge]

- For a continuous distribution of charge in finite volume $V \Rightarrow$
  rate of work done $= \int_V \vec{J}\cdot\vec{E}\, d^3x$

  { Jackson describes this as a conversion of electromagnetic energy into mechanical energy or thermal energy. For energy to be conserved, this amount of energy must be lost by the energy in the electromagnetic field within V. }

Let's look more carefully at this relation:

$$\int_V \vec{J}\cdot\vec{E}\, d^3x = \frac{1}{4\pi}\int_V \left(\nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t}\right)\cdot\vec{E}\, d^3x$$

$$= \frac{1}{4\pi}\int_V d^3x \Big[\underbrace{-\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}_{-\frac{1}{2}\frac{\partial}{\partial t}\vec{E}^2} + \underbrace{\vec{E}\cdot(\nabla\times\vec{B})}_{-[\nabla\cdot(\vec{E}\times\vec{B}) - \vec{B}\cdot(\nabla\times\vec{E})]}\Big]$$

$$\underbrace{\nabla\times\vec{E}}_{\text{by Faraday's law} = -\frac{\partial\vec{B}}{\partial t}} \qquad + \frac{1}{2}\frac{\partial}{\partial t}\vec{B}^2$$

$$\int_V \vec{J}\cdot\vec{E}\, d^3x = \frac{1}{4\pi}\int_V -\left[\frac{1}{2}\frac{\partial}{\partial t}(\vec{E}^2+\vec{B}^2) + \nabla\cdot(\vec{E}\times\vec{B})\right] d^3x$$

Define $\mathcal{E} = \frac{1}{8\pi}(\vec{E}^2+\vec{B}^2)$ = total energy density

$\vec{P} = \frac{1}{4\pi}(\vec{E}\times\vec{B})$ = Poynting vector

Now we have:

$$\int_V \vec{J}\cdot\vec{E}\, d^3x = -\int_V \left[\frac{\partial\mathcal{E}}{\partial t} + \nabla\cdot\vec{P}\right] d^3x$$

$\therefore$ The time rate of change of electromagnetic energy in V plus the energy flowing through S (boundary of V) equals the negative of the work done by the fields on the sources in V. Energy is conserved!!

- How should we view $\left[\int_V \vec{J}\cdot\vec{E}\, d^3x\right]$?

  This is the rate at which energy is converted from field energy to particle energy. We can then look at conservation of energy for the composite particle/field system.

Amina Kinkhabwala

Conservation of energy for the combined system:

$$\frac{dE}{dt} = \frac{d}{dt}\left(E_{mech}+E_{field}\right) = -\oint_S \hat{n}\cdot\vec{P}\,da$$

$$\text{where } \begin{cases} \frac{dE_{mech}}{dt} = \int_V \vec{J}\cdot\vec{E}\,d^3x \\ \frac{dE_{field}}{dt} = \frac{d\mathcal{E}}{dt} \end{cases} \quad \checkmark$$

- Similarly, we can derive a relation for conservation of momentum for our composite field/particle system.

Consider the total electromagnetic force on a charged particle:

$$\vec{F} = q\left(\vec{E}+\hat{v}\times\vec{B}\right)$$

Again, look at collection of charged particles in volume V.
By Newton's 2^{nd} law:

$$\frac{d\vec{P}_{mech}}{dt} = \int_V \Big(\underbrace{\rho}_{\frac{1}{4\pi}(\nabla\cdot\vec{E})}\vec{E} + \underbrace{\vec{J}}_{\frac{1}{4\pi}\left[(\nabla\times\vec{B})-\frac{\partial\vec{E}}{\partial t}\right]}\times\vec{B}\Big)d^3x$$

$$\frac{d\vec{P}_{mech}}{dt} = \frac{1}{4\pi}\int_V\Big[(\nabla\cdot\vec{E})\vec{E} - \vec{B}\times(\nabla\times\vec{B}) + \underbrace{\vec{B}\times\frac{\partial\vec{E}}{\partial t}}_{-\frac{\partial}{\partial t}(\vec{E}\times\vec{B})+\vec{E}\times\underbrace{\frac{\partial\vec{B}}{\partial t}}_{-(\nabla\times\vec{E})}}\Big]d^3x$$

$$\frac{d\vec{P}_{mech}}{dt} = \frac{1}{4\pi}\int_V\Big[(\nabla\cdot\vec{E})\vec{E} - \vec{B}\times(\nabla\times\vec{B}) - \vec{E}\times(\nabla\times\vec{E}) - \frac{\partial}{\partial t}(\vec{E}\times\vec{B})\Big]d^3x$$

