1a. An electric charge $q$ is located at the origin of coordinates. What is the electric flux through the plane $z = d$? Explain all steps.

1b. What is the flux through a specific part of the $z = d$ plane: a disc of radius $\sqrt{3}d$ with center on the $z$ axis?

Solution a. The flux lines coming out of a point charge come out radially in all directions. Therefore, just as many come out in the positive $z$ direction as in the negative $z$ direction. Consequently, exactly half the total flux will pass through the plane $z = d$, $\Phi_E = 2\pi q$ in cgs units or $\Phi_E = q/2\varepsilon_0$ in SI units.

b. For a disc of radius $R$ centered on the $z$ axis in the plane $z = d$ the flux is $\Phi_E = \int dS \cdot \vec{E} = \int_0^{2\pi} d\phi \int_0^R \rho d\rho \ q d/\sqrt{d^2 + \rho^2} = 2\pi q d(1/d - 1/\sqrt{d^2 + R^2})$, which for $R = \sqrt{3}d$ is $q\pi$ in cgs units, using $\rho d\rho = d\rho^2/2$. Alternately, because the flux comes out equally in all directions, the flux into any surface is proportional to the solid angle subtended by that surface at the location of the charge. For a whole sphere around the charge, that solid angle is $4\pi$. For a disc in the plane $z = d$ with radius $R = \sqrt{3}d$, the opening angle of a cone from the charge to the edge of the disc is $60^\circ = \pi/3$ rad. This gives a solid angle $\Omega = 2\pi \int_0^{\pi/3} d\theta \sin\theta = 2\pi (1 - \cos(\pi/3)) = \pi$. Again we conclude the flux through the disc is $1/4$ of the total flux out of the charge.

2. A conducting sphere centered on the origin has electric potential $\phi = 0$. Far away, there is a uniform electric field $\vec{E} = E_0 \hat{\vec{x}}$. Obtain the electric potential $\phi(x, y, z)$ and the electric field $\vec{E}(x, y, z)$ everywhere, assuming that only on the surface of the sphere is there nonzero electric charge density, and $\phi(x, y, 0) = 0$.

Solution The ‘external’ potential may be written in spherical polar coordinates as $\phi_{\text{ext}} = -E_0 z = -E_0 r \cos\theta$. Because the boundary condition $\phi(R) = 0$ (where $R$ is the radius of the sphere) is independent of angle, the entire function $\phi(\vec{r})$ must either be independent of angle or proportional to $\cos\theta$. The part independent of angle outside the sphere must be $\phi_{\ell=0} = A + B/r$, as these are the $\ell = 0$ solutions of the Laplace equation. However, this can only satisfy the condition $\phi(x, y, 0) = 0$ for $A = B = 0$, which in turn means there is no net electric charge on the sphere (by the Gauss law for the net electric flux out of the sphere). Therefore, inside the
sphere we have $\phi = 0$, and outside we have $\phi = -E_0(r - R^3/r^2)\cos \theta$, because this is the only combination of radial functions allowed by the Laplace equation for $\ell = 1$, and vanishing for $r = R$. We may write this in the easier form $\phi = -E_0 z(1 - R^3/r^3)$. Then we get inside the sphere $\vec{E} = 0$, and outside the sphere $E = -\nabla \phi = E_0(\hat{z} + (3\hat{r} \cdot \hat{r} - \hat{z})R^3/r^3)$, that is, a uniform electric field plus an electric dipole field corresponding to dipole moment $\vec{d} = R^3E_0\hat{z}$. This makes sense: the constant external field pushes positive charge to the positive $z$ side of the sphere, and negative charge to the negative $z$ side.

3. [Corrected as on the board during the exam:] A source located at the origin of coordinates in its rest frame emits red light ($\lambda = 780$ nm) in all directions. The source is moving towards a receiver with velocity $\vec{\nu} = v\hat{z}$, so that in the source rest frame the receiver moves with velocity $-\vec{\nu}$ along the line $x = 2$ m, $y = 0$. For what value of $\gamma = 1/\sqrt{1 - (v/c)^2}$ would light coming out in the $x$ direction in the source rest frame appear violet ($\lambda = 390$ nm) for the receiver? Would that ray be received before, after, or at the same time as the source passed the receiver? Explain.

