1 Reminder

In the 10/10 lecture we began a discussion of what linear and angular momentum are possible in the following two cases:

\[ \text{CASE I } \quad q \quad \text{and} \quad g \]
\[ \text{CASE II } \quad q \quad \text{and} \quad \vec{\mu} \]

where \( q \) is the familiar electric point charge, \( \vec{\mu} \) is the magnetic moment from a current loop and \( g \) is the currently unobserved magnetic monopole.

2 Angular momentum

The electric field for point charge at origin \( \vec{r} = 0 \),

\[ \vec{E} = \frac{q\vec{r}}{r^3} \]

The angular momentum density is defined as,

\[ \vec{l} = \vec{r} \times \vec{S} = \frac{1}{\mu_0}\vec{r} \times (\vec{E} \times \vec{B}) \]

where \( \vec{S} \) is the Poynting vector. The net angular momentum \( \vec{L} \) is found by integrating,

\[ \vec{L} = \frac{1}{\mu_0} \int d^3r \left( \vec{r} \times \left( \vec{E} \times \vec{B} \right) \right) \]
\[ \begin{aligned}
&= \frac{q}{\mu_0} \int d^3 \left( \vec{r} \times \left( \vec{r} \times \vec{B} \right) \right) \\
&= \frac{q}{\mu_0} \int d^3 \left( \vec{r} \cdot \vec{B} \right) - \left( \vec{r} \right) \vec{B} \\
&= \frac{q}{\mu_0} \int d^3 r \left( \frac{r^2}{r^3} \left( \vec{r} \cdot \vec{B} \right) - \frac{r^2}{r^3} \vec{B} \right) \\
&= \frac{q}{\mu_0} \int d^3 r \left( \frac{\vec{r} \cdot \vec{B} - \vec{B}}{r} \right)
\end{aligned} \]

where \( \vec{\hat{r}} = \frac{\vec{r}}{r} \). We have used only ordinary vector algebra above, now we invoke a dirty trick, namely:

\[ \left( \vec{B} \cdot \nabla \right) \vec{\hat{r}} = \left( \vec{B} \cdot \nabla \right) \frac{\vec{r}}{r} = \left[ \left( \vec{B} \cdot \nabla \right) \frac{1}{r} \right] \frac{1}{r} + \left[ \left( \vec{B} \cdot \nabla \right) \frac{1}{r} \right] \vec{r} \]

Knowing that \( \nabla \frac{1}{r} = -\frac{\vec{r}}{r^2} \), we will obtain:

\[ \vec{L} = -\frac{q}{\mu_0} \int d^3 r \left( \vec{B} \cdot \nabla \right) \vec{\hat{r}} \]

Integrating by parts we observe the boundary term vanishes,

\[ \vec{L} = \frac{q}{\mu_0} \int d^3 r \left( \nabla \cdot \vec{B} \right) \]

Now we may explicitly find \( \vec{L} \) for both cases.

**Case I** If \( \vec{B} \) is from the magnetic monopole at \( \vec{a} \) then in analogy to Gauss Law we have \( \nabla \cdot \vec{B} = \mu_0 \rho_m \) where \( \rho_m = g \delta^3 (\vec{r} - \vec{a}) \). Thus,

\[ \vec{L} = q \int d^3 r \vec{r} \delta^3 (\vec{r} - \vec{a}) = q g \vec{a} \]

\[ \begin{array}{c|c|c|c|c|c|c}
\vec{p} & q \vec{g} & q \vec{\mu} & \vec{L} & \sqrt{1} \\
\hline
\end{array} \]

**Case II** We know that if \( \vec{B} \) is from \( \vec{\mu} \) then \( \nabla \cdot \vec{B} = 0 \). Therefore,

\[ \vec{L} = \frac{q}{\mu_0} \int d^3 r \left( \nabla \cdot \vec{B} \right) = 0 \]
\[
\begin{array}{c|cc}
\bar{p} & q, g & q, \bar{J} \\
L & \sqrt{L} & 0 \\
\end{array}
\]

3 Linear momentum

Now we want to find what is happening with \( \bar{p} \) for both cases.

