Lienard–Wiechert Potentials

We want to determine the potentials for the field produced by a charge moving along a trajectory \( \vec{r} = \vec{r}_0(t) \).

1. The Landau and Lifshitz Method

We say that the field at the point of observation, \( \vec{A} \), has a time \( \hat{t} \).

However, this field is determined by the motion of the charge at time \( \hat{t}' \).

* Note: \( \hat{t}' \) earlier than \( \hat{t} \) necessarily.

\( \hat{t} \) and \( \hat{t}' \) are related by:

\[
\hat{t} = \hat{t}' + \frac{R(\hat{t}')}{c}
\]

where \( R(\hat{t}') = |\vec{r} - \vec{r}(\hat{t})| \).

In the charge's rest frame, the potentials are simply the Coulomb potential:

\[
\Phi = \frac{\rho}{R(\hat{t}')}; \quad \vec{A} = 0
\]
Goal: To find the potentials in an arbitrary reference system.

Technique: Find a 4-vector which for $\vec{v} = 0$ coincides with the Coulomb potential.

To begin, we note that we can rearrange our relation between $t$ and $t'$ to give:

$$R(t') = C(t - t')$$

and thus we can write:

$$\phi = \frac{q_0}{C(t - t')}$$

We also note that,

$$A^\mu = (\phi, \vec{A})$$ is a 4-vector and,

$$R^\mu = (C(t - t'), \vec{R})$$ is a 4-vector

and $R^\mu$ is a "light-like" 4-vector for which $|\text{time component}| = |\text{space component}|$

and,

$$A^\mu R_\mu = -q_0$$

$$R_\mu R^\mu = 0$$
We now go to our arbitrary reference system by employing:

\[ \phi = \gamma \phi'(t - t', \mathbf{r} - \mathbf{r}') \]

\[ \mathbf{A} = \gamma \mathbf{A}'(t - t', \mathbf{r} - \mathbf{r}') \]

We notice that in this frame,

\[ \mathbf{A}^\mu \mathcal{R}_\mu = -\bar{\phi} \bar{\mathcal{R}} + \bar{\mathbf{A}} \cdot \mathbf{R} \]

\[ = -\phi \text{ as before} \]

Thus,

\[ -\bar{\phi} \bar{\mathcal{R}} (1 - \bar{\mathbf{A}} \cdot \mathbf{R}) = -\phi \]

\[ \Rightarrow \bar{\phi} = \left( \frac{\phi}{\bar{\mathcal{R}} - \bar{\mathbf{A}} \cdot \mathbf{R}} \right) \]

and,

\[ \bar{\mathbf{A}} = \left( \frac{\bar{\mathbf{A}} \phi}{\bar{\mathcal{R}} - \bar{\mathbf{A}} \cdot \mathbf{R}} \right) \]
Jackson goes for the more "brute force" method of directly integrating:

\[ \phi = \int d^3r' \frac{a}{|r - r'|} \delta^{(3)}(r' - R(0,t')) \]

where \( t' = t - \frac{R(t')}{c} \) as before.

In order to evaluate this integral, we need to use the trick:

\[ \delta(g(x)) = \frac{\delta(x-a)}{|\frac{dg}{dx}|_{x=a}} \]

To convince ourselves of this, we consider the following:

\[ \int dg(x) f(g(x)) \delta(g(x)) = f(a) \quad \text{if} \quad g(x) = 0 \quad \text{at} \quad x = a \]

So now,

\[ \int dx \frac{dg(x)}{dx} f(x) \delta(g(x)) = f(a) \]

\[ = \int dx \delta(g(x)) \frac{dg(x)}{dx} f(x) \]

\[ = \int dg(x) \delta(g(x)) f(x) \quad \text{which is exactly what we have above} \]
So we conclude,

\[ \delta(g(x)) = \frac{\delta(x-a)}{dg \, dx} \]

But we have to take care here. Even if \( \frac{dg}{dx} \bigg|_{x=a} \) is negative, we still want a positive multiple of \( f(a) \). So we make the further requirement:

\[ \delta(g(x)) = \frac{\delta(x-a)}{\left| \frac{dg}{dx} \bigg|_{x=a} \right|} \]

Armed with this tool, we are in the position to do some evaluating.

Starting back with:

\[ \Phi = \int d^3r' \frac{a_e}{|\bar{r} - \bar{r}'|} \delta^{(3)}(\bar{r}' - \bar{r}_0(t')) \]

We choose coordinates such that \( \bar{r}' = \bar{r} \hat{z} \)

\[ \Rightarrow \Phi = \int d^3\bar{r}' \frac{a_e}{|\bar{r} - \bar{r}'|} \delta \left( \bar{z}' - z_0(t') \right) \]
Now employing our trick from above:

\[
\delta(z' - z_0(t')) = \frac{\delta(z - z_0)}{\left| \frac{1}{d\tau (z - z_0(t - \frac{R(t')}{c})} \right|}
\]

Now we want to evaluate this denominator:

\[
\frac{d}{dz} \left( z - z_0(t - \frac{R(t')}{c}) \right) = 1 - \beta \frac{d}{dz} \left( t - \frac{R(t')}{c} \right)
\]

\[
= 1 + \frac{\beta}{c} \frac{d}{dz} \left( \frac{z - z'}{1 - \beta \hat{R}_z} \right)
\]

\[
= 1 - \beta \hat{R}_z
\]

Now gathering everything we have learned, we can write:

\[
\phi = \int dz' \frac{q_0}{|\hat{R}|} \frac{\delta(z - z_0)}{1 - \beta \hat{R}_z}
\]

which leads us to:

\[
\phi = \left( \frac{q_0}{R - \beta R} \right) \quad \text{and easily to:} \quad \hat{R} = \left( \frac{\beta q_0}{R - \beta R} \right)
\]