PHY 505

Hour Exam No. 2

Solutions

1. There is never any field inside the smaller shell $(r < R_1)$.

(a) (i) Put a charge q on R_1, no charge on R_2

The field is radial & by Gauss's Law.

$$E(r) = \begin{cases} K\, q/r^2 & r > R_2 \\ \dfrac{Kq}{\kappa_2 r^2} & R_1 < r < R_2\,, \end{cases} \qquad k = 1/4\pi\epsilon_0$$

The Potential $\quad V(r) = \int_r^\infty E(r')\,dr'$

and $\quad V(R_2) = \dfrac{Kq}{R_2} = C_{21}^{-1}\, q$

$$V(R_1) = Kq\left[\frac{1}{R} + \frac{1}{\kappa_2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right] = C_{11}^{-1}\, q$$

(ii) Put a charge q on R_2, no charge on R_1

$$E(r) = \begin{cases} Kq/r^2\,, & r > R_2 \\ 0 & r < R_2 \end{cases}$$

and $\quad V(R_2) = \dfrac{kq}{R_2} = C_{22}^{-1}\, q$

$$V(R_1) = \frac{Kq}{R_2} = C_{12}^{-1}\, q$$

That is, $\quad C_{11}^{-1} = K\left\{\dfrac{1}{R_2} + \dfrac{1}{\kappa_2}\left\{\dfrac{1}{R_1} - \dfrac{1}{R_2}\right\}\right\}$

$$C_{12}^{-1} = C_{21}^{-1} = C_{22}^{-1} = K/R_2$$

b) $$U = \frac{1}{2} \sum_{i,j} Q_i C^{-1}_{ij} Q_j$$

with $Q_1 = -Q$, $Q_2 = 2Q$

$$U = \frac{Q^2}{2} \left[C^{-1}_{11} - 4 C^{-1}_{12} + 4 C^{-1}_{22} \right]$$

$$= \frac{kQ^2}{2} \left\{ \frac{1}{R_2} + \frac{1}{K_2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right\}$$

Extra Credit:

with $Q_1 = -Q$, $Q_2 = 2Q$

$$E(r) = \begin{cases} \dfrac{kQ}{r^2} & r > R_2 \\ -\dfrac{kQ}{K_2 r^2} & R_1 < r < R_2 \end{cases}$$

Hence, $$U = \frac{1}{2} \left\{ \int_{R_1}^{R_2} \epsilon \left[\frac{kQ}{K_2 r^2} \right]^2 4\pi r^2 dr + \int_{R_2}^{\infty} \epsilon_0 \left[\frac{kQ}{r^2} \right]^2 4\pi r^2 dr \right\}$$

$$= \frac{kQ^2}{2} \left[\frac{1}{K_2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{1}{R_2} \right]$$

in agreement with the result of (b).

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$$\bar{E}_0 = E_0 \hat{x}$$

$$\& \quad \phi_0 = -E_0 x = -E_0 r\cos\theta$$

We can expand the potential,

$r > a \qquad \phi_1 = -E_0 r\cos\theta + B_0 \ln r$
$$+ \sum_1^\infty \left(B_n \cos(n\theta) + D_n \sin(n\theta) \right) / r^n$$

$r < a \qquad \phi_2 = A_0 + \sum_1^\infty \left(A_n \cos(n\theta) + C_n \sin(n\theta) \right) r^n$

(We have set an arbitrary constant equal to zero in ϕ_1)

Boundary Conditions $\qquad \phi_1(a,\theta) = \phi_2(a,\theta)$
$$\epsilon_0 \frac{\partial \phi_1}{\partial r}\Big|_a = \epsilon \frac{\partial \phi_2}{\partial r}\Big|_b$$

These equations hold independently for the coefficients of each trigonometric term. Except for $\cos\theta$ terms, there are sets of 2 homogeneous equations in 2 unknowns with only the trivial zero solution

For the $\cos\theta$ terms.

