Conservation of Energy and Momentum for System of Charged Particles and EM Fields:

- For a single charge, the rate of work done by $E \cdot \vec{B} = q \vec{v} \cdot \vec{E}$
  
  $[E \cdot \nabla \Rightarrow \text{does no work on the charge}]$

For a continuous distribution of charge in finite volume $V \Rightarrow$

rate of work done $= \iiint_V \vec{J} \cdot \vec{E} \, d^3x$

Jackson describes this as a conversion of electromagnetic energy into mechanical energy or thermal energy. For energy to be conserved, this amount of energy must be lost by the energy in the electromagnetic field within $V$.

Let's look more carefully at this relation:

\[
\iiint_V \vec{J} \cdot \vec{E} \, d^3x = \frac{1}{2} \iiint_V \left( \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \right) \cdot \vec{E} \, d^3x \\
= \frac{1}{2} \iiint_V 
\left[ \vec{E} \cdot \vec{\nabla} E + \vec{E} \cdot (\vec{\nabla} \times \vec{B}) \right] \\
- \frac{1}{2} \iiint_V \left( \nabla \cdot \vec{E} \right) \vec{B} - \vec{B} \cdot \left( \nabla \times \vec{E} \right) \] 

by Faraday's law $= \frac{\partial B}{\partial t}$

\[
\iiint_V \vec{J} \cdot \vec{E} \, d^3x = \frac{1}{2} \iiint_V \left[ \vec{E} \cdot \vec{\nabla} E + \vec{E} \cdot (\vec{\nabla} \times \vec{B}) \right] \\
+ \frac{1}{2} \iiint_V \left( \nabla \cdot \vec{E} \right) \vec{B} - \vec{B} \cdot \left( \nabla \times \vec{E} \right) \] 

Define $\vec{E} = \frac{1}{2} \left( \vec{E}^2 + \vec{B}^2 \right)$ = total energy density

$\vec{B} = \frac{1}{2\mu} (\vec{E} \times \vec{B})$ = Poynting vector

Now we have:

\[
\iiint_V \vec{J} \cdot \vec{E} \, d^3x = - \iiint_V \left[ \frac{\partial E}{\partial t} + \nabla \cdot \vec{B} \right] \, d^3x 
\]

The time rate of change of electromagnetic energy in $V$ plus the energy flowing through $S$ (boundary of $V$) equals the negative of the work done by the fields on the sources in $V$. Energy is conserved!

How should we view $\left[ \iiint_V \vec{J} \cdot \vec{E} \, d^3x \right]$?

This is the rate at which energy is converted from field energy to particle energy. We can then look at conservation of energy for the composite particle/field system.
Conservation of energy for the combined system:

$$\frac{dE}{dt} = \frac{d}{dt} (E_{\text{mech}} + E_{\text{field}}) = - \oint_{S_0} \mathbf{v} \cdot \mathbf{F} \, d\mathbf{a}$$

where

$$\frac{dE_{\text{mech}}}{dt} = \int_{S_0} \mathbf{F} \cdot \mathbf{E} \, d^3x$$

$\checkmark$

Similarly, we can derive a relation for conservation of momentum for our composite field/particle system.

Consider the total electromagnetic force on a charged particle:

$$\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Again, look at collection of charged particles in volume $V$.

By Newton's 2nd law:

$$\begin{align*}
\frac{d\mathbf{p}_{\text{mech}}}{dt} &= \int_{V} \left( \rho \mathbf{E} + \frac{\partial \mathbf{E}}{\partial t} \right) \, d^3x \\
\frac{d\mathbf{p}_{\text{mech}}}{dt} &= \int_{V} \left( \nabla \times \mathbf{E} - \mathbf{B} \times (\nabla \times \mathbf{E}) + \frac{\partial \mathbf{E}}{\partial t} \right) \, d^3x \\
\frac{d\mathbf{p}_{\text{mech}}}{dt} &= \int_{V} \left( \nabla \times \mathbf{B} - \mathbf{E} \times (\nabla \times \mathbf{B}) - \frac{\partial \mathbf{E}}{\partial t} \right) \, d^3x
\end{align*}$$

Let's add $\mathbf{B} \cdot \nabla \mathbf{B} = 0$ to the right hand side:

$$\begin{align*}
\frac{d\mathbf{p}_{\text{mech}}}{dt} &= \int_{V} \nabla \times \mathbf{B} \, d^3x \\
\frac{d\mathbf{p}_{\text{mech}}}{dt} &= \int_{V} \mathbf{B} \times (\nabla \times \mathbf{B}) \, d^3x
\end{align*}$$

From this we see that:

$$\mathbf{F}_{\text{mech}} = \frac{q}{2} \int_{V} (\mathbf{E} \times \mathbf{B}) \, d^3x$$

Let's simplify the right hand side of:

