Lecture notes: Classical electrodynamics 10/22/01

André Butz

Discussion of the first problem in the midterm exam

problem: A light ray is traveling in the +x-direction with frequency 2000 Hz. A viewer who sees the original frame as moving with velocity in the +z direction $v_z = 0.8c$ observes the light ray. What is its frequency and direction according to this viewer? You may specify the direction in terms of the tangent of the angle to the x-axis.

given: $\omega = 2\pi \cdot 2000\,Hz$, $\vec{k}' = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

boost: $\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$ where $\gamma = \frac{1}{\sqrt{1 - \frac{v_z^2}{c^2}}}$, $\beta = \frac{v_z}{c}$

We want to discuss two ways of solving this problem:

- Knowing the kinematic energy-momentum four-vector we can derive a four-vector k^μ which includes frequency ω and wave vector $\vec{k}$:

$$p^\mu = \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix} \quad \text{with } E \text{: energy}, \ \vec{p} \text{: momentum}$$

$$\Longrightarrow \quad k^\mu = \frac{p^\mu}{\hbar} = \begin{pmatrix} \frac{\omega}{c} \\ \vec{k} \end{pmatrix}$$

This four-vector k^μ undergoes the boost Λ:

$$k^{\mu'} = \Lambda k^\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \frac{\omega}{c} \\ k \\ 0 \\ \gamma\beta \frac{\omega}{c} \end{pmatrix}$$

In the frame of the viewer the frequency is ω' and the wave-vector $\vec{k}'$:

$$\underline{\omega' = \gamma \cdot \omega}$$

$$\vec{k}' = \begin{pmatrix} k \\ 0 \\ \gamma\beta \frac{\omega}{c} \end{pmatrix} \Longrightarrow \underline{\tan\theta = \frac{\gamma\beta \frac{\omega}{c}}{k} = \gamma\beta}$$

where θ is the angle to the x axis. Remember, that photons have a linear dispersion relation: $\omega = ck$.

- Another way of solving the problem is to consider two points in four-space that are separated by one wave length λ:

$$x_1^\mu = (0,0,0,0) \quad , \quad x_2^\mu = (\lambda, \lambda, 0, 0)$$

$$\Longrightarrow \Delta x^\mu = x_2^\mu - x_1^\mu = (\lambda, \lambda, 0, 0)$$

The four-vector Δx^μ undergoes the boost Λ:

$$\Delta x^{\mu'} = \Lambda \Delta x^\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\lambda \\ \lambda \\ 0 \\ \beta\gamma\lambda \end{pmatrix}$$

The space part yields the angle θ to the x-axis:

$$\underline{\tan\theta = \frac{\beta\gamma\lambda}{\lambda} = \gamma\beta}$$

To get the frequency as seen by the viewer we consider the light ray as a electromagnetic wave described by $\vec{E} = \vec{E}_0 \cdot e^{i k x^\mu}$ (where $k x^\mu = -\omega t + \vec{k}\vec{x}$) at a point $x^\mu = (0, 2, 0, 0)$.

$\Longrightarrow x^{\mu'} = \Lambda x^\mu = (0, 2, 0, 0)$

As we know that scalar and especially scalar products are invariant under boosts we can construct the following equation:

$$k_\mu x^\mu = 0 + k_x \cdot 2 + 0 + 0 = k'_\mu x'^\mu = 0 + k_x' 2 + 0 + 0$$

$$\Longrightarrow k_x = k_x' = |\vec{k}'| \cos\theta = \frac{\omega'}{c} \cos\theta =$$

$$= \frac{\omega'}{c} \cdot \sqrt{\frac{1}{1 + \tan^2\theta}} = \frac{\omega'}{c} \sqrt{\frac{1}{1 + \gamma^2\beta^2}} =$$

$$= \frac{\omega'}{c} \sqrt{\frac{1}{1 + \frac{\beta^2}{1-\beta^2}}} = \frac{\omega'}{c} \sqrt{\frac{1}{\frac{1-\beta^2+\beta^2}{1-\beta^2}}} =$$

$$= \frac{\omega'}{c} \sqrt{1-\beta^2} = \frac{\omega'}{c} \cdot \frac{1}{\gamma} = k_x = k = \frac{\omega}{c}$$

with $k_x = k = \frac{\omega}{c}$ as $\vec{k} = (k, 0, 0)$.

