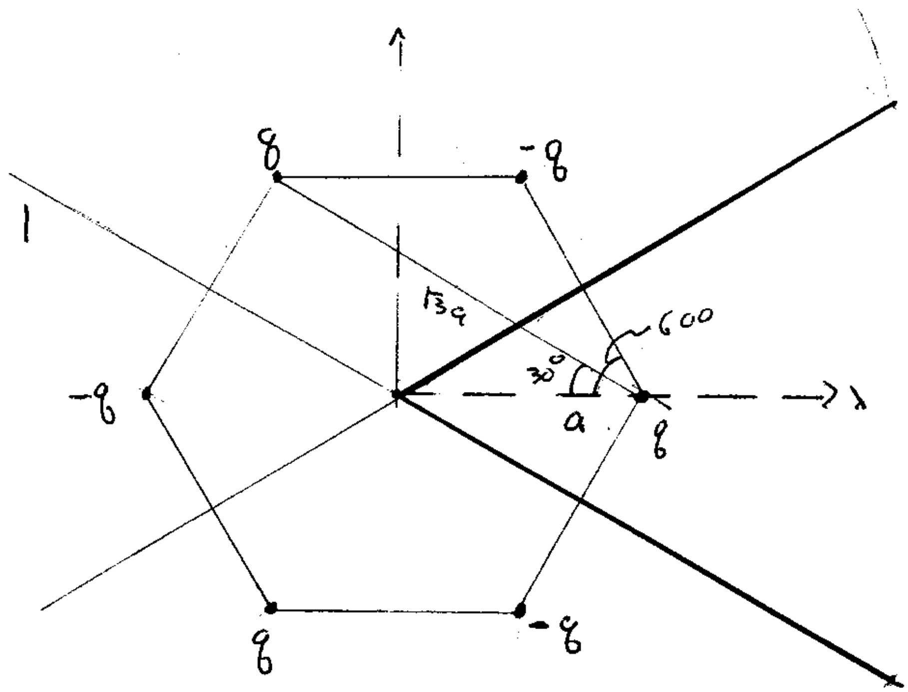
PHY 505
FINAL EXAM
SOLUTIONS




Boundary Conditions can be solved by placing image charges at corners of hexagon as shown

Net force is towards center along x axis.

i.e. $F_x = Kq^2 \left[ -\frac{1}{(2a)^2} + 2\frac{1}{(\sqrt{3}a)^2}\cos 30^0 \right.$

$\left. - 2\frac{1}{(a)^2}\cos 60^0 \right]$ , $K = \frac{1}{4\pi\epsilon_0}$

$= \frac{Kq^2}{a^2}\left[ -\frac{5}{4} + \frac{\sqrt{3}}{3} \right]$

2. As $|\vec{x}| \to \infty$, $\vec{\mathcal{E}} \to \vec{\mathcal{E}}_0 = \mathcal{E}_0 \hat{z}$

and $\phi \to \phi_0 = -\mathcal{E}_0 z = -\mathcal{E}_0 r \cos\theta$,

Bdy Conditions, $\phi_{out}(r=a,\theta) = \phi_{in}(r=a,\theta)$ (1)

$$\epsilon \left.\frac{\partial \phi_{out}}{\partial r}\right|_{r=a} = \epsilon_0 \left.\frac{\partial \phi_{in}}{\partial r}\right|_{r=a} \quad (2)$$

Expanding the potentials in Legendre Polynomials (Axial symmetry), the only non-vanishing terms will be those $\sim P_1(\cos\theta) = \cos\theta$. So we can write

$$\phi_{out} = -\mathcal{E}_0 r\cos\theta + A/r^2 \cos\theta$$

$$\phi_{in} = B r\cos\theta$$

(1) $-\mathcal{E}_0 a + A/a^2 = Ba$

(2) $-\epsilon\{\mathcal{E}_0 + 2A/a^3\} = \epsilon_0 B$

Solution: $A = -\dfrac{(\epsilon-\epsilon_0)}{(2\epsilon+\epsilon_0)} \mathcal{E}_0 a^3$

$$B = \frac{-3\epsilon}{(2\epsilon+\epsilon_0)} \mathcal{E}_0$$

$$\therefore \quad \phi_{out} = -\mathcal{E}_0 z - \frac{(\epsilon-\epsilon_0)}{(2\epsilon+\epsilon_0)} \mathcal{E}_0 \frac{a^3 z}{r^3}$$

$$\phi_{in} = \frac{-3\epsilon}{(2\epsilon+\epsilon_0)} \mathcal{E}_0 z$$

Ans

$$\vec{\mathcal{E}}_{out} = \mathcal{E}_0\left(\hat{z} + \frac{\epsilon-\epsilon_0}{2\epsilon+\epsilon_0}\left[\frac{a^3}{r^3}\hat{z} - \frac{3a^3 z}{r^4}\hat{r}\right]\right)$$