Let's add $\vec{B}(\nabla\cdot\vec{B})=0$ to the right hand side:

$$\frac{d}{dt}\Big[\vec{P}_{mech} + \frac{1}{4\pi}\int_V(\vec{E}\times\vec{B})d^3x\Big] = \frac{1}{4\pi}\int_V d^3x\Big[(\nabla\cdot\vec{E})\vec{E} - \vec{E}\times(\nabla\times\vec{E}) + (\nabla\cdot\vec{B})\vec{B} - \vec{B}\times(\nabla\times\vec{B})\Big]$$

From this we see that: $\vec{P}_{field} = \frac{1}{4\pi}\int_V(\vec{E}\times\vec{B})\,d^3x$

Let's simplify the right hand side of

Consider

$$\left[\vec{a}\times(\nabla\times\vec{a})\right]_i = \varepsilon_{ijk}a_j\left(\varepsilon_{klm}\partial_l a_m\right)$$
$$= \underbrace{\varepsilon_{ijk}\varepsilon_{lmk}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}}\left(a_j\partial_l a_m\right)$$
$$= \underbrace{a_j\partial_i a_j}_{\frac{1}{2}\partial_i a_j^2} - a_j\partial_j a_i$$
$$\Rightarrow \vec{a}\times(\nabla\times\vec{a}) = \tfrac{1}{2}\vec{\nabla}\vec{a}^2 - (\vec{a}\cdot\vec{\nabla})\vec{a}$$

Now consider:

$$\left[(\nabla\cdot\vec{a})\vec{a} - \vec{a}\times(\nabla\times\vec{a})\right]_i = a_i\partial_j a_j - \tfrac{1}{2}\partial_i a_j^2 + a_j\partial_j a_i$$
$$= \partial_j(a_i a_j) - \tfrac{1}{2}\partial_i a_j^2$$

Amina Kinkhabwala

So far we have: $\left[\frac{d}{dt}(\vec{p}_{mech}+\vec{p}_{field})\right]_i = \frac{1}{4\pi}\int_V d^3x \left[\partial_j(E_iE_j)+\partial_j(B_iB_j)-\frac{1}{2}\partial_i E_j^2-\frac{1}{2}\partial_i B_j^2\right]$

Now we define $\overleftrightarrow{T}$, the Maxwell stress tensor, by:

$$T_{ij} = \frac{1}{4\pi}\left[E_iE_j+B_iB_j-\frac{\delta_{ij}}{2}(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B})\right]$$

$$\Rightarrow \boxed{\frac{d}{dt}(\vec{p}_{mech}+\vec{p}_{field})_i = \int_V \partial_j T_{ij}\, d^3x}$$

As a side note: Dyadic notation for 2nd rank tensors:

1. Dyadic $\overleftrightarrow{T}$ defined by $(\overleftrightarrow{T})_{ij} \equiv T_{ij}$
2. Dyadic (or outer) product of 2 vectors: $(\vec{A}\vec{B})_{ij} \equiv A_i B_j$
3. $(\vec{A}\cdot\overleftrightarrow{T})_i = A_j T_{ji}$
4. $(\overleftrightarrow{T}\cdot\vec{A})_i = T_{ij}A_j$
5. Divergence: $(\nabla\cdot\overleftrightarrow{T})_i = \partial_j T_{ji}$
6. Divergence from the left: $(\overleftrightarrow{T}\cdot\overleftarrow{\nabla})_i = \partial_j T_{ij}$
7. Identity: $(\overleftrightarrow{I})_{ij} = \delta_{ij}$

∴ Our result can also be written as:

$$\frac{d}{dt}(\vec{p}_{mech}+\vec{p}_{field}) = \int_V (\vec{\nabla}\cdot\overleftrightarrow{T})\, d^3x = \oint_S da\, \hat{n}\cdot\overleftrightarrow{T}$$