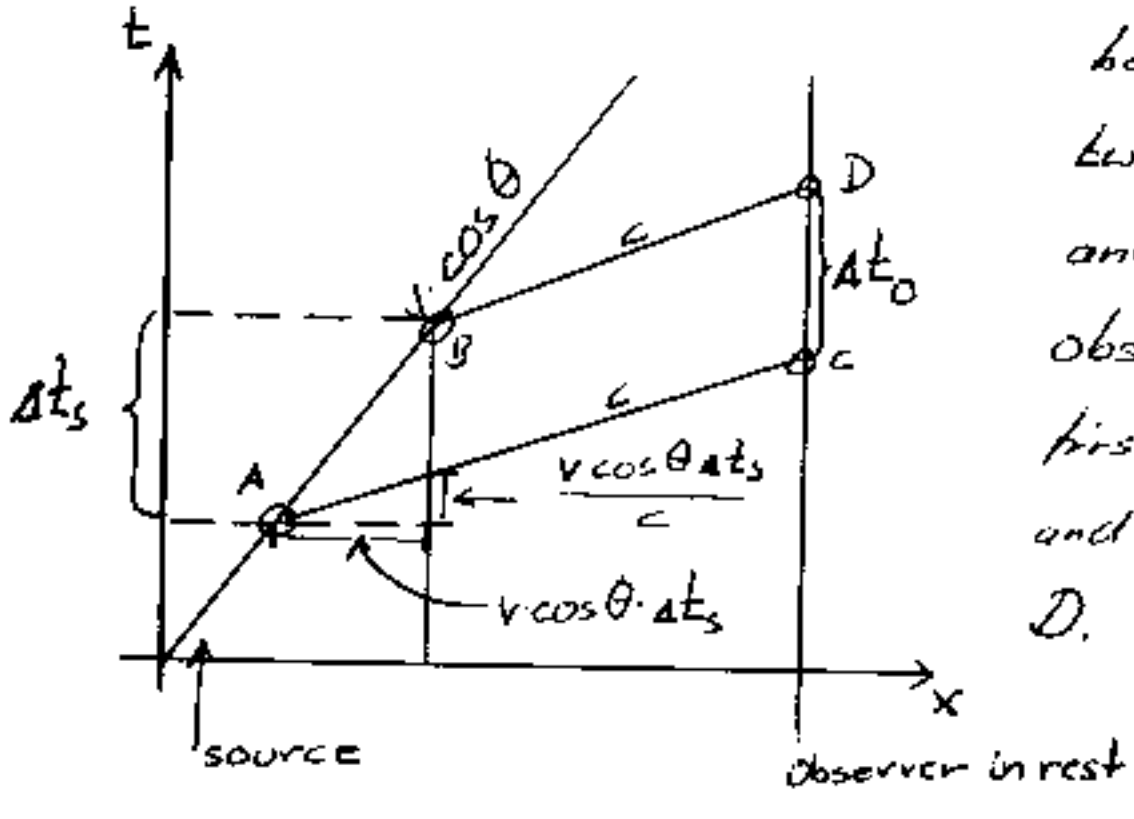
$$\underline{\underline{\omega' = \gamma \cdot \omega}}$$

Doppler effect

classical (soundwaves). The speed of the sound in air is c.

- source in motion, observer in rest w.r.t. the air:

  Consider the observer at rest w.r.t the air and the source moving with velocity $\vec{v}$ relative to the air. The component of $\vec{v}$ which is parallel to the line of sight between source and observer is $v \cdot \cos\theta$. Let the x axis be the Line of sight. The source is a bell which is struck twice, at event A and event B. The observer hears the first stroke at event C and the second at event D.



Δt_0 is the time interval between D and C.
Δt_s is the time interval between B and A.