$(\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

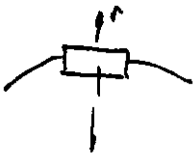
$$\vec{\mathcal{E}}_{in} = \frac{3\epsilon}{2\epsilon+\epsilon_0}\mathcal{E}_0 \hat{z}$$

$$\sigma_p = \epsilon_0\left(\vec{\mathcal{E}}_{out}\cdot\hat{r} - \vec{\mathcal{E}}_{in}\cdot\hat{r}\right)\Big|_a = \frac{-3(\epsilon-\epsilon_0)}{(2\epsilon+\epsilon_0)} \epsilon_0 \mathcal{E}_0 \cos\theta$$

Alternatively, $\vec{P}_{out} = (\epsilon - \epsilon_0)\vec{E}_{out}$

{ $\vec{P}_{in} = 0$ since it is a vacuum! }

From $\rho_p = -\nabla\cdot\vec{P}$, Gauss' law gives:



$$\sigma_p = -\vec{P}_{out}\cdot\hat{r}\,\big|_{r=a}$$

$$= -\frac{3(\epsilon-\epsilon_0)}{(2\epsilon+\epsilon_0)}\,\epsilon_0 E_0 \cos\theta$$

3a) (i) $\int \vec{B} \cdot \vec{H}\, d^3x \;=\; \int \nabla \times \vec{A} \cdot \vec{H}\, d^3x$

Vector Identity: $\nabla \times \vec{A} \cdot \vec{H} = \nabla \cdot (\vec{A} \times \vec{H}) + \vec{A} \cdot \nabla \times \vec{H}$

No free charges: $\nabla \times \vec{H} = 0$

localized magnetization $\lim_{|\vec{x}| \to \infty} H \sim 1/r^2$

$\lim_{|\vec{x}| \to \infty} \sim 1/r$

Then Gauss Theorem $\Rightarrow \int \nabla \cdot (\vec{A} \times \vec{H})\, d^3x = 0$

(ii) No free charges: $\nabla \times \vec{H} = 0$, write $\vec{H} = -\nabla \Phi_m$

$\int \vec{B} \cdot \vec{H}\, d^3x = -\int \vec{B} \cdot \nabla \Phi_m\, d^3x$

Use $\nabla \cdot (B \Phi_m) = B \cdot \nabla \Phi_m = \Phi_m \underbrace{\nabla \cdot B}_{0}$

etc.

b) $\vec{B} = \mu_0 (\vec{H} + \vec{M})$, no free charges, let $\vec{H} = -\nabla \Phi_m$

$0 = \nabla \cdot B = \mu_0 (-\nabla^2 \Phi_m + \nabla \vec{M})$

$$\Rightarrow \quad \Phi_m(\vec{x}) = -\frac{1}{4\pi} \int \frac{\nabla \cdot M(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x$$

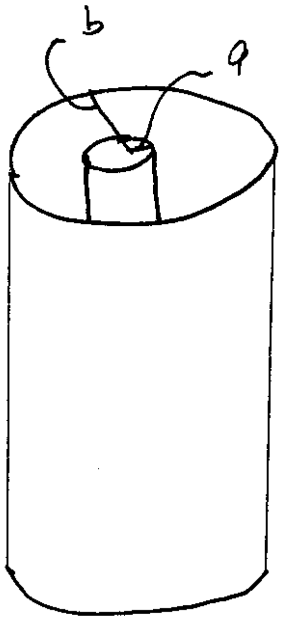
$$= \frac{1}{4\pi} \int_S \frac{\hat{n} \cdot M}{|\vec{x} - \vec{x}'|}\, ds$$

but $\vec{M} \perp \hat{n}$ at surface of cylinder $\Rightarrow \vec{M} = 0$

Hence $\vec{H} = 0$ everywhere

$$\vec{B} = \begin{cases} \mu_0 \vec{M} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

4




$$W = \frac{1}{2} L I^2 = \frac{1}{2\mu_0} \int B^2 \, d^3x$$

By Amperes Law

$$\vec{B} = B \hat{\phi}$$

$$B = \begin{cases} \frac{\mu_0 I}{2\pi} \, r/a^2 & r < a \\ \frac{\mu_0 I}{2\pi} \frac{1}{r} & a < r < b \\ 0 & r > b \end{cases}$$

Hence, Energy stored/unit length $\quad \mathcal{W} = \frac{1}{2} \mathcal{L} I^2$

$$= \frac{1}{(2\mu_0)} \left( \frac{\mu_0 I}{2\pi} \right)^2 \left\{ \int_0^{2\pi} d\theta \left[ \int_0^a (r/a^2)^2 \, r\,dr + \int_a^b \left(\frac{1}{r}\right)^2 r\,dr \right] \right\}$$

$$= \frac{I^2}{2} \left[ \frac{\mu_0}{2\pi} \left[ \frac{1}{4} + \ln(b/a) \right] \right]$$

Ans $\quad \mathcal{L} = \frac{\mu_0}{8\pi} \left[ 1 + 4 \ln(b/a) \right]$

This can also be solved using $\quad W = \frac{1}{2} \int \vec{J} \cdot \vec{A} \, d^3x \quad$ with more work, but

$$W = \frac{\mu_0}{8\pi} \iint \frac{\vec{J}(x) \cdot \vec{J}(x')}{|\vec{x} - x'|} \, d^3x \, d^3x'$$

runs into problems because of the singular current density $\vec{J} = -\frac{I}{(2\pi b)} \delta(r-b) \hat{z}$ in the outer shell.