$$-E_0 a + B_1/a = A_1 a$$

$$-E_0 - B_1/a^2 = \kappa A_1 \qquad \kappa = \epsilon/\epsilon_0$$

or $\qquad A_1 a^2 - B_1 = -E_0 a^2$
$$\kappa A_1 a^2 + B_1 = -E_0 a^2$$

Solution $\qquad A_1 = \frac{-2}{\kappa+1} E_0$

$$B_1 = \frac{\kappa-1}{\kappa+1} E_0 a^2$$

Therefore $\phi_1 = -E_0 r\cos\theta + \frac{\kappa-1}{\kappa+1} E_0 \, a^2/r \, \cos\theta$

$$= -\vec{E}_0\cdot\vec{r} + \frac{\kappa-1}{\kappa+1} a^2 \frac{\vec{E}_0\cdot\vec{r}}{r^2}$$

$$\phi_2 = \frac{-2}{\kappa+1}\vec{E}_0\cdot\vec{r}$$

And $\vec{E} = -\nabla\phi$ gives

$$\vec{E}_1 = \vec{E}_0\left[1 - \frac{\kappa-1}{\kappa+1}\left(a^2/r^2\right)\right] + \frac{2(\kappa-1)}{(\kappa+1)}\frac{a^2(\vec{E}_0\cdot\vec{r})\vec{r}}{r^4}$$

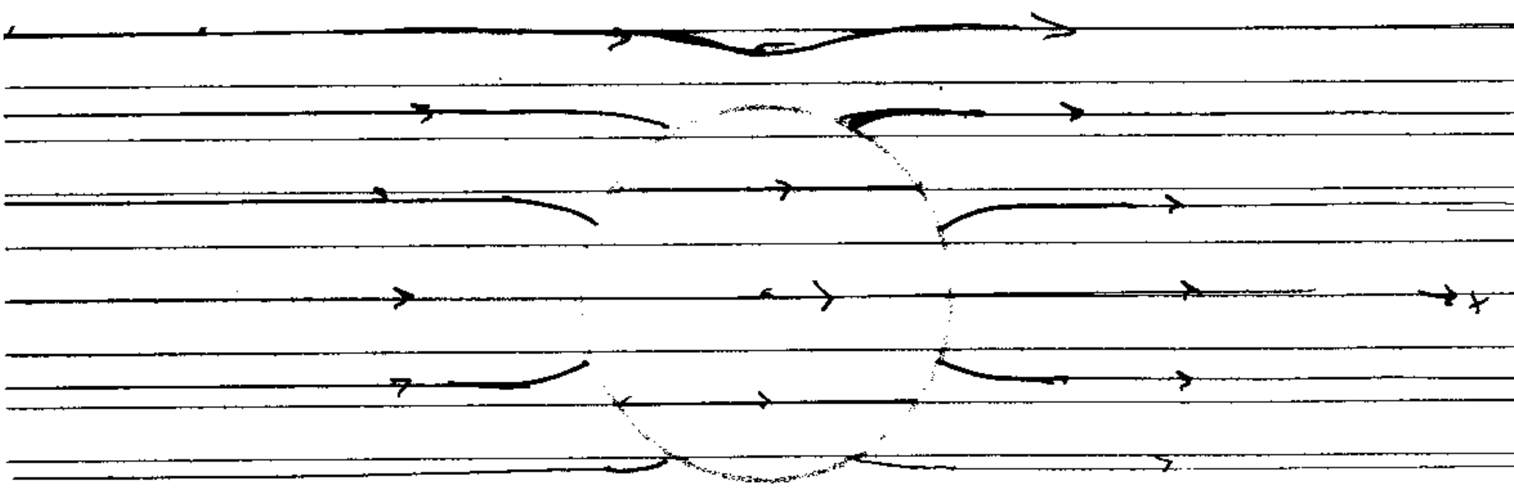
$$E_2 = \frac{2}{\kappa+1}\vec{E}_0$$

b) The Polarization Charge can be determined from

$$(\hat{r}\cdot\vec{E}_1 - \hat{r}\cdot E_2)_{r=a} = \sigma_p/\epsilon_0$$

$$\Rightarrow \sigma_p = \frac{2(\kappa-1)}{(\kappa+1)}\epsilon_0 E_0 \cos\Theta$$

Extra Credit:



$|E_2| < |E_0|$. There are pos. surface charges on $x>0$ half of sphere which act as sources for fieldlines & negative charges on $x<0$ which act as sinks (attractors for fieldlines.