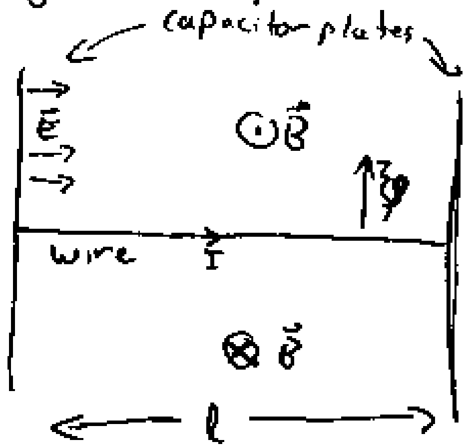
Matthew Eardley

E & M Notes

10/10/01 part 2

The following thought experiment from Feynmann illustrates the flow of energy and power of the Electromagnetic field.

fig. 1



- Consider a pair of capacitor plates connected by a wire as shown. The $\vec{E}$-field between the plates induces a current in the wire. The wire has a finite resistance, so the current flow causes it to heat up. Since the wire is dissipating heat, power ~~must be~~ must be flowing into the wire. Where does it come from?

  → the ~~E field~~ EM field

- In units where $\boxed{c=1}$,

Poynting vector: $\vec{\wp} = \dfrac{\vec{E}\times\vec{B}}{4\pi}$ $\quad$ E-field: $E_x = \dfrac{V}{\ell}$ $\quad V = IR \quad I = \dfrac{V}{R}$

The current in the wire creates an azimuthal $\vec{B}$-field as shown in fig. 1.

$$\vec{\nabla}\times\vec{B} = 4\pi\vec{J} \xRightarrow{\text{integrate over wire}} \cancel{2}\cancel{\pi}\rho B_\varphi = \overset{2}{\cancel{4\pi}} I$$

ρ = vertical distance from wire

$$B_\varphi = \frac{2I}{\rho}$$

• What is the rate that EM power is going into the wire?

$$B_\varphi = \frac{2I}{\rho} = \frac{2V}{\rho R}$$

$$\frac{1}{4\pi}\vec{E}\times\vec{B} = \frac{1}{4\pi}\frac{V}{l}\frac{2V}{\rho R}$$ } this is the power coming into a "can" of radius ρ surrounding the wire

total pwr. per unit length $= \cancel{2}\pi\cancel{\rho}\frac{V}{l}\frac{\cancel{2}V}{\cancel{\rho}R}\frac{1}{\cancel{4}\pi}$

$$= \frac{V^2}{Rl}$$

total power flowing into wire:

$$\boxed{P_{total} = \frac{V^2}{R} = IV}$$

• This is exactly the same as the equation for the power consumption of a resistor with a current going through it.

• Thus we find that power is "coming in" from the EM field. We have local conservation of energy: the power going into the wire is dissipated by heat.

10 / 12 / 01

We will now discuss ~~the~~ some general properties of the Electromagnetic Energy-Momentum $[(EM)^2]$ Tensor.

- The $(EM)^2$ tensor is symmetric

$$T^{\mu\nu}_{(EM)} = T^{\nu\mu}_{(EM)}$$

in addition, the total Energy Momentum field (including terms for the EM field and for mechanical interactions) is Symmetric:

$$T^{\mu\nu}_{total} = T^{\nu\mu}_{total}$$

- This follows from considerations of special relativity and classical mechanics. Also, since we know that the EM field components of the tensor are symmetric, the <u>mechanical</u> components of the EM tensor must be symmetric as well (and thus the total tensor is symmetric).

- From this it follows that the Poynting vector and the momentum density are equal:

$$\vec{p} = \vec{\wp}$$

MF (3)

• Imagine ~~source~~ a system in static equilibrium
(no internal motion)

$\boxed{c=1}$ [note: in this case we consider $T^{\alpha\nu}_{tot}$, the total energy-momentum tensor)

total mass: $M = \int d^3r\, T^{00}(x)$ (total mass of the system is equal to it's energy)

$$M\vec{R} = \int d^3r\, T^{00}(x)\,\vec{r}$$

$$\begin{cases} x \equiv (x, y, z, t) \\ \vec{r} \equiv (x, y, z) \end{cases}$$

• what is $\frac{d}{dt} M\vec{R}$?

