CLASSICAL ELECTRODYNAMICS

PHYSICS 505 Homework Solutions 7 DECEMBER 2001

NOTE: For all three of these problems, the results are accurate only if the ratio of size of source to wavelength is small, i.e., $ka \ll 1$. The more complicated, and quite interesting, case when this is not true will be an important subject next semester, involving spherical Bessel functions, realistic antenna design, etc.

1. In class the radiation dipole magnetic field for a point charge oscillating in the z direction with frequency ω and amplitude a was obtained using $\vec{B}_{\rm rad} = \hat{r} \times \vec{E}_{\rm rad}$. Instead, obtain the result using $\vec{B} = \nabla \times \vec{A}$.

Solution: $\vec{B}_{\rm rad} = (1/r)$ part of $\nabla \times \vec{A}_{\rm rad}$.

$$\vec{A}_{\rm rad} = (1/r)e^{-i(\omega t - kr)} \int d^3r' e^{-i\vec{k}\cdot\vec{r}\,'} i(\omega/c)\rho_\omega(\vec{r}\,')\vec{r}\,'$$

$$\approx ik\vec{d}_\omega e^{-i(\omega t - kr)}/r \ .$$

$$\vec{B}_{\rm rad} = -k\vec{k} \times \vec{d}_\omega e^{-i(\omega t - kr)}/r \ .$$

$[\vec{d}_\omega = qa\hat{z};\ k = \omega/c;\ \vec{k} = k\hat{r}]$

2. For a point charge oscillating with $z = ae^{-i\omega t}$, obtain the electric quadrupole moment contribution to the scalar and vector potentials, and find what frequency or frequencies it has.

Solution: By symmetry about the z axis, only the Legendre polynomial with $\ell = 2$ and $m = 0$ is relevant: Expand $e^{-i\vec{k}\cdot\vec{r}'}$ to obtain the first nonvanishing power of $\vec{k} \cdot \vec{r}\,'$ which would contribute upon integration with the factor $P_2(\cos\theta')$. This means we may replace $e^{-i\vec{k}\cdot\vec{r}\,'}$ with $-(\vec{k} \cdot \vec{r}\,')^2/2$, because a factor independent of angle, or one proportional to $\cos\theta'$, is orthogonal to $P_2(\cos\theta')$. This yields

$$\phi_{\rm rad} = \int d^3r' q\delta^{(3)}(\vec{r}\,' - \vec{a}e^{-i\omega t})(-(\vec{k} \cdot \vec{r}')^2/2)$$

$$= -2k^2a^2e^{-2i(\omega t - kr)}q(\hat{r} \cdot \hat{z})^2/r \ ,$$

where the factor 2 in the exponent and also in the vector $\vec{k}$ is there because integration of the charge density with $(r')^2 P_2$ has picked out the oscillation frequency 2ω, so the wavenumber has changed also to twice its value for the dipole. Let us use the latter as the unit from now on. If one used $a\hat{z}\cos\omega t$ to describe the oscillation, then a zero frequency piece would appear in $\phi_{\rm rad}$, but it would not contribute to the radiation field strengths.

The trickiest part of this whole assignment is the calculation of the vector potential $\vec{A}_{\rm rad}$. Recall that for the dipole part, in $\vec{A}$ one did an integration by parts and to lowest order in k got only the constant, unit, contribution from $e^{-i\vec{k}\cdot\vec{r}\,'}$. It stands to reason that the quadrupole part should involve the first-order term in the exponential, $-i\vec{k}\cdot\vec{r}$. To get this, write $J_i r_j = (1/2)(J_i r_j + J_j r_j) - \vec{J}\cdot\vec{r}\delta_{ij}/3$ +antisymmetric part+trace part. As long as there is no magnetic moment contribution, the antisymmetric part will give zero in the integral, and the trace part gives a piece of $\vec{A}$ parallel to $\vec{r}$ which will not contribute to the field strength. Thus the calculation is reduced again using integration by parts to

$$\vec{A}_{\rm rad} = -2k^2 a^2 e^{-2i(\omega t - kr)} q\hat{z}\hat{r}\cdot\hat{z}/r \ ,$$

omitting a part proportional to $\hat{r}$.

3. Obtain the quadrupole contributions to the radiation field strengths $\vec{E}_{\rm rad}$ and $\vec{B}_{\rm rad}$, and the angular distribution of the radial power density.

Solution:

$$\vec{B}_{\rm rad} = \nabla \times \vec{A}_{\rm rad} = 4i\hat{r} \times \hat{z}(\hat{z}\cdot\hat{r}) q k^3 a^2 e^{-2i(\omega t - kr)}/r$$

and

$$\vec{E}_{\rm rad} = -\hat{r} \times \vec{B}_{\rm rad} = 4i(\hat{z} - \hat{r}(\hat{z}\cdot\hat{r}))\hat{z}\cdot\hat{r} q k^3 a^2 e^{-2i(\omega t - kr)}/r \ .$$

The angular distribution of radial power, including a factor 1/2 for time averaging of real fields, then is

$$r^2 \mathcal{P}\cdot\hat{r} = 8cq^2k^6a^4\sin^2\theta\cos^2\theta/4\pi \ ,$$

and the total power is $P = 16cq^2k^6a^4/15$.

In comparing with textbook accounts, recall that the quadrupole oscillation frequency 2ω here is twice the frequency of the position oscillation $z = ae^{-i\omega t}$, so that one should substitute $2k$ for the k in the book. Care also is needed in applying the definition of quadrupole moment Q, which involves a choice of convention.