Case II We find the total momentum from integrating the momentum density (the Poynting vector)

\[
\bar{P}_{\text{total}} = \int d^3r \frac{1}{\mu_0} \bar{E} \times \bar{B}
\]

\[
\bar{E}(\bar{r}) = \frac{q \bar{r}}{r^3} = -q \nabla \frac{1}{r}
\]

\[
\bar{P}_{\text{total}} = \int d^3r \frac{1}{\mu_0} \left( -q \nabla \frac{1}{r} \right) \times \bar{B}
\]

Integrate by parts and assume that the boundary term vanishes,

\[
\bar{P}_{\text{total}} = q \int d^3r \frac{1}{r \mu_0} \left( \nabla \times \bar{B} \right)
\]

Since we are considering only the static situation, \( \nabla \times \bar{B} = \mu_0 \bar{J} \), therefore

\[
\bar{P}_{\text{total}} = q \int d^3r \frac{\bar{J}}{r}
\]

Using the following Taylor expansion:

\[
\frac{q}{r} = q \left( \frac{1}{R} - \frac{\bar{R}(\bar{r} - \bar{R})}{R^3} + \cdots \right)
\]

Then,

\[
\bar{P}_{\text{total}} = - \int d^3r \bar{E}(\bar{R}) \cdot (\bar{r} - \bar{R}) \cdot \bar{J}(\bar{r})
\]
The integral of the first term in the expansion vanishes since:

$$\vec{P}_0 = \int d^3r \frac{q}{\vec{R}} \vec{J} (\vec{r}) = \int d^3r \frac{q}{r} \vec{J} (\vec{r}) \nabla \vec{r} = 0$$

Changing the variable \( \vec{r} = \vec{R} \rightarrow \vec{r} \) in the expression for the total momentum we obtain:

$$\vec{P}_{\text{total}} = - \int d^3r \vec{E} (\vec{R} = 0) \cdot \vec{r} \vec{J} (\vec{r})$$

where we used the fact that \( \vec{J} (\vec{r}) = \vec{J} (\vec{r} - \vec{R}) \). Written in component form the total momentum is,

$$(\vec{P}_{\text{total}})_j = - E_i \int d^3r \cdot (r_i J_j (\vec{r}))$$

Now we try to get something for nothing,

$$r_i J_j = \frac{1}{2} (r_i J_j + r_j J_i) + \frac{1}{2} (r_i J_j - r_j J_i)$$

$$\int \frac{d^3r}{2} (r_i J_j + r_j J_i) = \frac{1}{2} \int d^3r \vec{J} \cdot \nabla (r_i r_j)$$

$$= - \frac{1}{2} \int d^3r ( \nabla \cdot \vec{J} ) r_i r_j$$

However the current flows in a particular geometry and is therefore not diverging, in other words we consider a static current so \( \nabla \cdot \vec{J} = 0 \) thus we find

$$(\vec{P}_{\text{total}})_j = - \frac{E_i}{2} \int d^3r (r_i J_j - r_j J_i)$$

$$\int d^3r (r_i J_j - r_j J_i) \equiv (\vec{\mu})_k$$

where \( i, j, k \) are in cyclic order. Therefore,

$$\vec{P}_{\text{total}} = \vec{E} \times \vec{\mu}$$

Previously we learn that for an isolated system at rest \( \vec{P}_{\text{total}} = 0 \). So why is \( \vec{E} \times \vec{\mu} \) nonzero, what momentum is balancing this one? Could it be from empty space? The answer is no because we only have electric and magnetic
fields. Could it be inside $q$? This seems unlikely because $q$ is very pointlike. Therefore, it must be some mechanical momentum inside the dipole $\vec{\mu}$. So, in order to fix the problem we say that the canonical momentum should have two terms, one mechanical and one describing the fields. If we are sitting still, $\vec{v} = 0$ so $\vec{P}_{\text{total}} = \vec{P}_{\text{mechanical}} + \vec{E} \times \vec{\mu}$ where