Solution The light ray is perpendicular to the direction of boost from source rest frame to receiver rest frame. Consequently, the frequency in the receiver frame $\omega' = \gamma(\omega + (v/c)k_z) = \gamma \omega$. Because $\lambda'/\lambda = 390/780 = 1/2$, using $\omega \lambda = 2\pi c$ we get $\omega'/\omega = 2 \Rightarrow \gamma = 2$. To answer the second part of the question, choose the origin of time in the source rest frame as the moment when the light ray hits the receiver, which also is the moment in the source rest frame when the receiver passes the source (that is, the moment when the distance between receiver and source is smallest). That specifies the event in the source frame, $(t = 0, x = 2$ m, $y = 0, z = 0$. Now we transform this four-vector to the receiver frame: $t' = \gamma(t - (v/c)z) = 0$, $x' = x = 2$ m, $y' = y = 0$, $z' = \gamma(z - vt) = 0$. Clearly this also is the moment in the receiver frame when the source passes the receiver. Thus the answer is that the light ray is received at the same time that the source passes the receiver. Note that in both the source frame and the receiver frame the ray is emitted by the source before it hits the receiver, but the amount of time between emission and absorption is not the same in the two frames, being clearly longer in the receiver frame, where the light travels a longer path.

4. Electric charges $\pm q$ are located at $x = \pm a$, and there is a uniform magnetic field $\vec{B} = B_0\hat{z}$ (i.e., $\vec{B}$ up from $x$-$y$ plane).
a. Show that because $\vec{E}$ is a conservative field ($\nabla \times \vec{E} = 0$), lines of flux of the Poynting power vector $\vec{P} \propto \vec{E} \times \vec{B}$ never begin or end.

b. Sketch the pattern of these lines of $\vec{P}$ in the $x$-$y$ plane. Include the line which passes through the origin.

Solution a. The local statement that flux lines have no end is $\nabla \cdot \vec{P} = 0$. Let's check this: $\nabla \cdot (\vec{E} \times \vec{B}) = (\nabla \times \vec{E}) \cdot \vec{B} = \vec{0} \cdot \vec{B} = 0$. This used the fact $\vec{B}$ = constant, so that $\nabla$ on $\vec{B}$ vanishes.

b. Sketch. Notice that the lines indeed do not stop or start anywhere. Lines of $\vec{E}$ are shown dashed, and the lines of $\vec{P}$ of course are perpendicular to them.
5. The Thomson scattering cross section of very low frequency photons on an electron at rest, \( \sigma(\omega = 0) = \frac{8\pi(e^2/mc^2)^2}{3} \) (cgs units), can be understood as due to dipole radiation by the electron moving under the influence of an incident electromagnetic wave. The radiated power from an oscillating dipole of amplitude \( \vec{d}_\omega \) is proportional to \( \omega^4 |\vec{d}_\omega|^2 \).

a. Obtain the amplitude of the oscillating electric dipole moment \( \vec{d}_\omega \) of the electron pushed by the field \( \vec{E}_0 e^{-i\omega t} \) [Hint: Use Newtonian mechanics for this; you should explain why that is justified]. Check that indeed the dipole frequency is the same as that of the electric field.

b. Use your result in (a) to confirm that \( \sigma(\omega = 0) \) is a finite, nonzero constant.

Solution a. The Newton force law is \( m\ddot{\vec{r}}/dt^2 = -e\vec{E}_0 e^{-i\omega t} \), with solution \( \vec{r} = e\vec{E}_0 e^{-i\omega t}/m\omega^2 \). To get the electric dipole moment, we use \( \vec{d} = \int d^3r\rho(\vec{r}; t) \), with \( \rho = -e\delta^{(3)}(\vec{r} - e\vec{E}_0 e^{-i\omega t}/m\omega^2) \), giving 
\[ \vec{d} = -e^2 (\vec{E}_0/m\omega^2)e^{-i\omega t} \equiv \vec{d}_\omega e^{-i\omega t}. \] Clearly the frequency of the dipole oscillation is the same as that of the applied field. To see whether this Newtonian calculation is right, calculate the velocity, \( \vec{v} = d\vec{r}/dt = -i\omega\vec{r} = -ie\vec{E}_0 e^{-i\omega t}/m\omega. \) If we choose \( \omega \), and then set \( \vec{E}_0 \) small enough, we can always make this velocity very small compared to the speed of light \( c \), and so the Newtonian calculation should be accurate.

b. We see that from the problem there is a factor \( \omega^4 \) which by itself would make the power go to zero as the frequency went to zero. On the other hand, \( |\vec{d}_\omega|^2 \) has a factor \( (1/\omega^2)^2 \), which by itself would make the radiated power blow up as the frequency went to zero. Because the product \( \omega^4/(\omega^2)^2 \) is independent of \( \omega \), the limit of the power is a finite, nonzero constant. As the Thomson cross section is the ratio of this power to the intensity of the incident wave, the cross section also has a finite value, as indicated in the problem.