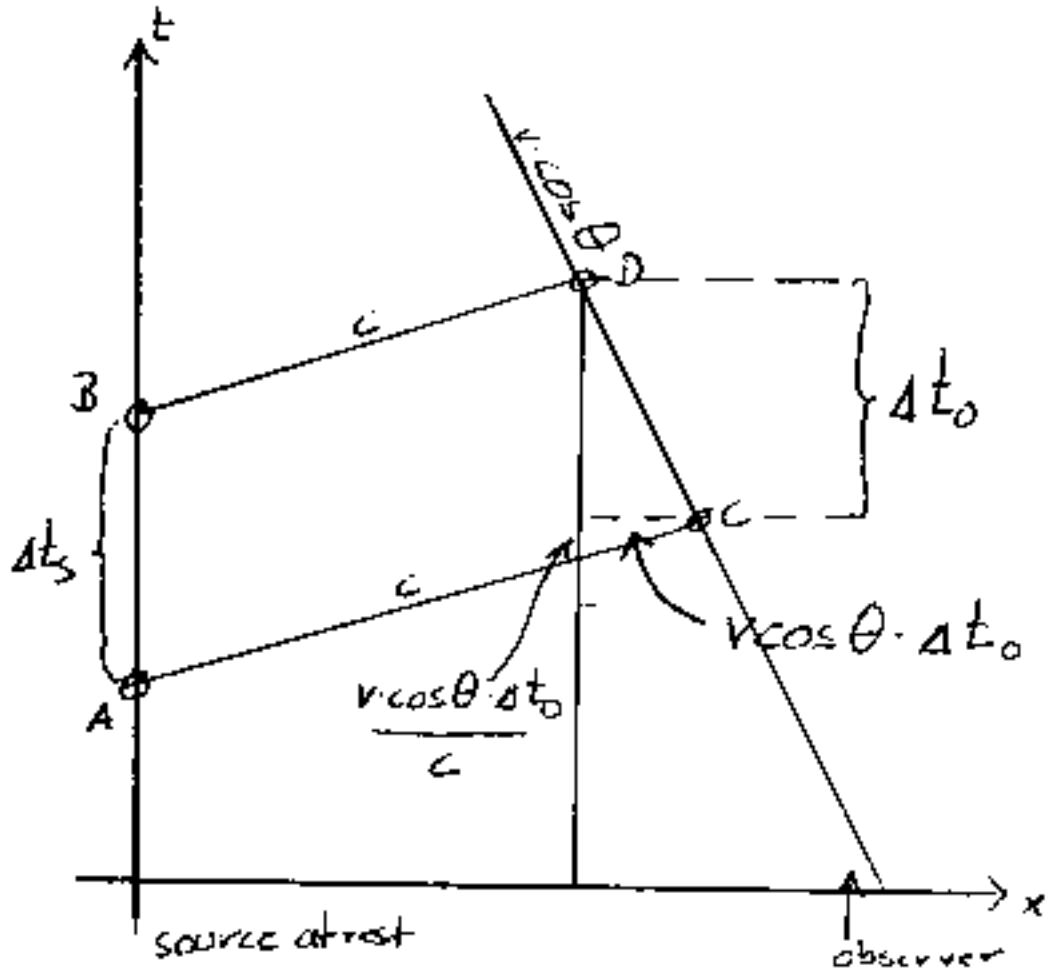
$$\Delta t_0 = \Delta t_s - \frac{v\cdot\cos\theta\,\Delta t_s}{c} = \left(1 - \frac{v}{c}\cos\theta\right)\Delta t_s$$

The period between strikes corresponds to the reciprocal of the frequency ν:

$$\nu_0 = \frac{1}{1 - \frac{v\cos\theta}{c}}\,\nu_s$$

- observer in motion, source in rest w.r.t. the air

We are considering the same ussumptions as in the preceding case. But now the source is at rest and the observer is moving with velocity $\vec{v}$.



$$\Delta t_S = \Delta t_0 + \frac{v \cdot \cos\theta \, \Delta t_0}{c} = \left(1 + \frac{v}{c}\cos\theta\right)\Delta t_0$$

$$\Longrightarrow \quad \nu_S = \frac{1}{1 + \frac{v}{c}\cos\theta}\,\nu_0$$

$$\Longrightarrow \quad \underline{\underline{\nu_0 = \left(1 + \frac{v}{c}\cos\theta\right)\nu_S}}$$

In our convention: $\theta < 90°$ if distance between source and observer becomes smaller.

$\theta > 90°$ if distance between source and observer becomes larger.

relativistic (light signals): The classical Doppler effect depends on the absolute speeds of the source and the observer relative to the air. The air is considered as a absolute frame of reference. In the case of light signals there exists no absolute medium. Only the relative velocity between observer and source can be relevant.

The observer moves with velocity $\vec{v}$ in z-direction. In the frame of the source the lightray is described by $k^\mu = \left(\frac{\omega}{c}, k_x, k_y, k_z\right)$. $k^{\mu'}$ is the lightray in the observer's frame.

$$k^{\mu'} = \Lambda \cdot k^\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \gamma\frac{\omega}{c} + \beta\gamma k_z \\ k_x \\ k_y \\ \beta\gamma\frac{\omega}{c} + \gamma k_z \end{pmatrix}$$

$$\Longrightarrow \frac{\omega'}{c} = \gamma\frac{\omega}{c} + \beta\gamma k_z = \gamma\frac{\omega}{c} + \beta\gamma |\vec{k}| \cos\theta =$$

$$= \gamma\frac{\omega}{c} + \beta\gamma\frac{\omega}{c}\cos\theta$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$, $\beta = \frac{v}{c}$, θ: angle of lightray with z-axis

$$\Longrightarrow \underline{\underline{\omega' = \omega\left(\frac{1 + \frac{v}{c}\cos\theta}{\sqrt{1 - \frac{v^2}{c^2}}}\right)}}$$

Note: Observations made at right angles ($\theta = 90°$) to the direction of motion show the non-zero transverse Doppler effect. Classically there should be no effect.