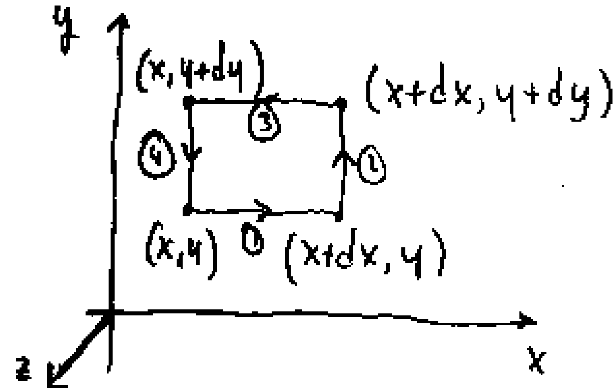
EM - HW 2

① Use Stokes theorem to calculate $\nabla \times \vec{f}$ in Cartesian coordinates.

Stokes theorem tells us

$$\oint_C \vec{f}(\vec{r}) \cdot d\vec{l} = \int_S \nabla \times \vec{f} \cdot d\vec{s}$$

where the contour C encircles the surface S.



It suffices to consider an infinitesimal surface. We'll calculate $[\nabla \times \vec{f}(x,y)]_z$ by considering the figure above, with area

$$d\vec{s} = (dx dy)\hat{z}$$

The surface integral gives (infinitesimally)

$$\int_S (\nabla \times \vec{f}) \cdot d\vec{s} = \hat{z} \cdot (\nabla \times \vec{f}) dx dy, \text{ where } \nabla \times \vec{f} \text{ is taken at } (x,y).$$

And the line integral, also infinitesimally

$$\int_{①} \vec{f} \cdot d\vec{l} = f_x(x,y)\, dx \quad , \quad \int_{②} \vec{f} \cdot d\vec{l} = f_y(x+dx, y)\, dy =$$

$$= \left(f_y(x,y)\, dy + \frac{\partial f_y}{\partial x}(x,y)\, dx\, dy \right)$$

$$\int_{③} \vec{f}\cdot d\vec{l} = -\left(f_x(x,y) + \frac{\partial f_x}{\partial y}(x,y)\,dy\right)dx$$

$$\int_{④} \vec{f}\cdot d\vec{l} = -f_y(x,y)\,dy$$

Summing all contributions

$$\oint \vec{f}\cdot d\vec{l} = \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)dx\,dy$$

So comparing to the surface integral - terms $\mathcal{O}(2)$ in differentials:

$$\left(\nabla\times\vec{f}\right)_z = \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

Take now properly oriented contours in the $(y\,z)$ and (x, z) planes: you can simply cyclically permute coordinates, as this respects orientations. This yields the other components of the curl, by the same reasoning we used before.

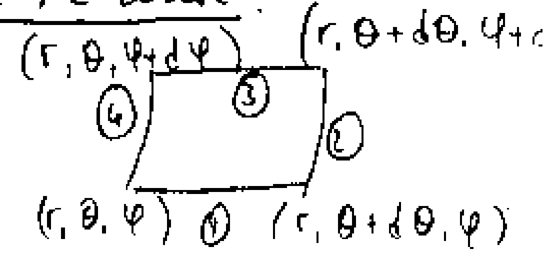
$$\left(\nabla\times\vec{f}\right)_x = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)$$

$$\left(\nabla\times\vec{f}\right)_y = \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)$$

② Spherical coordinates:

Take first an infinitesimal area with $r = \text{const}$

$$d\vec{s} = \hat{r}\, r^2 \sin\theta\, d\theta\, d\varphi$$



The surface integral then yields

$$\int_S (\nabla\times\vec{f})\cdot d\vec{s} = \hat{r}\cdot(\nabla\times\vec{f})\, r^2 \sin\theta\, d\theta\, d\varphi \quad (\text{evaluated at } (r,\theta,\varphi))$$

On the other hand, for the line integral, use

$$d\vec{l} = \hat{r}\, dr + \hat{\theta}\, r\, d\theta + \hat{\varphi}\, r \sin\theta\, d\varphi$$

$$\vec{f}\cdot d\vec{l} = f_r\, dr + r f_\theta\, d\theta + r\sin\theta f_\varphi\, d\varphi$$

$$\equiv F_r\, dr + F_\theta\, d\theta + F_\varphi\, d\varphi$$

Then

$$\int_{①} \vec{f}\cdot d\vec{l} = F_\theta(r,\theta,\varphi)\, d\theta$$

$$\int_{②} \vec{f}\cdot d\vec{l} = F_\varphi(r,\theta+d\theta,\varphi)\, d\varphi = F_\varphi(r,\theta,\varphi)\, d\varphi + \left.\frac{\partial F_\varphi}{\partial\theta}\right|_{r,\theta,\varphi} d\theta\, d\varphi$$