Consider

$$\begin{align*}
\nabla \times (\mathbf{v} \times \mathbf{a}) &= \mathbf{v} \times (\mathbf{a} \cdot \nabla) \\
&= \mathbf{v} \times \mathbf{a} \cdot \nabla \\
&= \mathbf{v} \times \mathbf{a} \cdot \delta x \, \delta y \, \delta z \\
&= \mathbf{v} \times \mathbf{a} \\
\Rightarrow \nabla \times (\mathbf{v} \times \mathbf{a}) &= \frac{1}{2} \nabla \times \nabla \times (\mathbf{v} \times \mathbf{a})
\end{align*}$$

Now consider:

$$\begin{align*}
\nabla \times (\mathbf{v} \times \mathbf{a}) &= \mathbf{a} \times \mathbf{v} - \frac{1}{2} \nabla (\mathbf{a} \times \mathbf{v}) + \mathbf{v} \times \mathbf{a} \\
&= \mathbf{a} \times \mathbf{v} - \frac{1}{2} \nabla (\mathbf{a} \times \mathbf{v}) + \mathbf{v} \times \mathbf{a}
\end{align*}$$
So far we have: \[ \frac{\partial}{\partial t} \left( \overline{\rho}_{\text{mech}} + \overline{\rho}_{\text{field}} \right) = \frac{\partial}{\partial x} \left[ \gamma (E_i E_i + B_i B_i) - \frac{1}{2} \nabla E_i^2 - \frac{1}{2} \nabla B_i^2 \right] \]

Now we define \( T_{ij} \), the Maxwell stress tensor, by:

\[ T_{ij} = \frac{\partial}{\partial x} \left[ E_i E_j + B_i B_j - \frac{E^2}{2} (\nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{B}) \right] \]

\[ \Rightarrow \frac{\partial}{\partial t} \left( \overline{\rho}_{\text{mech}} + \overline{\rho}_{\text{field}} \right)_i = S_{ij} \partial_j T_{ij} \, d^3x \]

As a side note: Dyadic notation for 2nd rank tensors:

1. Dyadic **\( T \)** defined by: \( (T)_{ij} = T_{ij} \)
2. Dyadic (or outer) product of 2 vectors: \( (\mathbf{A} \cdot \mathbf{B})_{ij} = A_i B_j \)
3. \( (\mathbf{A} \cdot T)_{ij} = A_i T_{ij} \)
4. \( (T \cdot \mathbf{A})_{ij} = T_{ij} A_i \)
5. Divergence: \( (\nabla \cdot T)_{ij} = \partial_j T_{ij} \)
6. Divergence from the left: \( (T \cdot \nabla)_{ij} = \partial_j T_{ij} \)
7. Identity: \( (I)_{ij} = \delta_{ij} \)

Our result can also be written as:

\[ \frac{\partial}{\partial t} \left( \overline{\rho}_{\text{mech}} + \overline{\rho}_{\text{field}} \right) = \int (\nabla \cdot \mathbf{T}) \, d^3x = \int \delta_{ij} \partial_j \mathbf{n} \cdot \mathbf{T} \]
The following thought experiment from Feynmann illustrates the flow of energy and power of the Electromagnetic field.

Consider a pair of capacitor plates connected by a wire as shown. The \( \vec{E} \)-field between the plates induces a current in the wire. The wire has a finite resistance, so the current flow causes it to heat up. Since the wire is dissipating heat, power must be flowing into the wire. Where does it come from?

- Do the \( \vec{B} \)-field, EM field

In units where \( \frac{c}{\lambda} = 1 \), \( \vec{E} \)-field

\[ \vec{\nabla} \times \vec{B} = \frac{\mu_{0}}{4\pi} \vec{J} \] \( \vec{E}_x = \frac{V}{l} \) \( V = IR \) \( I = \frac{V}{R} \)

The current in the wire creates an azimuthal \( \vec{B} \)-field as shown in fig. 1.

\( D \) = distance from wire

\[ B_0 = \frac{2I}{D} \]
What is the rate that EM power is going into the wire?

\[ B_{eq} = \frac{2V}{R} \]

\[ \frac{1}{4\pi} E \times B = \frac{1}{4\pi} \frac{V}{e} \frac{2V}{\partial R} \] (this is the power coming into a "can" of radius \( R \) surrounding the wire)

Total power per unit length = \( 2\pi R \frac{V}{e} \frac{2V}{\partial R} \)

\[ P = \frac{V^2}{RL} \]

Total power flowing into wire:

\[ P_{total} = \frac{V^2}{R} = IV \]

This is exactly the same as the equation for the power consumption of a resistor with a current going through it.

Thus we find that power is "coming in" from the EM field. We have local conservation of energy: the power going into the wire is dissipated by heat.
We will now discuss the some general properties of the Electro-magnetic Energy-Momentum \((EM)^{a}\) Tensor.

1. The \((EM)^{a}\) tensor is symmetric
   
   \[ T^{\mu \nu}_{(EM)} = T^{\nu \mu}_{(EM)} \]

   In addition, the total Energy Momentum field (including terms for the EM field and for mechanical interactions) is symmetric:
   
   \[ T^{\mu \nu}_{\text{total}} = T^{\nu \mu}_{\text{total}} \]

2. This follows from considerations of special relativity and classical mechanics. Also, since we know that the EM field components of the tensor are symmetric, the mechanical components of the EM tensor must be symmetric as well (and thus the total tensor is symmetric).

