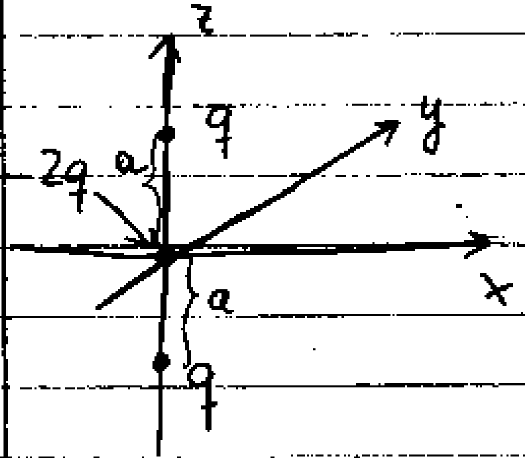
EM- HW 10

Problem 1:



Method 1: using the general expansion

$$\phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} b_{lm} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

with

$$b_{lm} = \int d^3x \, Y^*_{lm}(\theta,\varphi) r^l \rho(\vec{x})$$

plug-in the charge distribution

$$\rho(\vec{x}) = q\,\delta(z-a)\delta(x)\delta(y) + q\,\delta(z+a)\delta(x)\delta(y) - 2q\,\delta(z)\delta(x)\delta(y)$$

and get the coefficients:

$b_{00} = 0$, as $Q = \sum q = 0$ (no monopole moment)

$b_{10} = b_{11} = 0$, as $P = \sum q_i \vec{r}_i = 0$ (no dipole moment)

$$b_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int d^3\vec{x}\,(2z^2 - x^2 - y^2)\Big[q\,\delta(z-a)\delta(x)\delta(y) + q\,\delta(z+a)\delta(x)\delta(y) - 2q\,\delta(x)\delta(y)\delta(z)\Big]$$

$$= \frac{1}{2}\sqrt{\frac{5}{4\pi}}\left(2qa^2 + 2qa^2\right) = \sqrt{\frac{5}{4\pi}}\,2qa^2$$

The rest of the b's are zero by the symmetry of the charge distribution.

Now, the potential reads

$$\phi(\vec{x}) = \frac{4\pi}{5}\sqrt{\frac{5}{4\pi}}\,2aq^2\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\frac{1}{r^3}$$

$$\boxed{\phi(\vec{x}) = \frac{qa^2(3z^2 - r^2)}{r^5}}$$

Method 2 :

Taylor expand

$$\varphi(\vec{r}) = \frac{q}{|\vec{r}-\vec{a}|} + \frac{q}{|\vec{r}+\vec{a}|} - \frac{2q}{|\vec{r}|}$$

where $\vec{a} = (0,0,a)$

Now,

$$\frac{1}{|\vec{r}-\vec{a}|} = \frac{1}{\sqrt{r^2 - 2\vec{a}\cdot\vec{r} + a^2}} =$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{a^2}{r^2} - \frac{2\vec{a}\cdot\vec{r}}{r^2}\right) + \frac{3}{8}\cdot\left(\frac{a^2}{r^2} - \frac{2\vec{a}\cdot\vec{r}}{r^2}\right)\right]$$

$$= \frac{1}{r}\left(1 + \frac{\vec{a}\cdot\vec{r}}{r^2} - \frac{a^2}{2r^2} + \frac{3}{8}\left(\frac{4(\vec{a}\cdot\vec{r})^2}{r^4}\right) + O\left(\frac{a^3}{r^3}\right)\right)$$

$$= \frac{1}{r}\left(1 + \frac{az}{r^2} - \frac{a^2}{2r^2} + \frac{3}{2}\frac{a^2z^2}{r^4} + \dots\right)$$

Similarly, changing $\vec{a} \rightarrow -\vec{a}$:

$$\frac{1}{|\vec{r}+\vec{a}|} = \frac{1}{r}\left(1 - \frac{az}{r^2} - \frac{a^2}{2r^2} + \frac{3}{2}\frac{a^2z^2}{r^4} + \dots\right)$$