$$\vec{P}_{\text{mechanical}} = -\vec{E} \times \vec{\mu}$$

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**Case I** In the lecture on 10/10 (set 7 of notes) we explicitly calculated that the linear momentum of a magnetic monopole was zero. We now offer another argument to that fact in the hope of gaining a better understanding, also it gives us an excuse to examine some neat mathematics.

$$\vec{E}' = \vec{E} \cos\alpha + \vec{B} \sin\alpha$$

$$\vec{B}' = \vec{B} \cos\alpha - \vec{E} \sin\alpha$$

$$q' = q \cos\alpha + g \sin\alpha$$

$$g' = g \cos\alpha - q \sin\alpha$$

It can be shown that if $\{\vec{E}, \vec{B}\}$ is the solution to the generalized Maxwell equations (those who include the magnetic monopole) then $\{\vec{E}', \vec{B}'\}$ also are solutions. The angle $\alpha$ is called the complexion (see "Gravitation", Misner, Thorne and Wheeler, page 109) and it is not a physical transformation. To complete our argument, we note that the momentum is not dependent on the complexion. Therefore, we are always free to choose $\alpha$ so that $g'$ is 0, so clearly the momentum for $g$ is 0.

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**Digression:** The following section diverges slightly from the in-class lecture, but we hope you will at least consider reading it :) . What does \( \alpha \) correspond to? We begin by stating the generalized Maxwell equations:

\[
\nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_0} \tag{1}
\]

\[
\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}_e \tag{2}
\]

\[
\nabla \cdot \vec{B} = \mu_0 \rho_m \tag{3}
\]

\[
-\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} + \vec{J}_m \tag{4}
\]

where \( \vec{J}_e \) and \( \vec{J}_m \) are the electric and magnetic current densities respectively generated by the moving electric and magnetic charge densities, \( \rho_e \) and \( \rho_m \). Now we consider a rotation with \( \alpha = 90^\circ \) from the initial state where we have electric current and charge density but no magnetic charge and current (the ordinary world). The duality rotation implies:

\[
\vec{E} \rightarrow \vec{E}' = \vec{B} \\
\vec{B} \rightarrow \vec{B}' = -\vec{E} \\
q \rightarrow q' = g = 0 \\
g \rightarrow g' = -q
\]

Then we examine how this transformation changes equations 1 and 2 including some factors of \( c \) (we leave for the reader this exercise).

\[
\nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_0} \rightarrow \nabla \cdot \vec{B} = 0 \\
\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}_e \rightarrow -\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t}
\]

So we have changed equations 1 and 2 into equations 3 and 4. We took the magnetic charge to be zero in the hope that the reader will recognize the equations above. This exchange also holds when there is no magnetic charge. It turns out that if we write the electromagnetic field tensor \( F^{\mu\nu} \) as a 2-form and the 4-current \( J^\mu \) as a 1-form we may again see the above phenomenon. In
the new formalism the rotation of $\alpha = 90^\circ$ shall be performed by an algebraic operation called the Hodge dual (denoted by $\ast$). The correspondance of this new language for electromagnetism are outlined in the appendix, but for now we just state the result.

$$\text{d}^* \mathbf{F} = 4\pi^* \mathbf{J} \longrightarrow \text{d} \mathbf{F} = 0$$

Equations 1 and 2 are equivalent to $\text{d}^* \mathbf{F} = 4\pi^* \mathbf{J}$ and equations 3 and 4 are synonymous to $\text{d} \mathbf{F} = 0$. This marks the end of the digression until the appendix, we hope that some of you can profit from our wandering.

*STAY TUNED FOR THE APPENDIX!!!!!!*