Boost shuttle

We start with power in the rest frame:

\[ \Phi = \frac{e^2}{c^3} \vec{a} \cdot \vec{a} \sin^3 \theta \]

(which actually is equal to \( r^2 \Phi \cdot \vec{a} \))

\[ \vec{a} = \frac{d\vec{x}}{dt} \]

1) \( \cos \theta = \vec{a} \cdot \vec{r} \)

2) \( \sin \theta = \vec{a} \times \vec{r} \)

3) \( a = \ddot{\vec{r}} \)


Dipole radiation formula

Now, transform to a frame where \( \vec{v} \) nears c

\[ \Phi' = \Phi \text{ in } (\text{EM}) \text{ tensor where } T^{\mu \nu} \propto k^\mu k^\nu \]

\[ T^{\mu \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\( \epsilon = \text{Energy density} = E^2 \)

\( \Pi = \text{Power density} \)

\( \xi = \text{Momentum density} \)

\( k^\mu = (\omega, 0, 0, \frac{\omega}{c}) \)

\( \epsilon \) transforms like a time component times another time component

\( \Pi \) transforms like a time component of a 4-vector

\[ T^\mu_\nu \rightarrow \gamma(1 + \beta \cos \theta) T^\mu_\nu \]

(\( \gamma = \sqrt{1 - \beta^2} \))

(from original frame of reference)

\( \theta \) changes into \( \tilde{\theta} \)
look into this & to 
\[ \Delta \theta \rightarrow \hat{\Delta} \theta \]

\[ \cos \theta = \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \]

this term we recognise from scalar, vector potentials and E-field

we now count the factors of \( \frac{1}{1 - \beta \cos \theta} \) we've acquired!

1) \( \sin^2 \theta \) term

2) \( \frac{1}{R^2} \rightarrow \frac{1}{r^2} \)

3) Lorentz transform on \( k^0 \) (i.e. boost on T^μμ)

This results in \( \frac{1}{1 - \beta \cos \theta} \) term in the power

\[ \rightarrow \frac{\Delta^2}{\Delta \theta} = \beta \frac{\Delta \theta}{\Delta \beta} \text{ for } \frac{\beta}{1 - \beta \cos \theta} \]

which cancels with \( \frac{1}{1 - \beta \cos \theta} \)

\[ \rightarrow \text{ Transformed power: } \bar{F}^2(\bar{F} \cdot \bar{F}) = \frac{e^2}{c^2} \frac{\Delta \theta}{\Delta \beta} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)} \]

This produces a very forward transform on the power

Find a typical angle: integrate over all angles to find

\[ \text{take } \cos \theta = \beta = 1 \]

now expand \( \cos \theta = \beta \rightarrow \cos \theta = 1 - \frac{\theta^2}{2} = \beta \Rightarrow -\frac{1}{2} \beta^2 \]

\[ \Rightarrow 1 - \frac{\theta^2}{2} = 1 - \frac{\Delta \theta}{2} \]

\[ \Rightarrow \theta = \frac{1}{\beta} \]

because: \( \beta = \frac{1}{\sqrt{1 - \beta^2}} \Rightarrow 1 - \beta^2 = \frac{1}{\beta^2} \Rightarrow \beta^2 = 1 - \frac{1}{\beta^2} \]

\[ \Rightarrow \beta = 1 - \frac{1}{\Delta \theta^2} \]
Let's look at the profile of the radiation pattern.

Circular pattern in forward direction

$\Delta \vec{E} \approx \frac{1}{r}$

All power forward.

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Now suppose $\vec{a} \parallel \vec{p}$ (Circular Motion).

This time, there's no zero for $r \parallel \vec{p}$.

$\Rightarrow$ we should get a peak in the rest frame.

"Trainlight Effect" $\Delta \vec{E} \parallel \vec{p}$

$\Rightarrow$ concentrates $E \parallel \vec{p}$

$\Rightarrow$ Synchrotron Radiation $\parallel \vec{p}$

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The power loss from the radiation field was troublesome to experimentors early in High Energy Physics.

Now, it is its own source of study (e^+e^-)

But not really relevant to e^- nowadays:

Compare $\vec{p}^+ \parallel \vec{p}^-$ power loss.

Power loss goes as $\frac{1}{\vec{p}^2}$

\[
\frac{\text{Rad } e^-}{\text{Rad } \vec{p}^+} \sim \left( \frac{m_p}{m_e} \right)^2 \sim 4 \times 10^6 \quad \text{(proton mass is more important)}
\]

\[e^+ \text{ at } \text{LEP, CERN}
\]

100 GeV for $e^- e^+$ used

4 x 10^8 GeV for $\vec{p}^+$ would be needed.
Total Radiated Power for Fast Moving Charge

\[ P = \frac{2}{3} \beta^2 e^2 \frac{\beta^2}{c^5} = \bar{P} \]

Lorentz transform is a good thing here.

Why? \( \frac{dE}{dt} \) is a Lorentz-Invariant Quantity

\( dE \cdot dt \) both in last (or first) spot on 4-vectors

transform in the same way.

The power does, however, get re-arranged in:

- frequency
- quantity

(See Jackson, chap 14)

Now, let's look at Photon - e\(^{-}\) scattering

very low frequency \( \rightarrow \) classical, non-relativistic

incoming beam: \( \vec{E} = E_x e^{-i (\omega t - k_x z)} \)

\[
\frac{\text{Power}}{\text{Area}} = \frac{\langle \vec{E} \times \vec{B} \rangle}{4\pi} = \frac{\langle |E_x|^2 \rangle}{4\pi} \frac{\omega^2}{2} \quad \text{(incoming power)}
\]

Take one e\(^{-}\) to scatter off of:

\[ \vec{F} = e \vec{E} \quad \text{(w/ e\(^{-}\) at } z = \infty \text{)} \]

\[ \vec{F}_x = e E_x e^{-i \omega t} \]

\[ = -m \omega^2 x e^{-i \omega t} \quad \Rightarrow \omega^2 x e^{-i \omega t} = \frac{e}{m} E_x e^{-i \omega t} \]
Dipole Radiation

\[ \vec{P} = \frac{(\omega^2 \overrightarrow{d} \times \overrightarrow{r})^2}{4\pi \epsilon_0^2} \]

\[ = \frac{(e^2 \vec{E}_x)^2 (\vec{r} \times \vec{r})^2}{4\pi \epsilon_0^2} \]

Now integrate to find total power outgoing

\[ \int d\Omega \frac{(e^2 \vec{E}_x)^2 (\vec{r} \times \vec{r})^2}{4\pi \epsilon_0^2} = \frac{\epsilon_0 \pi}{3} \left( \frac{e^2}{m^2 c^2} \right) \]

Compared to incoming power = \( \frac{c \vec{E}_x^2}{4\pi} \)

Defines classical electron radius

dipole moment has the property:

\[ \omega^2 \vec{J} = \vec{p} \left( \frac{e \vec{E}_x}{m} \right) e^{-i\omega t} \]

\[ = \frac{e^3 \vec{E}_x}{m} e^{-i\omega t} \]

because:

1st \( e \) from dipole moment  
2nd \( e \) from response to external field

Thompson scattering \( \frac{1}{\sigma} \)