Adding up $\frac{q}{|\vec{r}-\vec{a}|} + \frac{q}{|\vec{r}+\vec{a}|} = \frac{2q}{r} + \frac{3qa^2z^2}{r^5} - \frac{a^2}{r^3}$

So that putting everything together

$$\boxed{\varphi(\vec{r}) = qa^2\left(\frac{3z^2}{r^5} - \frac{r^2}{r^5}\right) = qa^2\,\frac{3z^2 - r^2}{r^5}}$$

Problem 2 :

Let's compute the $\vec{E}$-field. Write the potential in spherical coordinates as

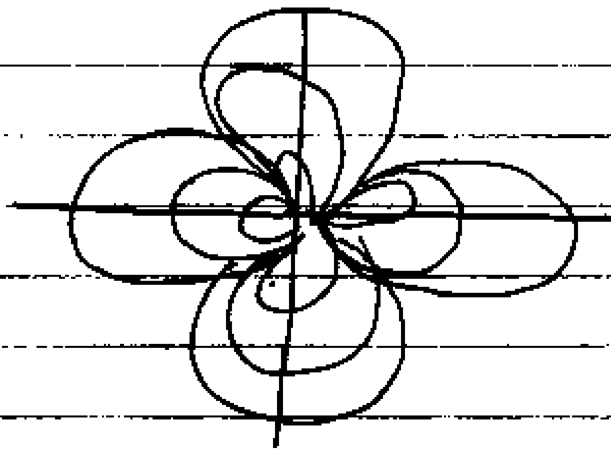
$$\phi(r, \theta, \varphi) = qa^2\,\frac{3\cos^2\theta - 1}{r^3}$$

then

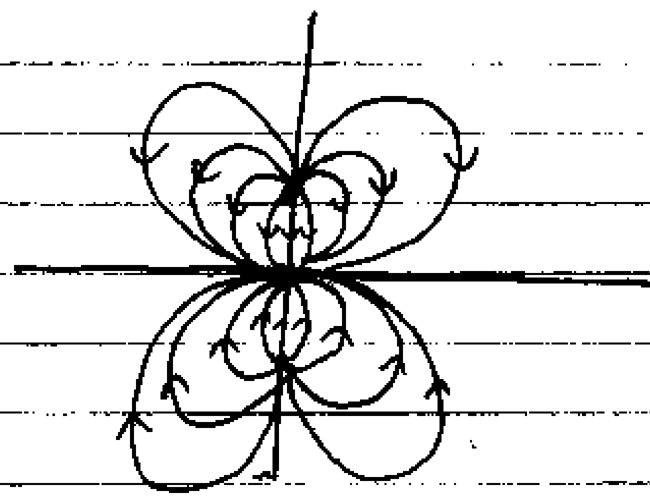
$$\vec{E} = -\nabla\phi = -\left(\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial\theta}\right)\left(\frac{qa^2}{r^3}\left(3\cos^2\theta - 1\right)\right)$$

$$\Rightarrow \boxed{\begin{aligned} E_r &= \frac{3qa^2}{r^4}\left(3\cos^2\theta - 1\right) \\ E_\theta &= \frac{6qa^2}{r^4}\sin\theta\cos\theta \\ E_\phi &= 0 \end{aligned}}$$

Equipotentials:



E-field lines:



Problem 3 :

The potential must satisfy Laplace's equation

$$\nabla^2 \phi(r, \theta, \varphi) = 0$$

but choosing the direction of $\vec{E}$-field in z-direction the configuation doesn't depend on φ. Therefore the general solution can be written as :

$$\phi(r,\theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\theta)$$

We can thus write

$$\phi_{in}(r,\theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\theta)$$

$$\phi_{out}(r,\theta) = \sum_{l=0}^{\infty} (C_l r^l + D_l r^{-(l+1)}) P_l(\cos\theta)$$

We must impose the following conditions

1) $\phi_{in}(r=0)$ is non-singular $\Rightarrow B_l = 0 \ \forall l$

2) $\phi_{out}(\infty, \theta) = -E_0 z = -E_0 r \cos\theta$

as this is the potential corresponding to $\vec{E}_0$ which at infinity is not affected by the presence of the sphere.