$$\int_{③} \vec{f}\cdot d\vec{l} = -F_\theta(r,\theta,\varphi+d\varphi)\, d\theta = -F_\theta(r,\theta,\varphi)\, d\theta - \left.\frac{\partial F_\theta}{\partial\varphi}\right|_{r,\theta,\varphi} d\varphi\, d\theta$$

$$\int_{④} \vec{f}\cdot d\vec{l} = -F_\varphi(r,\theta,\varphi)\, d\varphi$$

So that

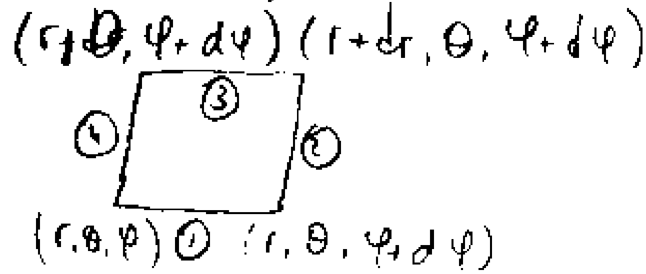
$$\oint \vec{f}\cdot d\vec{l} = \left(\frac{\partial F_\varphi}{\partial\theta} - \frac{\partial F_\theta}{\partial\varphi}\right) d\theta\, d\varphi$$

Comparing the surface and line integrals, and writing F in terms of the original f:

$$(\nabla\times\vec{f})_r = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta\, f_\varphi) - \frac{\partial}{\partial\varphi}(r f_\theta)\right]$$

$$\boxed{(\nabla\times\vec{f})_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, f_\varphi) - \frac{\partial}{\partial\varphi}(f_\theta)\right]}$$

Take now $\theta = \text{const}$.



$$d\vec{s} = r\sin\theta\, dr\, d\varphi\, \hat{\theta}$$

$$\int(\nabla\times\vec{f})\cdot d\vec{s} = (\nabla\times\vec{f})_\theta\, r\sin\theta\, dr\, d\varphi$$

The line integral gives

$$\int_{①}\vec{f}\cdot d\vec{l} = F_\varphi(r,\theta,\varphi)\, d\varphi$$

$$\int_{②}\vec{f}\cdot d\vec{l} = F_r(r,\theta,\varphi+d\varphi)\, dr = F_r(r,\theta,\varphi)\, dr + \left.\frac{\partial F_r}{\partial\varphi}\right|_{r,\theta,\varphi} d\varphi\, dr$$

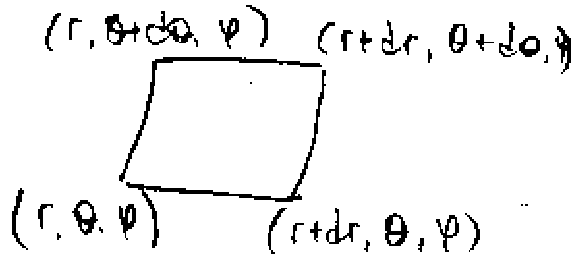
$$\int_{③}\vec{f}\cdot d\vec{l} = -F_\varphi(r+dr,\theta,\varphi)\, d\varphi = -F_\varphi(r,\theta,\varphi)\, d\varphi - \left.\frac{\partial F_\varphi}{\partial r}\right|_{r,\theta,\varphi} dr\, d\varphi$$

$$\int_{④}\vec{f}\cdot d\vec{l} = -F_r(r,\theta,\varphi)\, d\varphi$$

$$\Rightarrow \oint\vec{f}\cdot d\vec{l} = \left(\frac{\partial F_r}{\partial\varphi} - \frac{\partial F_\varphi}{\partial r}\right) dr\, d\varphi$$

$$\Rightarrow \boxed{(\nabla\times\vec{f})_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial f_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\sin\theta\, f_\varphi)\right]}$$

Finally, take $\varphi = \text{const}$:



$$d\vec{s} = r\,dr\,d\theta\,\hat{\varphi}$$

$$\int (\nabla\times\vec{f})\cdot d\vec{s} = (\nabla\times\vec{f})_\varphi\, r\,dr\,d\theta$$