Poynting vector

$$\frac{d}{dt} M = \int d^3r \frac{\partial}{\partial t} T^{00} = \int d^3r (-\vec{\nabla}\cdot\vec{\wp})$$

from the divergence theorem over some surface S,

$$\frac{d}{dt} M = \int d^2\vec{S}\cdot(-\vec{\wp})$$

If we choose the surface S sufficiently far away, $\vec{\wp} = 0$ on the surface.

thus

$\frac{d}{dt} M = 0 \Rightarrow$ thus total mass energy is conserved for an isolated system)

• Therefore from conservation of local energy, we get conservation of total energy.

So

$$\frac{d}{dt} M\vec{R} = \int d^3r \left(\frac{\partial}{\partial t} T^{00}\right)\vec{r}$$

since surface integral equals zero.

$$= \int d^3r \left[\underbrace{(-\partial_i \wp_i r_j)}_{=0} + \wp_i \partial_i r_j\right]$$

$$= \int d^3r (\vec{\wp}\cdot\vec{\nabla})\,\vec{r}$$

since $\partial_i r_j = \delta_{ij}$

momentum density

$$\frac{d}{dt} M\vec{R} = \int d^3r\, \vec{\wp} = \int d^3r\, \vec{p} = \vec{p}$$ ← total momentum

thus we see $\boxed{\frac{d}{dt} M\vec{R} = M\frac{d}{dt}\vec{R} = \vec{p}}$

10/13/11 cont'd

$$\frac{d}{dt} M\vec{R} = \vec{p}$$

integrating,

$$M\vec{R} = \vec{p}\,t + M\vec{R}_0$$

[$\vec{R}_0$ is the center of mass location at $t=0$]

$$\underbrace{M\vec{R} - \vec{p}\,t} = \text{Const.}$$

- this is the generator of <u>boosts</u> from poisson brackets

Similarly, $\vec{p}$ is the generator of translations

ang. momentum $\vec{L}$ is the generator of rotations.

- Usually, ~~for~~ ~~tr~~ if $\vec{G}$ is a generator of a conserved quantity,

$$\cancel{\frac{d}{dt} G = i[G,H]} \quad \boxed{\hbar = 1}$$

$$\frac{d}{dt} G = i\,[G, H] = 0$$

in the quantum mechanics analog of Poisson ~~to bra~~ brackets, the commutator.

but in this case $M\vec{R} - \vec{p}\,t$ does <u>not</u> commute with the Hamiltonian because of its explicit time dependence.

Instead,

$$\frac{d}{dt} G = i\,[G, H] + \frac{\partial G}{\partial t} = 0$$

IE S

We will now be discussing only the Electromagnetic components of the Energy-Momentum tensor.

J.J Thompson, who discovered the momentum of the EM-field, provides the following thought experiment:

Consider two systems; I. an electric charge and a magnetic "charge", and II. an electric charge and a magnetic momentum. (Both systems at rest w.r.t. each other)

I. $q \bullet \qquad\qquad \bullet\, g$

II. $q \bullet \qquad\qquad \bullet\, \vec{\mu}$

What, if anything, are $\vec{p}$ and $\vec{L}$ for I.?

$$\begin{cases} q \text{ @ } \vec{r} = 0 \\ g \text{ @ } \vec{r} = \vec{a} \end{cases}$$

since $\hat{r} \times \hat{r} = 0$

momentum density $\rightarrow$ $$\vec{p} = \frac{\vec{E} \times \vec{B}}{4\pi} = qg\, \frac{\vec{r} \times (\cancel{\vec{r}} - \vec{a})}{|\vec{r}|^3 |\vec{r} - \vec{a}|^3} = qg\, \frac{\vec{r} \times (-\vec{a})}{|\vec{r}|^3 |\vec{r} - \vec{a}|^3}$$

$\vec{a} = (0, 0, a)$

{ thus followed an intense discussion of magnetic monopoles }

$\Rightarrow$ the x, y components of $\vec{p}$ vanish because the component along the direction between the particles (the z-direction) is the only important one.

So $\vec{p} \propto \hat{z} \times (-a\hat{z}) = 0$

So for case I the total linear momentum of the EM field is zero (we will find that the angular momentum is not zero).