1. Using the fact that the electromagnetic field may be written as a covariant antisymmetric tensor function under Lorentz transformations, work out the effect of a pure boost in the z direction on all components of E and B.

2. For a perfect circuit consisting of a resistanceless battery which maintains its voltage, connected by perfectly conducting wires to either end of a resistive length of wire, sketch the sources and sinks, and the lines of flow, of power in the system. Assuming definite cross sections for the wire and the battery, and uniform current density in both, compute the Poynting vector as a function of distance from the centerline of the wire and the centerline of the battery. Use symbolic, not numerical, values for the parameters of the system.

3. For a body at rest (with no internal motions) in a frame specified by xyzt, compute the energy-momentum tensor function versus x'y'z't', in which the first frame is seen as moving with specified rapidity.
Practice Exam Solutions

1. \[
F_{\mu \nu} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & B_3 & -B_2 \\
-E_2 & B_3 & 0 & B_1 \\
E_3 & B_2 & -B_1 & 0
\end{pmatrix}
\]

\[
(F_{\mu \nu})' = \Lambda \Lambda^{-1} F_{\mu \nu} \Lambda^{-1} \Lambda^{-1}
\]

\[
\Lambda \Lambda^{-1} = \begin{pmatrix}
\cosh y & 0 & 0 & \sinh y \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh y & 0 & 0 & \cosh y
\end{pmatrix}
\]

\[
(F_{01})' = (\cosh y \times 1) F_{01} + (\sinh y \times 1) F_{31}
\]

\[
E_1' = (\cosh y \times E_1) + \sinh y B_2
\]

\[
E_2' = \gamma (E_2 - B_1) = \gamma (E_1 + B_2)
\]

\[
E_3' = (\cosh y \times \cosh y) F_{03} - (\sinh y \times \sinh y) F_{30}
\]

\[
= E_3 - \frac{1}{2}
\]
\[ F'_{12} = (1 \times 1) F_{12} \]
\[ B'_3 = B_3 \]
\[ F'_{13} = -B'_2 = (1 \times \cos h y) F_{13} + (2 \times \sin h y) F'_{10} \]
\[ = \gamma (-B_2 - u E_1) \]
\[ B'_2 = \gamma (B_2 + u E_2) \]
\[ B'_1 = \gamma (B_1 - u E_1) \]

\[ \text{RESISTOR \ sink} \]
\[ \text{Battery \ source} \]

Power flux lines (Actually go out of plane and back)

\[ -2 - \]
Wire or battery with uniform current density.

Power = \frac{E \times B}{4\pi}

Treat E as constant

= \frac{V}{\ell} (-\lambda)

B \varphi = \frac{2 \pi I (r^2 / R^2)}{r}

If wire has radius R,
then \ B \varphi = \frac{2 \pi I (r^2 / R^2)}{r}

\text{for } r \leq R

= \frac{2 \pi I}{r}
\text{for } r > R.

Inside wire \ B \varphi = \frac{2 \pi I R}{4\pi R}

\Theta = \frac{V}{\ell} \cdot \frac{2 \pi I R}{4\pi R}

Total power inside \ r

= 2\pi r \ell \Theta (r) = \frac{IV}{R} r^2

exactly right for resistive loss.

-3-
Another sketch - cross section with battery and resistor perpendicular to plane, current out for battery and in for resistor into plane.

(2 dimensional dipole pattern)
3. In rest frame, $T_{ij} = T_{i0} = 0$. $|T_{ij}| \ll T_{i0}$, so neglect all but $T_{00} = \mathcal{E}(x, y, z)$, independent of $t$. Make boost in $z$ direction.

In lab frame

$$\mathcal{T}^1_{\mu\nu}(x) = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} T^1_{\mu\nu}(x)$$

$x = \Lambda^{-1}_{\mu} x' \quad x = x' \quad y = y' \quad z = \gamma (z' - \nu t')$

because of inverse $\Lambda^{-1}$

$$E'(x') = \gamma v E(x', y', z' - \nu t')$$

$$P(x') = p(x') = \gamma E(x', y', z' - \nu t')$$

$T_{ij}(x') = 0$ unless $i = j = 3$

$$T_{33}(x') = \gamma v^2 E(x', y', z' - \nu t')$$

Note Lorentz contraction $\Rightarrow$

$E' = \gamma E$