We can let $\vec{A} = A_z(r)\,\hat{z}$ with $B_\varphi = -\frac{\partial A_z}{\partial r}$

Hence, $A_z = -\frac{\mu_0 I}{4\pi}\, r^2/a^2 + C, \quad r < a$

$$A_z = -\frac{\mu_0 I}{2\pi}\{\ln(r/a) + 1/2\} + C \qquad a < r < b$$

$$A_z = -\frac{\mu_0 I}{2\pi}\{\ln(b/a) + 1/2\} + C \qquad r > b$$

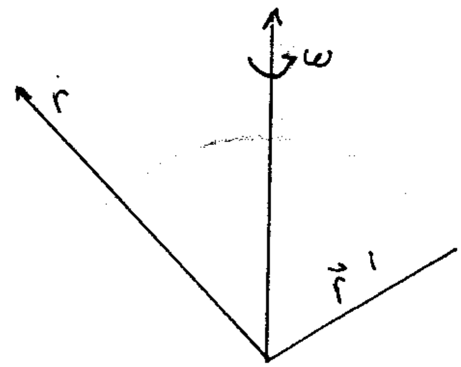
$$W = \frac{1}{2}\int \bar{J}\cdot\bar{A}\,dS = \frac{I^2}{2}\Big\{ \frac{\mu_0}{2\pi}\int d\ell \int r\,dr\, \frac{\delta(r-b)}{2\pi b}\cdot$$

$$\{\ln(b/a) + 1/2\}$$

$$- \frac{\mu_0}{4\pi}\frac{1}{\pi a^2}\int d\ell \int_0^a r\,dr\,(r/a)^2 \Big\}$$

$$= \frac{I^2}{2}\Big\{\frac{\mu_0}{2\pi}\{\ln(b/a) + \tfrac{1}{4}\}\Big\} \quad ✓$$

Extra Credit Problem




$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|}$$

Here we have a surface current.

$$\vec{K} = \sigma\, \vec{\omega} \times \vec{r}'\, \delta(r' - a)$$

$$\sigma = Q / 4\pi a^2$$

Write: $\vec{r}' = (a \sin\theta' \cos\phi',\ a \sin\theta' \sin\phi',\ a \cos\theta')$

$$\vec{\omega} \times \vec{r}' = \omega a \{ -\sin\theta' \sin\phi' \,\hat{i} + \sin\theta' \cos\phi' \,\hat{j} \}$$

Note: $Y_{1\pm1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}$

& $$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi)\, Y_{lm}^*(\theta', \phi')$$

Inserting these results in Biot-Savart Law & using the orthonormality of spherical harmonics

$$\vec{A} = \left(\frac{\mu_0}{4\pi}\right) \frac{Q\omega a}{3} \frac{r_<}{r_>^2} \sin\theta \{ -\sin\phi\, \hat{i} + \cos\phi\, \hat{j} \}$$

Using cylindrical coord. $(\rho, \phi, z)$

$$\hat{\phi} = -\sin\phi\, \hat{i} + \cos\phi\, \hat{j}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\,\frac{Q\omega}{3}\begin{cases} \rho/a\ \hat{\varphi}, & r < a \\ \dfrac{a^2\rho}{[\rho^2+z^2]^{3/2}}\ \hat{\varphi}, & r > a \end{cases}$$

$$\vec{B} = \nabla \times \vec{A} = -\hat{\rho}\,\frac{\partial A_\varphi}{\partial z} + \hat{z}\,\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\varphi)$$

$r < a$, $\quad \vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{2Q\omega}{3a}\hat{z}$ $\quad$ constant field

$r > a$ $\quad \vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{Q\omega a^2}{3}\left[\hat{\rho}\left[\frac{3\rho z}{[\rho^2+z^2]^{5/2}}\right] + \hat{z}\left[\frac{2}{[\rho^2+z^2]^{3/2}} - \frac{3\rho^2}{[\rho^2+z^2]^{5/2}}\right]\right]$

$$= \left(\frac{\mu_0}{4\pi}\right)\frac{Q\omega a^2}{3}\left[\hat{\rho}\left[\frac{3\rho z}{[\rho^2+z^2]^{5/2}}\right] + \hat{z}\left[\frac{2z^2-\rho^2}{[\rho^2+z^2]^{5/2}}\right]\right]$$

Note: Dipole moment of sphere $\vec{m} = \frac{Q\omega a^2}{3}\hat{z}$

$\vec{B}$ above is equal to the dipole potential

$$\vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}$$