3. From this it follows that the Poynting vector and the momentum density are equal:
   
   \[ \vec{S} = \rho \vec{v} \]
Imagine a system in static equilibrium (no internal motion).

Note: in this case we consider $T^{\mu\nu}$, the total energy-momentum tensor.

Total mass: $M = \int d^3 r \ T^{00}(x)$  
(total mass of the system is equal to its energy)

$$M \dot{R} = \int d^3 r \ T^{00}(x) \ \dot{R}$$

where $x \equiv (x, y, z, t)$

$\vec{R} \equiv (x, y, z)$

What is $\frac{d}{dt} M \dot{R}$?

$$\frac{d}{dt} M = \int d^3 r \ \frac{\partial}{\partial t} T^{00} = \int d^3 r \ (-\nabla \cdot \vec{E})$$

From the divergence theorem over some surface $S$:

$$\frac{d}{dt} M = \int d^2 S \cdot (-\vec{E})$$

If we choose the surface $S$ sufficiently far away, $\vec{E} = 0$ on the surface.

Thus,

$$\frac{d}{dt} M = 0 \implies \text{total mass energy is conserved}
$$

Therefore, from conservation of local energy, we get conservation of total energy.

So,

$$\frac{d}{dt} M \dot{R} = \int d^3 r \ (\frac{\partial}{\partial t} T^{00}) \ \dot{R}$$

Since surface integrals equals zero:

$$= \int d^3 r \ [\nabla \cdot (-i \beta_i \ n_j) + \beta_i \ n_i \ \gamma_j]$$

$$= \int d^3 r \ (\vec{E} \cdot \vec{E}) \ \dot{R}$$

Since $\partial_i r_j = \delta_{ij}$, momentum density

$$\frac{d}{dt} M \dot{R} = \int d^3 r \ \dot{\vec{E}} = \int d^3 r \ \dot{\rho} = \dot{\rho} \ \leq \text{total momentum}$$

Thus, we see $\frac{d}{dt} M \dot{R} = M \frac{d}{dt} \dot{R} = \dot{\rho}$

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CONT'D
\[
\frac{d}{dt} M \dot{R} = \dot{p} \\
\text{integrating,} \\
M \dot{R} = \dot{p} t + M \dot{R}_0.
\]

\[
M \ddot{R} - \dot{p} t = \text{const.}
\]

This is the generator of boosts from Poisson brackets.

Similarly, \( \dot{p} \) is the generator of translations.

Any momentum \( \mathcal{P} \) is the generator of rotations.

• Usually, \( \dot{\mathcal{P}} \) is if \( \mathcal{P} \) is a generator of a conserved quantity.

\[
\dot{\mathcal{P}} = \mathcal{L}_{\mathcal{P}} H = \mathcal{E}_{\mathcal{P}}
\]

\[
\frac{d}{dt} \mathcal{P} = i \{ \mathcal{P}, H \} = 0
\]

In the quantum mechanics analog of Poisson brackets, the commutator.

But in this case \( M \ddot{R} - \dot{p} t \) does not commute with the Hamiltonian because of its explicit time dependence.

Instead,

\[
\frac{d}{dt} \mathcal{P} = i \{ \mathcal{P}, H \} + \frac{\mathcal{E}_{\mathcal{P}}}{2} = 0
\]
We will now be discussing only the Electromagnetic Components of the Energy-Momentum tensor.

J.J. Thompson, who discovered the momentum of the EM-field, provides the following thought experiment:

Consider two systems: I. an electric charge and a magnetic charge, and II. an electric charge and a magnetic moment (both systems at rest w.r.t. each other)

\[ \text{I. } q \quad \cdot \quad g \]

\[ \text{II. } q \quad \cdot \quad \vec{m} \]

What, if anything, are \( \vec{p} \) and \( \vec{L} \) for I.? \[
\begin{align*}
\dot{\vec{r}}/\vec{v} = 0 \\
\{ \begin{array}{c}
\dot{q} = 0 \\
\dot{g} = \vec{v} = \vec{a}
\end{array}
\end{align*}
\]

\( \vec{p} = \frac{\vec{E} \times \vec{\theta}}{4\pi} = q \vec{g} \frac{x \times (\vec{r} - \vec{a})}{|\vec{r} - \vec{a}|^3} = q \vec{g} \frac{x \times (\vec{r} - \vec{a})}{|\vec{r} - \vec{a}|^3} = q \vec{g} \frac{x \times (\vec{r} - \vec{a})}{|\vec{r} - \vec{a}|^3}
\]

\[ \vec{a} = (0, 0, a) \]

\{ thus followed an intense discussion of magnetic monopoles \}

\( \Rightarrow \) the \( x, y \) components of \( \vec{p} \) vanish because the component along the direction between the particles (the \( z \)-direction) is the only important one.

\( \Rightarrow \vec{p} \propto \vec{x} \times (-a \hat{z}) = 0 \)

\( \Rightarrow \) for case I the total linear momentum of the EM-field is zero (we will find that the angular momentum is not zero).