The line integral is

$$\int_{①} \vec{f}\cdot d\vec{\ell} = F_r(r,\theta,\varphi)\,dr$$

$$\int_{②} \vec{f}\cdot d\vec{\ell} = F_\theta(r+dr,\theta,\varphi)\,d\theta = F_\theta(r,\theta,\varphi)\,d\theta + \left.\frac{\partial F_\theta}{\partial r}\right|_{r,\theta,\varphi} dr\,d\theta$$

$$\int_{③} \vec{f}\cdot d\vec{\ell} = -F_r(r,\theta+d\theta,\varphi)\,dr = -F_r(r,\theta,\varphi)\,dr - \left.\frac{\partial F_r}{\partial \theta}\right|_{r,\theta,\varphi} dr\,d\theta$$

$$\int_{④} \vec{f}\cdot d\vec{\ell} = -F_\theta(r,\theta,\varphi)\,d\theta$$

$$\Rightarrow \oint \vec{f}\cdot d\vec{\ell} = \left(\frac{\partial F_\theta}{\partial r} - \frac{\partial F_r}{\partial \theta}\right) dr\,d\theta$$

$$\Rightarrow \boxed{(\nabla\times\vec{f})_\varphi = \frac{1}{r}\left(\frac{\partial}{\partial r}(r f_\theta) - \frac{\partial f_r}{\partial \theta}\right)}$$

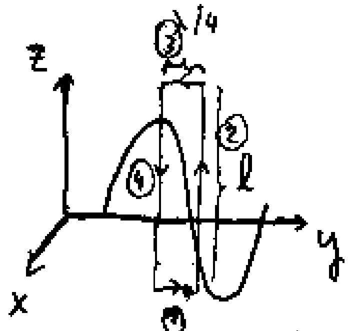
3) Take the configuration:

$$\vec{E} = \hat{z} E_0 \cos(\omega t - ky)$$
$$\vec{B} = \hat{x} B_0 \cos(\omega t - ky)$$

Now check the Maxwell equation

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_s \vec{B}\cdot d\vec{a}$$

Take now a contour in the (y z) plane:



$$\oint_c \vec{E}\cdot d\vec{l} = -E_0 \underbrace{\int \cos\omega t\, dz}_{4} + E_0 \underbrace{\int \cos(\omega t - \pi/2)\, dz}_{2} \quad \left(\text{use } k = \frac{2\pi}{\lambda}\right)$$

$$= E_0 l(-\cos\omega t + \sin\omega t)$$

Compare now to the surface integral obtained from the magnetic field ($d\vec{s}$ points in $+\hat{x}$ direction for our chosen orientation)

$$-\int \frac{\partial \vec{B}}{\partial t}\, d\vec{s} = \int B_0 \omega \sin(\omega t - ky)\, dy\, dz = \frac{B_0 \omega l}{k}\left(+\cos(\omega t - ky)\right)\Big|_0^{\lambda/4}$$

$$= \frac{B_0 \omega l}{k}(\sin\omega t - \cos\omega t)$$

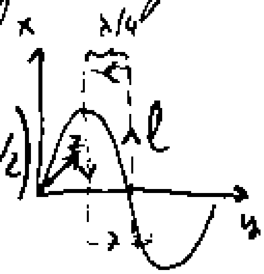
Comparing both results:

$$\frac{B_0 \omega l}{k} = E_0 l \implies \boxed{\frac{E_0}{B_0} = \frac{\omega}{k}}$$

We now need another expression involving E_0, B_0 and c. For this we use:

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0\epsilon_0 \int_s \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$

The line integral for the magnetic field yields (xy-plane)



$$\oint \vec{B}\cdot d\vec{l} = -B_0\int dx \cos\omega t + B_0\int dx \cos(\omega t - \pi/2)$$

$$= B_0 l\,(\sin\omega t - \cos\omega t)$$

The surface integral gives (now for our chosen contour $d\vec{a}$ points in $-\hat{z}$ direction)

$$\mu_0\epsilon_0\int_s \frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} = -\mu_0\epsilon_0\int dx\,dy\,\omega E_0(-\sin(\omega t - ky))$$

$$= \mu_0\epsilon_0 l\omega E_0 \frac{1}{k}\cos(\omega t-ky)\Big|_0^{\lambda/4}$$

$$= \frac{\mu_0\epsilon_0 l\omega E_0}{k}(\sin\omega t - \cos\omega t)$$

$$\Rightarrow \boxed{B_0=\mu_0\epsilon_0\left(\frac{\omega}{k}\right)E_0}$$

Using our previous result:

$$\mu_0\epsilon_0\left(\frac{\omega}{k}\right)^2=1 \Longrightarrow \boxed{v\equiv \frac{\omega}{k}=\frac{1}{\sqrt{\mu_0\epsilon_0}}\equiv c}$$