This means $C_1 = -E_0$, all the rest zero.

So we have

$$\phi_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\phi_{out}(r,\theta) = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos\theta)$$

Now impose

a) $\phi_{in}(R) = \phi_{out}(R)$

b) $\epsilon \left.\frac{\partial \phi_{in}}{\partial r}\right|_{r=R} = \epsilon_0 \left.\frac{\partial \phi_{out}}{\partial r}\right|_{r=R}$ $\quad (\vec{D}_{in}^{\perp} = \vec{D}_{out}^{\perp})$

From (a) we get:

(A)

$$\boxed{\begin{aligned} A_l R^l &= \frac{D_l}{R^{l+1}}, \quad l \neq 1 \\ A_1 R &= -E_0 R + \frac{D_1}{R^2}, \quad l = 1 \end{aligned}}$$

From (b) we get

$$\text{(B)}\quad \boxed{\begin{aligned} &\epsilon\, l A_l R^{l-1} = -\epsilon_0 \frac{(l+1)}{R^{l+2}} D_l \,, \quad l \neq 1 \\ &\epsilon A_1 = \epsilon_0 \left(-E_0 - \frac{2D_1}{R^3} \right) , \quad l = 1 \end{aligned}}$$

Putting both conditions together:

$$\left.\begin{aligned} \text{(B)} &\Rightarrow D_l = \frac{-\epsilon}{\epsilon_0}\left(\frac{l}{l+1} R^{2l+1} \right) A_l \\ \text{(A)} &\Rightarrow D_l = \left(R^{2l+1} \right) A_l \end{aligned}\right\} l \neq 1$$

This implies $A_l = D_l = 0$ for $l \neq 1$

So we consider $l = 1$ (define $\epsilon_r = \epsilon / \epsilon_0$)

$$\begin{aligned} \text{(B)} &\Rightarrow \boxed{\epsilon_r A_1 = -E_0 - \frac{2D_1}{R^3}} \\ \text{(A)} &\Rightarrow \boxed{A_1 = -E_0 + \frac{D_1}{R^3}} \end{aligned}$$

Which gives

$$A_1 = -\frac{3E_0}{2+\epsilon_r}$$

$$B_1 = E_0\left(\frac{\epsilon_r - 1}{2+\epsilon_r}\right)R^3$$

$\Rightarrow$

$$\boxed{\begin{aligned}\phi_{in} &= \frac{-3E_0}{2+\epsilon/\epsilon_0}\, r\cos\theta \quad , \; r \le R \\ \phi_{out} &= -E_0 r\cos\theta - E_0\left(\frac{1-\epsilon/\epsilon_0}{2+\epsilon/\epsilon_0}\right)\frac{R^3}{r^2}\cos\theta \\ & \qquad r \ge R\end{aligned}}$$

The Electric field is then

$$\boxed{\begin{aligned}\vec{E}_{in}(r,\theta) &= -\nabla\phi_{in} = \frac{3E_0}{2+\epsilon_r}\cos\theta\,\hat{r} - \frac{3}{\epsilon_r+2}E_0 r\sin\theta\,\hat{\theta} \\ \vec{E}_{out}(r,\theta) &= \left(E_0\cos\theta - 2\,\frac{\epsilon_r - 1}{\epsilon_r+2}E_0\frac{R^3}{r^3}\cos\theta\right)\hat{r} + \\ & \quad + \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}E_0\frac{R^3}{r^3}\sin\theta - E_0 r\sin\theta\right)\hat{\theta}\end{aligned}}$$

So that the electric dipole moment is

$$\boxed{P = \frac{\epsilon_r - 1}{\epsilon_r + 2}E_0 R^3}$$ (take $\frac{1}{r^3}$ terms)