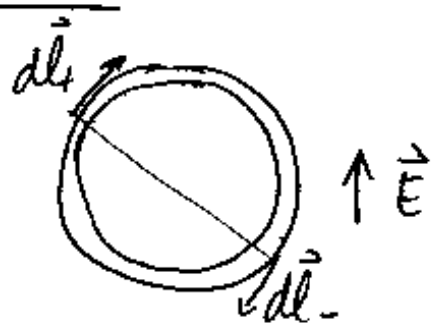
EM - HW 9.

Problem 1: Thomson momentum.



For an infinitesimal path dl, one has for N_+ charges with velocity $\vec{u}_+$ in dl_+ and N_- charges with velocity $\vec{u}_-$ in dl_-:

$$I = \frac{qN_+}{dl_+}u_+ = \frac{qN_-}{dl_-}u_- \Rightarrow N_\pm u_\pm = \frac{I\,dl}{q}$$

For the momentum: $\vec{p} = \gamma m \vec{u}$

$$dp = \gamma_+ m N_+ u_+ - \gamma_- m N_- u_- = \frac{m I dl}{q}(\gamma_+ - \gamma_-)$$

$(\gamma_+ \neq \gamma_-)$

The energy gain(loss) by a charge is equal to the work done (consumed) by the electric field

$$d(\gamma mc^2) = qE\sin\theta\, dl \Rightarrow (\gamma_+ - \gamma_-)mc^2 =$$

$$= \pi R q E \sin\theta$$

So,

$$dp = \frac{m I dl}{q} \frac{\pi R q E \sin\theta}{mc^2} = \frac{\pi R^2 I E \sin\theta\, d\theta}{c^2}$$

$$\Rightarrow \boxed{p = \frac{\pi R^2 I E}{c^2}} \text{ (for positive charges only)}$$

Problem 2:

$$r^2 Y_{lm}(\theta,\varphi) = \begin{cases} f_{lm}(r,z)(x+iy)^m & m>0 \\ f_{lm}(r,z)(x-iy)^{|m|} & m<0 \end{cases}$$

Consider the vector space of all degree 2 polynomials in 3 variables. You can choose a basis

$$\{x^2, y^2, z^2, xy, xz, yz\}$$

and therefore see that its dimension is 6. If you impose that Laplace equation must be obeyed, this gives a constraint and therefore reduces the dimension to 5. Hence we must find 5 polynomials. Its easier to work in variables

$$\begin{cases} u = x+iy \\ v = x-iy \\ z \end{cases} \Rightarrow \{u^2, v^2, z^2, uv, uz, vz\}$$

The laplace operator is, in the new variables:

$$\nabla^2 = 4\partial_u \partial_v + \partial_z^2$$

and so it is straighforward to check that our 5 polynomials are

u^2

v^2

$uv - 2z^2$

uz

vz

They are orthogonal in our natural basis. So, if we use

$$x \pm iy = r \sin\theta e^{\pm i\varphi} \quad ; z = r\cos\theta$$

we immediately get

$$u^2 = (x+iy)^2 = r^2 \sin^2\theta e^{2i\varphi} = r^2 Y_{22}$$

$$v^2 = (x-iy)^2 = r^2 \sin^2\theta e^{-2i\varphi} = r^2 Y_{2,-2}$$

$$uv - 2z^2 = r^2 \sin^2\theta - 2r^2\cos^2\theta$$

$$= r^2(1 - 3\cos^2\theta) = r^2 Y_{2,0}$$

$$uz = r^2 \sin\theta\cos\theta\, e^{i\varphi} = r^2 Y_{2,1}$$

$$vz = r^2 \sin\theta\cos\theta\, e^{-i\varphi} = r^2 Y_{2,-1}$$

where everything has been written modulo normalization constants

Problem 3:

To get the normalizations use

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \, |Y_\ell^m(\theta,\varphi)|^2 = 1$$

Using the results from problem 2 and performing some straightforward integrals:

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$$

$$Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{\pm 2i\varphi}$$